Monte Carlo methods for Option Pricing in the Black and Scholes model

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Preliminaries

1. Write a `Scilab` function computing the empirical mean (`Mean`, the empirical variance `Variance` of an array of numbers.

   Check they are equal to the ones prefined in `Scilab` : `mean`, `variance`.

   Correction

2. Write a function which sample a vector of gaussian independant random variables with mean 0 and variance 1. Note that this function already exists in `Scilab` (`x=rand(1,n,"gauss")`).

   Draw the histogram of the vector and compare it with the law of gaussian random variable with mean 0 and variance 1.

   Correction

3. We want to compute using a Monte-Carlo method $\mathbf{E}(e^{\beta G})$ where G is gaussian with mean 0 and variance 1. We recall that $\mathbf{E}(e^{\beta G}) = \exp(\beta^2/2)$.

   Compute using simulation $\mathbf{E}(e^{\beta G})$ for $\beta = 2, 4, 6, 8, 10 \ldots$. Give a confidence intervall in each case. For whihc value of β can you safely use a Monte-Carlo method ?

   Correction

The Black et Scholes model We consider the Black et Scholes model :

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

In what follows, we will assume that $S_0 = 100$, $\sigma = 0.3$ (annual volatility) and $r = 0.05$ (exponential riskless interest rate).

1. Draw an histogram of the law of W_T and S_T ($T = 1$, $\sigma = 0.3$, $r = 0.05$).

Correction

2. We want to compute the price of a call with strike $K = 100$. Compute this price using a Monte-Carlo method with a number of drawing equal to $N = 1000$,1000,10000. Give a confidence interval in each cases.

Correction

3. We will now use the random variable S_T as a control variate. Check that $\mathbf{E}(S_T) = e^{rt}$ (give a financial argument for this result ?).

   Write a program using S_T as a control variate. Compare the precision of this method with the previous one using various values for K and S_0.

   How is this method related to the call-put arbitrage formula ?

Correction

4. Wa assume that we want to compute a call option with strike K where S_0 is small with respect to K.

   Show that the relative precision of the computation decrease whith S_0/K. Take $S_0 = 100$ and $K = 100$, 150, 200, 250. What happen when $K = 400$?

Correction

5. Prove either directly or by using Girsanov theorem that

$$\mathbf{E}\left(f(W_T)\right) = \mathbf{E}\left(e^{-\lambda W_T - \frac{\lambda^2 T}{2}} f(W_T + \lambda T)\right).$$

   In the case of a call option where $S_0 = 100$ and $K = 150$, propose a value for λ which possibly reduce the simulation variance.

Correction