The multidimensional Black-Scholes model

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Basket options and control variates

We consider a $d$-dimensional basket model.

In order to correlate the assets we assume that $(W^1_t, \ldots, W^d_t, t \geq 0)$ is a vector of independent Brownian motions, that $\Sigma$ is a $d \times d$ matrix, and we define $\sigma_i$ by

$$\sigma_i = \sqrt{\sum_{j=1}^{d} \Sigma^2_{ij}},$$

and $\bar{W}^i$ by

$$\bar{W}^i_t = \frac{[\Sigma W^i_t]}{\sigma_i} = \frac{\sum_{j=1}^{d} \Sigma_{ij} W^j_t}{\sigma_i}.$$

$(\bar{W}^i_t, t \geq 0)$ is then a Brownian motion and we assume that each of the $d$ assets has a price $S^i_t$ given by a Black-Scholes model driven by the Brownian motion $\bar{W}^i$

$$\frac{dS^i_t}{S^i_t} = r dt + \sigma_i d\bar{W}^i_t, S^i_0 = x_i.$$

In the numerical examples we will set $d = 10$ and $x_i = 100$.

Note that $E(\bar{W}^i_t \bar{W}^j_t) = \rho_{ij} t$, where

$$\rho_{ij} = \frac{\sum_{k=1}^{d} \Sigma_{ik} \Sigma_{jk}}{\sigma_i \sigma_j}.$$

In the numerical examples, we will assume that $\rho$ is given by $\rho^0$ where $\rho_{ij}^0 = 0.5$ for $i \neq j$ and $\rho_{ii}^0 = 1$.

1. Propose a simulation methods for the vector $(\bar{W}^1_T, \ldots, \bar{W}^d_T)$ and $(S^1_T, \ldots, S^d_T)$.

   Solution

2. We consider a basket call option on an index $I_t$ given by

   $$I_t = a_1 S^1_t + \cdots + a_d S^d_t.$$
where $a_i > 0$ and $\sum_{i=1}^d a_i = 1$ (in numerical applications we will take $a_1 = \cdots = a_d = 1/d$).

Compute, using a Monte-Carlo method, the price of a call whose payoff is given at time $T$ by

$$(I_T - K)_+,$$

and give an estimate of the error for various values of $K$ ($K = 0.8 I_0$, $K = I_0$, $K = 1.2 I_0$, $K = 1.5 I_0$).

Do the same computation for an index put whose payoff is given by $(K - S_T)_+$.

**Solution**

3. Prove that $E(I_T) = I_0 \exp(rT)$. How to use $I_T$ as a control variate? Relate this method to the call-put arbitrage relation. Test the efficiency of the method for various values of $K$.

**Solution**

4. When $r$ and $\sigma$ are small, justify the approximation of $\log(I_t/I_0)$ by

$$Z_T = \frac{a_1 S_0^1}{I_0} \log(S_1^1/S_0^1) + \cdots + \frac{a_d S_0^d}{I_0} \log(S_d^d/S_0^d).$$

Prove that $Z_T$ is Gaussian with mean

$$T \sum_{i=1}^d \frac{a_i S_0^i}{I_0} \left( r - \frac{\sigma_i^2}{2} \right)$$

and variance $T \frac{1}{I_0^2} \sum_{i=1}^d \sum_{j=1}^d J_i \rho_{ij} J_j$

where $J_i = a_i S_0^i \sigma_i$.

We recall the following formula (Black-Scholes formula, exercise)

$$E \left( \left( e^{Z_T} - K \right)_+ \right) = e^{E(Z)_+ + \frac{1}{2} \text{Var}(Z)} N \left( d + \sqrt{\text{Var}(Z)} \right) - KN(d)$$

where $d = \frac{E(Z) - \log(K)}{\sqrt{\text{Var}(Z)}}$.

Use this formula to give an explicit expression to $E \left( \left( e^{Z_T} - K \right)_+ \right)$ and propose a control variate technique for the computation of the call option.

Compare this method to the standard one for different values of $K$.

**Solution**
**Black-Scholes model and importance sampling**

We consider now the one-dimensional Black-Scholes model

\[ S_t = S_0 \exp \left( (r - \frac{\sigma^2}{2}) t + \sigma W_t \right). \]

Let \( S_0 = 100 \), \( \sigma = 0.3 \) (annual volatility) and \( r = 0.05 \) (annual exponential interest rate).

1. We are interested in the computation of the price of a call option when \( K \) is large with respect to \( S_0 \).

   Prove, using simulation, that the relative precision of the computation decrease when \( K \) increase. Take \( S_0 = 100 \) and \( K = 100, 150, 200, 250 \). What happen when \( K = 400 \) ?

   **Solution**

2. Prove that :

   \[ \mathbb{E} (f(W_T)) = \mathbb{E} \left( e^{-\lambda W_T - \frac{\lambda^2 T}{2}} f(W_T + \lambda T) \right). \]

   Assume \( S_0 = 100 \) and \( K = 150 \), propose a value for \( \lambda \) allowing to reduce variance. Check it simulation.

   **Solution**