ARBITRARY FUNCTIONS PRINCIPLE AND DIRICHLET FORMS

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$$(\Omega, \mathcal{A}, \mathbb{P}) \xrightarrow{Y, Y_n} (E, \mathcal{F})$$

$\mathcal{D}$ an algebra of bounded functions from E into $\mathbb{R}$ or $\mathbb{C}$, dense in $L^2(E, \mathcal{F}, \mathbb{P}_Y)$ containing the constants

$(\alpha_n)_{n \in \mathbb{N}}$ a sequence of positive numbers

$$\text{(H1)} \qquad \begin{cases} \forall \varphi \in \mathcal{D}, \text{ there exists } \overline{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad s.t. \quad \forall \chi \in \mathcal{D} \\ \lim_{n \to \infty} \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)] = \mathbb{E}_Y[\overline{A}[\varphi]\chi]. \end{cases}$$

the expectation $\mathbb{E}_Y$ being relative to the law $\mathbb{P}_Y$.

$$\text{(H2)} \qquad \begin{cases} \forall \varphi \in \mathcal{D}, \text{ there exists } \underline{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad s.t. \quad \forall \chi \in \mathcal{D} \\ \lim_{n \to \infty} \alpha_n \mathbb{E}[(\varphi(Y) - \varphi(Y_n))\chi(Y_n)] = \mathbb{E}_Y[\underline{A}[\varphi]\chi]. \end{cases}$$

$$\text{(H3)} \quad \begin{cases} \forall \varphi \in \mathcal{D}, \text{ there exists } \widetilde{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad s.t. \quad \forall \chi \in \mathcal{D} \\ \lim_{n \to \infty} \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y))] = -2\mathbb{E}_Y[\widetilde{A}[\varphi]\chi]. \end{cases}$$

$$\text{(H4)} \quad \begin{cases} \forall \varphi \in \mathcal{D}, \text{ there exists } \bcancel{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad s.t. \quad \forall \chi \in \mathcal{D} \\ \lim_{n \to \infty} \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) + \chi(Y))] = 2\mathbb{E}_Y[\bcancel{A}[\varphi]\chi]. \end{cases}$$

As soon as two of hypotheses (H1) (H2) (H3) (H4) are fulfilled (with the same algebra $\mathcal{D}$ and the same sequence α_n), the other two follow thanks to the relations

$$\widetilde{A} = \frac{\overline{A} + \underline{A}}{2} \qquad \bcancel{A} = \frac{\overline{A} - \underline{A}}{2}.$$

$\overline{A}$ is called *the theoretical bias operator*.
$\underline{A}$ is called *the practical bias operator*.

Because of the property

$$< \widetilde{A}[\varphi], \chi >_{L^2(\mathbb{P}_Y)} = < \varphi, \widetilde{A}[\chi] >_{L^2(\mathbb{P}_Y)}$$

$\widetilde{A}$ is called *the symmetric bias operator*.

By the fact that most often $A\!\!\!\backslash$ is a first order operator (cf. prop.2 and 3 below) $A\!\!\!\backslash$ is called *the singular bias operator*

The following theorem is the core of our framework (cf. [Bou-06]):

Theorem 1. *Under hypothesis* (H3)

a) the limit

$$\widetilde{\mathcal{E}}[\varphi, \chi] = \lim_n \frac{\alpha_n}{2} \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y)] \qquad \varphi, \chi \in \mathcal{D}$$

defines a closable positive bilinear form whose smallest closed extension is denoted $(\mathcal{E}, \mathbb{D})$.

