The multidimensional Black-Scholes model

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Basket options and control variates

We consider a d-dimensional basket model.

In order to correlate the assets we assume that $(W_t^1, \ldots, W_t^d, t \geq 0)$ is a vector of independent Brownian motions, that Σ is a $d \times d$ matrix, and we define σ_i by

$$\sigma_i = \sqrt{\sum_{j=1}^{d} \Sigma_{ij}^2},$$

and $\bar{W}^i$ by

$$\bar{W}_t^i = \frac{[\Sigma W_t]_i}{\sigma_i} = \frac{\sum_{j=1}^d \Sigma_{ij} W_t^j}{\sigma_i}.$$

$(\bar{W}_t^i, t \geq 0)$ is then a Brownian motion and we assume that each of the d assets has a price S_t^i given by a Black-Scholes model driven by the Brownian motion $\bar{W}^i$

$$\frac{dS_t^i}{S_t^i} = r dt + \sigma_i d\bar{W}_t^i, S_0^i = x_i.$$

In the numerical examples we will set $d = 10$ and $x_i = 100$.

Note that $\mathbf{E}\left(\bar{W}_t^i \bar{W}_t^j\right) = \rho_{ij} t$, where

$$\rho_{ij} = \frac{\sum_{k=1}^d \Sigma_{ik}\Sigma_{jk}}{\sigma_i \sigma_j}$$

In the numerical examples, we will assume that ρ is given by ρ^0 where $\rho_{ij}^0 = 0.5$ for $i \neq j$ and $\rho_{ii}^0 = 1$.

1. Propose a simulation methods for the vector $(\bar{W}_T^1, \ldots, \bar{W}_T^d)$ and $(S_T^1, \ldots, S_T^d)$.

Solution

2. We consider a basket call option on an index I_t given by

$$I_t = a_1 S_t^1 + \cdots + a_d S_t^d.$$

where $a_i > 0$ and $\sum_{i=1}^d a_i = 1$ (in numerical applications we will take $a_1 = \cdots = a_d = 1/d$).

Compute, using a Monte-Carlo method, the price of a call whose payoff is given at time T by

$$(I_T - K)_+,$$

and give an estimate of the error for various values of K ($K = 0.8I_0$, $K = I_0$, $K = 1.2I_0$, $K = 1.5I_0$).

Do the same computation for an index put whose payoff is given by $(K - S_T)_+$.

Solution

3. Prove that $\mathbf{E}(I_T) = I_0 \exp(rT)$. How to use I_T as a control variate ? Relate this method to the call-put arbitrage relation. Test the efficiency of the method for various values of K.

Solution

4. When r and σ are small, justify the approximation of $\log(I_t/I_0)$ by

$$Z_T = \frac{a_1 S_0^1}{I_0} \log(S_t^1/S_0^1) + \cdots + \frac{a_d S_0^d}{I_0} \log(S_t^d/S_0^d).$$

Prove that Z_T is Gaussian with mean

$$T \sum_{i=1}^d \frac{a_i S_0^i}{I_0} \left(r - \sigma_i^2/2\right) \text{ and variance } T \frac{1}{I_0^2} \sum_{i=1}^d \sum_{j=1}^d J_i \rho_{ij} J_j$$

where $J_i = a_i S_0^i \sigma_i$.

We recall the following formula (Black-Scholes formula, exercise)

$$\mathbf{E}\left(\left(e^Z - K\right)_+\right) = e^{\mathbf{E}(Z) + \frac{1}{2}\text{Var}(Z)} N\left(d + \sqrt{\text{Var}(Z)}\right) - KN(d)$$

where $d = \frac{\mathbf{E}(Z) - \log(K)}{\sqrt{\text{Var}(Z)}}$.

Use this formula to give an explicit expression to $\mathbf{E}\left(\left(e^{Z_T} - K\right)_+\right)$ and propose a control variate technique for the computation of the call option.

Compare this method to the standard one for different values of K.

Solution

Black-Scoles model and importance sampling

We consider now the one-dimensional Black-Scholes model

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

Let $S_0 = 100$, $\sigma = 0.3$ (annual volatility) and $r = 0.05$ (annual exponential interest rate).

1. We are interested in the computation of the price of a call option when K is large with respect to S_0.

   Prove, using simulation, that the relative precision of the computation decrease when K increase. Take $S_0 = 100$ and $K = 100, 150, 200, 250$. What happen when $K = 400$?

Solution

2. Prove that :

   $$\mathbf{E}\left(f(W_T)\right) = \mathbf{E}\left(e^{-\lambda W_T - \frac{\lambda^2 T}{2}} f(W_T + \lambda T)\right).$$

   Assume $S_0 = 100$ and $K = 150$, propose a value for λ allowing to reduce variance. Check it simulation.

Solution