Pricing and hedging bonds and options in the Vasicek model

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Sampling the short rate The Vasicek model is presented in Chapter **??**. The short rate r_t follows the stochastic differential equation

$$dr(t) = a\,(b - r(t))\,dt + \sigma dW_t, \tag{1}$$

where a, b, σ are positive constants and W is a standard Brownian motion under $\mathbf{P}$. In computer experiments, one can choose $a = 10/\text{year}$, $r_0 = b = 0.05/\text{year}$, $\sigma = 0.1/\sqrt{\text{year}}$.

1. Show that r_h follows a Gaussian distribution with mean $b + e^{-ah}(r_0 - b)$ and variance $\sigma^2 \frac{1-e^{-2at}}{2a}$
2. What is the conditional distribution of r_{t+h} given $r_t = r$?
3. Explain how to sample exactly the vector $(r_{kh}, 0 \le k \le N)$.
4. Implement the suggested algorithm and plot the trajectory $(r_{kh}, 0 \le k \le N)$ for $h = 1$ hour, $h = 1$ day, $h = 1$ week and $N = 100$.

Sampling the zero-coupon bond dynamics We denote by $P(t,T)$ the price at time t of a zero-coupon with maturity date T. We assume that $\mathbf{P}$ is a probability under which all the discounted bond price

$$\tilde{P}(t,T) = e^{-\int_0^t r_s ds} P(t,T),$$

are martingales.

1. We know (see chapter **??**) that the zero coupon bonds can be written as

$$P(t,T) = \exp\left[-(T-t)R(T-t, r(t))\right]$$

with :

$$R(\theta, r) = R_\infty - \frac{1}{a\theta}\left((R_\infty - r)\left(1 - e^{-a\theta}\right) - \frac{\sigma^2}{4a^2}\left(1 - e^{-a\theta}\right)^2 \right).$$

and $R_\infty = \lim_{\theta\to\infty} R(\theta, r) = b - \frac{\sigma^2}{2a^2}$.
Sample the discretised trajectory price of a bond with maturity $T = 1$ $(P(kh,T), 0 \le k \le N$ where $h = 1$ day and is such that $Nh = 1$ year.

Pricing a zero-coupon bond option Here, we consider a call option in the Vasicek model, with maturity θ on a zero-coupon bond with maturity T, $T > \theta$. We want to implement a hedging strategy for this option.

1. Show, using the results of Chapter **??**, that

$$\frac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} = \sigma_s^T dW_t$$

where

$$\sigma_s^T = -\sigma \frac{1 - e^{-a(T-s)}}{a}.$$

2. Show using Proposition **??** that the price C_t of the call option at time t, is given by

$$C_t = P(t,\theta) B\left(t, \frac{P(t,T)}{P(t,\theta)}\right),$$

with

$$B(t,x) = xN(d_1(t,x)) - KN(d_2(t,x)).$$

where N is the cumulative normal distribution function,

$$d_1(t,x) = \frac{\log(x/K) + (\Sigma^2(t,\theta)/2)}{\Sigma(t,\theta)} \quad \text{and} \quad d_2(t,x) = d_1(t,x) - \Sigma(t,\theta).$$

and

$$\Sigma^2(t,\theta) = \int_t^\theta \left(\sigma_s^T - \sigma_s^\theta\right)^2 ds.$$

Implement this formula and plot the option price at time 0 as a function of the strike K.

3. Using Exercise **??** of this chapter, show that

$$C_t = P(t,T)H_t^T + P(t,\theta)H_t^\theta,$$

$H_t^T = N\left(d_1\left(t, \frac{P(t,T)}{P(t,\theta)}\right)\right)$ and $H_t^\theta = -KN\left(d_2\left(t, \frac{P(t,T)}{P(t,\theta)}\right)\right)$.

Implement these formulae and plot the values of H_0^T and H_0^θ as a function of the strike K.

Give a perfect hedging portfolio for the call option using only zero coupon bonds with maturity T and zero coupon bonds with maturity θ.

4. We are interested in studying discrete approximation of this perfect hedging portfolio in which the quantity $\bar{H}_s^T$ of zero coupon bonds with maturity T remains constant on the interval $[kh, (k+1)h]$ and equal to H_{kh}^T. $\bar{H}_s^\theta$, the quantity of zero coupon bonds with maturity θ, is determined using the discrete self-financing condition at times kh.

For a given h (successively chosen to be $h = 1$ day, $h = 1$ week, $h = 1$ month) sample the residual risk of this approximated hedging portfolio. Plot a histogram of the residual risk and study the values of its mean and its variance when h decreases to 0.