b) $(\mathcal{E}, \mathbb{D})$ *is a Dirichlet form*

c) $(\mathcal{E}, \mathbb{D})$ *admits a square field operator* Γ *satisfying*
$\forall \varphi, \chi \in \mathcal{D}$

$$\Gamma[\varphi] = \widetilde{A}[\varphi^2] - 2\varphi\widetilde{A}[\varphi]$$

$$\mathbb{E}_Y[\Gamma[\varphi]\chi] = \lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2(\chi(Y_n) + \chi(Y))/2]$$

d) $(\mathcal{E}, \mathbb{D})$ *is local if and only if* $\forall \varphi \in \mathcal{D}$

$$\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^4] = 0.$$

Explaining example. Considering for Y a Brownian motion B indexed by $[0,1]$ as random variable with values in $\mathcal{C}([0,1])$ and taking for Y_ε the approximation $Y_\varepsilon = B + \sqrt{\varepsilon}W$ where W is an independent standard Brownian motion, we may apply the theorem with $\mathcal{D}$ the linear combinations of functions $\varphi(B) = e^{i\int_0^1 f\,dB}$ with regular f say $\mathcal{C}^1_b$.

We have with $\chi(B) = e^{i\int_0^1 g\,dB}$

$$\begin{aligned}\mathbb{E}[(e^{i\int_0^1 f\,dY_\varepsilon} - e^{i\int_0^1 f\,dY})(e^{i\int_0^1 g\,dY_\varepsilon} - e^{i\int_0^1 g\,dY})]\\ = \mathbb{E}[e^{i\int(f+g)\,dY}]\mathbb{E}[(e^{i\sqrt{\varepsilon}\int f dW} - 1)(e^{i\sqrt{\varepsilon}\int g dW} - 1)]\end{aligned}$$

so that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}[(\varphi(Y_\varepsilon)-\varphi(Y)(\chi(Y_\varepsilon)-\chi(Y)] = (-\int_0^1 fg\,dt)e^{-\frac{1}{2}\int_0^1(f+g)^2dt}$$

what may be written $-2 < \widetilde{A}[\varphi], \chi >$ with

$$\widetilde{A}[\varphi] = e^{i\int f\,dB}[-\frac{i}{2}\int f\,dB - \frac{1}{2}\int f^2dt]$$

as seen by an elementary calculation. Hypothesis (H3) is satisfied. The theorem yields the well known Ornstein-Uhlenbeck structure on the Wiener space.

We shall say that an operator B from $\mathcal{D}$ into $L^2(\mathbb{P}_Y)$ is a *first order operator* if it satisfies

$$B[\varphi\chi] = B[\varphi]\chi + \varphi B[\chi] \qquad \forall \varphi, \chi \in \mathcal{D}$$

Proposition 2. *Under* (H1) *to* (H4)

a) the theoretical variance $\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 \psi(Y)]$ *and the* practical variance $\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 \psi(Y_n)]$ *exist and we have* $\forall \varphi, \chi, \psi \in \mathcal{D}$

$$\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y))\psi(Y)] = \mathbb{E}_Y[-\underline{A}[\varphi\psi]\chi + \underline{A}[\psi]\varphi\chi - \overline{A}[\varphi]\chi\psi]$$
$$\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y))\psi(Y_n)] = \mathbb{E}_Y[-\overline{A}[\varphi\psi]\chi + \overline{A}[\psi]\varphi\chi - \underline{A}[\varphi]\chi\psi]$$

b) These two variances coincide if and only if $\cancel{A}$ *is a first order operator, and then are equal to* $\mathbb{E}_Y[\Gamma[\varphi]\psi]$.

Proposition 3. *Under* (H1) *to* (H4) *If there is a real number* $p \geq 1$ *s.t.*

$$\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 |\psi(Y_n) - \psi(Y)|^p] = 0 \quad \forall \varphi, \psi \in \mathcal{D}$$

then $\cancel{A}$ *is first order.*

In particular under (H1) to (H4), if the locality condition of theorem 1 is fulfilled then $\cancel{A}$ is a first order operator.

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Proposition 4. *Under* (H3). *If the form* $(\mathcal{E}, \mathbb{D})$ *(cf. theorem 1) is local, then the* principle of asymptotic error calculus *is valid on*

$$\widetilde{\mathcal{D}} = \{F(f_1, \ldots, f_p) \ : \ f_i \in \mathcal{D}, \quad F \in \mathcal{C}^1(\mathbb{R}^p, \mathbb{R})\}$$

$$\begin{aligned} i.e.\ \lim_n \alpha_n \mathbb{E}[(F(f_1(Y_n), \ldots, f_p(Y_n)) - F(f_1(Y), \ldots, f_p(Y))^2] \\ = \mathbb{E}_Y[\textstyle\sum_{i,j=1}^p F'_i(f_1, \ldots, f_p) F'_j(f_1, \ldots, f_p) \Gamma[f_i, f_j]]. \end{aligned}$$

This result proves a commutation of limits : the asymptotic quadratic error estimated on a function in $\widetilde{\mathcal{D}}$ may be directly obtained using the functional calculus applied on $\mathcal{D}$.

Examples.

0. Preliminary example.

Let $(\mathcal{E}, \mathbb{D})$ be a Dirichlet form on the Hilbert space $L^2(E, \mathcal{F}, m)$ where m is a probability measure and let (P_t) be the strongly continuous contraction semi-group associated with $(\mathcal{E}, \mathbb{D})$.

Let us suppose that the quasi-regularity assumption is fulfilled so that we may construct a Markov process Y_t with P_t as transition semi-group (cf. [Ma-Ro-92] chapter IV §3), and let us suppose also that the domain $\mathcal{D}A$ of the generator $(A, \mathcal{D}A)$ contains an algebra $\mathcal{D}$ of bounded functions with constants dense in L^2. Then for $f \in \mathcal{D}$, the approximate forms

$$\mathcal{E}_t[f] = \frac{1}{t} < f - P_t f, f >_{L^2(m)} = \frac{1}{2t}\mathbb{E}_m[(f(Y_0) - f(Y_t))^2]$$

do converge (increasingly) when $t \downarrow 0$ to $\mathcal{E}[f] = - < Af, f >$. Hence hypothesis (H3) is fulfilled when Y_0 is approximated by Y_t under $\mathbb{P}_m$.

Here, as easily seen, we have

$$\overline{A}[f] = \underline{A}[f] = \widetilde{A}[f] = A[f] \qquad A\!\!\!\backslash[f] = 0 \qquad \forall f \in \mathcal{D}$$

The above properties of Dirichlet forms hold either for local or non-local forms. Since $\frac{1}{2t} \uparrow +\infty$ we see that the hypothesis (H3) may be satisfied with $\alpha_n \uparrow +\infty$ the limit form being nevertheless non-local (cf. (e) of theorem 1).

1. Typical formulae of finite dimensional error calculus.

1.a. Let us consider a triplet of real random variables (Y, Z, T) and a real random variable G independent of (Y, Z, T) centered with variance one. We are interested in the approximation Y_ε of Y given by

$$Y_\varepsilon = Y + \varepsilon Z + \sqrt{\varepsilon} T G.$$

In the multidimensional case, Y is with values in $\mathbb{R}^p$ as Z, T is a $p \times q$-matrix and G is independent of (Y, Z, T) with values in $\mathbb{R}^q$, centered, square integrable, such that $\mathbb{E}[G_i G_j] = \delta_{ij}$.

Operator $\overline{A}$.

Proposition. *If Z and T are square integrable, if φ is $\mathcal{C}^2$ bounded with bounded derivatives of first and second orders ($\varphi \in \mathcal{C}_b^2$) and if χ is bounded,*

$$\frac{1}{\varepsilon}\mathbb{E}[(\varphi(Y_\varepsilon) - \varphi(Y))\chi(Y)] \to \mathbb{E}_Y[\overline{A}[\varphi]\chi]$$

where $\overline{A}[\varphi](y) = \mathbb{E}[Z|Y = y]\varphi'(y) + \frac{1}{2}\mathbb{E}[T^2|Y = y]\varphi''(y)$.

In the multidimensional case

$$\overline{A}[\varphi](y) = \mathbb{E}[Z^t|Y = y]\nabla\varphi(y) + \frac{1}{2}\sum_{ij}\mathbb{E}[(TT^t)_{ij}|Y = y]\varphi''_{ij}(y).$$

1.b. Polya's urn, tails of martingales.

Let us consider the Polya's urn in its simplest configuration with two colors, one ball added each time, and an initial composition of one white ball and one black ball.

The ratio X_n of white balls after the n-th drawing satisfies

$$X_{n+1}(n+3) = X_n(n+1) + 1_{U_{n+1} \leq X_n}$$

where U_n is a r.v. uniformly distributed on $[0,1]$ independent of $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$, i.e.

$$X_{n+1} = X_n + \frac{1}{n+3}(1_{U_{n+1} \leq X_n} - X_n).$$

Let X be the (a.s. and $L^p,\ 1 \leq p < +\infty$) limit of the bounded martingale X_n, we study the approximation of X by X_n.

Taking for $\mathcal{D}$ the functions of class $\mathcal{C}^3$ on $[0,1]$ vanishing at 0 and 1, we obtain

$$\lim_n n\mathbb{E}[(\varphi(X) - \varphi(X_n)\chi(X_n)] = \frac{1}{12}\mathbb{E}[\varphi''(X)\chi(X)]$$

$$\lim_n n\mathbb{E}[(\varphi(X) - \varphi(X_n)^2] = \frac{1}{6}\mathbb{E}[\varphi'^2(X)].$$

Hence (H1) to (H4) are fulfilled $\underline{A}[\varphi] = \frac{1}{12}\varphi''$ and $\widetilde{A}[\varphi] = \frac{1}{12}\varphi''$ so that $\underline{A} = \overline{A}$ and $\not{A} = 0$. The limit error structure is the uniform error structure on $[0,1]$.

More generally, this kind of asymptotic behavior appears, under regularity assumptions, for the approximation between a martingale and its limit.

2. Natural inaccuracy of the Brownian motion simulated by the Donsker theorem.

We begin with the simplest case of one dimensional marginal laws which is here nothing else than the central limit theorem.

2.a. Natural inaccuracy in the central limit theorem.

Let be $S_p = \sum_{i=1}^p V_i$ where the randon variables V_i are i.i.d. centered with variance σ^2. We consider two indices m and n linked by the relation $n = n(m) = m + k(m)$ with $\theta\sqrt{m} \leq k(m) \leq \frac{1}{\theta}\sqrt{m}$ for a $\theta \in]0,1[$.

We consider the mutual approximation of $\frac{1}{\sqrt{m}}S_m$ and $\frac{1}{\sqrt{n}}S_n$ (which is an obvious extension of our framework).

For $\overline{A}$ we have to study

$$\alpha(m)\mathbb{E}[(\varphi(\frac{1}{\sqrt{m}}S_m) - \varphi(\frac{1}{\sqrt{n}}S_n))\chi(\frac{1}{\sqrt{n}}S_n)]$$

with $\alpha(m) = \frac{m}{k(m)}$ (so that $\theta\sqrt{m} \leq \alpha(m) \leq \frac{1}{\theta}\sqrt{m}$). For the algebra $\mathcal{D}$ we take the linear combinations of imaginary exponentials.

Proposition. *Suppose the V_i's possess a third order moment, then hypotheses* (H1) *to* (H4) *are fulfilled and for $\varphi \in \mathcal{D}$*

$$\overline{A}[\varphi](x) = \underline{A}[\varphi](x) = \widetilde{A}[\varphi](x) = \frac{1}{2}\sigma^2\varphi'' - \frac{1}{2}x\varphi'.$$

The Dirichlet form is the Ornstein-Uhlenbeck form on $\mathbb{R}$ (endowed with the normal law $\mathcal{N}(0,\sigma^2)$).

2.b. The Donsker case.

Let the V_i's be as before and

$$X_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} V_k + (nt - [nt] V_{[nt]+1} \right)$$

for $t \in [0,1]$, $[nt]$ denoting the entire part of nt.

For the algebra $\mathcal{D}$ we take the linear combinations of exponential of the form $\varphi(X) = \exp\{iX(f)\}$ where $X(f) = \int_0^1 f(s)\,dX(s)$ and with $f \in \mathcal{C}^1$ in order that $\int_0^1 f(s)\,dX(s)$ may be defined as $X(1)f(1) - X(0)f(0) - \int_0^1 X(s)df(s)$ for the general coordinate process $X(s)$ on $\mathcal{C}[0,1]$. As easily seen the algebra $\mathcal{D}$ is dense in $L^2(\mathcal{C}([0,1]),\mu)$ μ being the Wiener measure.

Thus we have $X_n(f) = \sqrt{n}\sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} f(s)\,ds\,V_{k+1}$ and for studying the operator $\overline{A}$ we have to look at

$$M_m = \alpha(m)\mathbb{E}[(\varphi(X_m) - \varphi(X_n))\chi(X_n)] = \alpha(m)\mathbb{E}[(e^{iX_m(f)} - e^{iX_n(f)})e^{iX_n(g)}]$$

We take as before $\alpha(m) = m/k(m)$.

Proposition. *Suppose the V_i's possess a third order moment, then hypotheses* (H1) *to* (H4) *are fulfilled. We have $\overline{A} = \underline{A} = \widetilde{A}$ on $\mathcal{D}$. The Dirichlet form is the Ornstein-Uhlenbeck form on the Wiener space (with a Brownian motion s.t. $< B >_t = \sigma^2 t$) normalized so that the square field operator satisfies $\Gamma[\int_0^1 h(s)\,dB_s] = \int_0^1 h^2(s)\,\sigma^2\,ds \quad \forall h \in L^2([0,1])$.*

Since the Dirichlet form is local, some limits are automatically obtained (proposition 4). Since $\not{A} = 0$, the theoretical and practical variances coincide.

3. Stochastic differential equations and Euler scheme.

Let $X = (X^i)_{i=1,\dots,d}$ be a continuous semi-martingale with values in $\mathbb{R}^d$ vanishing at zero defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. For $t \in [0,1]$ we consider the q-dimensional s.d.e.

$$dY_t = f(Y_t)\,dX_t \qquad Y_0 = y_0 \tag{1}$$

where $y_0 \in \mathbb{R}^q$, f is $\mathcal{C}^1$ from $\mathbb{R}^q$ into $\mathbb{R}^{q+d}$ with at most linear growth ($|f(x)| \leq K(1+|x|)$ denoting $|.|$ the norms on $\mathbb{R}^k$). It is known that (1) has a unique strong solution. We study the resolution of (1) by the Euler scheme :

$$dY_t^n = f(Y^n_{\frac{[nt]}{n}})\,dX_t \qquad Y_0^n = y_0$$

where $[nt]$ is the entire part of nt.

We denote $U_t^n = Y_t^n - Y_t$ the error process. U^n as process with values in $\mathcal{C}([0,1])$ tends to zero in probability (as soon as f is locally Lipschitz with at most linear growth [Ja-Pro-98]).

It is supposed that $X = M + A$ where M is a continuous local martingale vanishing at zero with values in $\mathbb{R}^d$ and A is a continuous finite variation adapted process vanishing at zero satisfying

$$\begin{array}{l} A_t^i = \int_0^t a_s^i\, ds \;\text{ with }\; \int_0^1 (a_s^i)^2\, ds < +\infty \text{ a.s.} \\ < M^i, M^j >_t = \int_0^t c_s^{ij}\, ds \;\text{ with }\; \int_0^1 (c_s^{ij})^2 ds < +\infty \text{ a.s.} \end{array}$$

then for every starting point y_0 and for all function f $\mathcal{C}^1$ with at most linear growth, the process $\sqrt{n}U^n$ converges in law on $\mathcal{C}([0,1])$ to the solution to

$$dU_t^i = \sum_{j=1}^{d} \sum_{k=1}^{q} \frac{\partial f^{ij}}{\partial x_k}(Y_t) \left[U_t^k\, dX_t^j - \sum_{\ell=1}^{d} f^{k\ell}(Y_t)\, dZ_t^{\ell j} \right], \qquad U_0^i = 0,$$

Z being given by

$$Z_t^{ij} = \frac{1}{2} \sum_{k,\ell=1}^{q} \int_0^t \sigma_s^{ik} \sigma_s^{j\ell}\, dW_s^{k\ell}$$

where W is a standard q^2-dimensional Brownian motion defined on an extension of the space independent of X and σ is a matrix of processes s.t. $(\sigma\sigma^t)^{ij} =< M^i, M^j >$ which exists as soon as $q \geq d$ case to which the question may be always reduced.

The result is due to Kurtz and Protter, see also [Ja-Pro-98]

In order to study the hypotheses (H1) to (H4) we consider the algebra $\mathcal{D}$ of the linear conbinations of functions φ defined on $\mathcal{C}([0,1])$ by

$$\varphi(Y) = e^{i<u_1, Y_{t_1}> + \cdots + i<u_r, Y_{t_r}>} \qquad u_\ell \in \mathbb{R}^q \qquad t_\ell \in [0,1] \qquad \ell = 1, \ldots, r$$

and the sequence $\alpha_n = n$.

a) Symmetric bias operator.

We study $n\mathbb{E}[(\varphi(Y^n)-\varphi(Y))^2]$.

Under natural hypotheses we have

$$n\mathbb{E}[(\varphi(Y^n)-\varphi(Y))^2] \to \mathbb{E}\left[\left(\sum_{j=1}^{q}\sum_{\ell=1}^{r} U^j_{t_\ell}\frac{\partial\varphi}{\partial y^j_{t_\ell}}(Y)\right)^2\right]. \tag{2}$$

Considering that X and W are defined on a product space whose samples are denoted ω and $\hat{\omega}$, formula (2) shows that if hypothesis (H3) is verified and if $n|U^n_t|^2$ is uniformly integrable, the limit Dirichlet form satisfies $Y_t \in \mathbb{D}$ and its square field operator satisfies $\Gamma[Y^j_t] = \hat{\mathbb{E}}[(U^j_t)^2]$.

In other words, the limit process $U(\omega,\hat{\omega})$ appears to be *a gradient* in the sense of Dirichlet forms of the process Y : we may write

$$(Y_t)^{\#} = \int_0^1 D_s Y_t \, dW_s = U_t(\omega,\hat{\omega})$$

and formula (2) follows by the chain rule.

The remaining question is whether the form defined on $\mathcal{D}$ by (2) is closable in $L^2(\mathcal{C}([0,1]),\mathbb{P}_Y)$. To this question we have yet only an answer in the simplest case where $q=1$. When

$$dY_t = a(Y_t,t)dB_t + b(Y_t,t)dt$$

with a, b $\mathcal{C}^1$ with at most linear growth, the process U is given by

$$U_t = N_t \int_0^t \frac{a(Y_s,s)a'_y(Y_s,s)}{\sqrt{2}N_s}\, dW_s$$

with

$$N_t = \exp\{\int_0^t a'_y(Y_s,s)dB_s - \frac{1}{2}\int_0^t a'^2_y(Y_s,s)ds + \int_0^t b'_y(Y_s,s)ds\}.$$

Let us denote $(\mathcal{E}^\theta_{ou}, \mathbb{D}^\theta_{ou})$ the Dirichlet form on the Wiener space of type Ornstein-Uhlenbeck with deterministic weight θ, and let us denote D^θ_{ou} its gradient operator defined with the auxiliary Hilbert space $L^2([0,1],dt)$. We have

Proposition. *If the coefficient a satisfies* $\mathbb{E}\int_0^1 a'^2_y(Y_s,s)ds < +\infty$ *and if* $a'^2_y(Y_s,s) \geq \theta(s) > 0$, *hypothesis* (H3) *is fulfilled. The asymptotic Dirichlet form is the image by* Y *of the form* $(\mathcal{E}_w, \mathbb{D}_w)$ *defined on the Wiener space by*

$$\mathbb{D}_w = \{F \in \mathbb{D}^\theta_{ou} \ : \ \int_0^1 \mathbb{E}[(D^\theta_{ou}[F](t))^2 \frac{a'^2_y(Y_t,t)}{\theta(t)}]dt < +\infty\}$$

$$\mathcal{E}_w[F] = \frac{1}{4}\int_0^1 \mathbb{E}[(D^\theta_{ou}[F](t))^2 \frac{a'^2_y(Y_t,t)}{\theta(t)}]dt.$$

The proof has been exposed at the Fifth Seminar on Stochastic Analysis, Random Fields and Application at Ascona in 2005 and will appear in the proceedings.

The form $(\mathcal{E}_w, \mathbb{D}_w)$ admits the square field operator

$$\Gamma_w[F] = \frac{1}{2}\int_0^1 (D^\theta_{ou}[F](t))^2 \frac{a'^2_y(Y_t,t)}{\theta(t)}dt.$$

The operator $\widetilde{A}$ is given by $\widetilde{A}[\varphi](y) = \mathbb{E}[A_w[\varphi(Y)]|Y=y]$ where $\xi_t = \frac{1}{2}a'^2_y(Y_t,t)$, and $A_w[\varphi(Y)] = -\frac{1}{2}\delta^\theta_{ou}[\frac{\xi}{\theta}D^\theta_{ou}[F]]$, and δ^θ_{ou} being the Skorokod stochastic integral operator associated with $(\mathcal{E}^\theta_{ou}, \mathbb{D}^\theta_{ou})$.

b) The theoretical bias operator.

The operator $\overline{A}$ involves an iterated gradient.

The main part of the calculation has been performed by Malliavin and Thalmaier ([Ma-Tha-03] and [Ma-Tha-05]) and we adopt their hypotheses : Y is solution of the s.d.e.

$$dY_t = a(Y_t)dB_t + b(Y_t)dt$$

where B is a $(d-1)$-dimensional Brownian motion and where the matrix a and the function b are $\mathcal{C}^\infty$ with bounded derivatives.

The operator $\overline{A}$ is given by $\lim_n n\mathbb{E}[(\varphi(Y^n) - \varphi(Y))\chi(Y)]$.

We can see that the operator $\overline{A}$ is the image by Y of a singular distribution operator on the Wiener space. For marginals of order one, it is carried by the diagonal of the second chaos and involves the four first derivatives of the coefficients a and b of the SDE.

II. The case of degenerated conditional laws.

$$(\Omega, \mathcal{A}, \mathbb{P}) \overset{Y, Y_n}{\longmapsto} (E, \mathcal{F})$$

Most often, when the Dirichlet form exists and does not vanish, the conditional law of Y_n given $Y = y$ is not reduced to a Dirac mass, and the variance of this conditional law yields the square field operator Γ. All the preceding examples display such situation. Similarly when the approximation is deterministic, i.e. when Y_n is a function of Y say $Y_n = \eta_n(Y)$, then *most often* the symmetric bias operator $\tilde{A}$ and the Dirichlet form vanish. For instance we have :

Proposition. *Suppose* $Y_n = \eta_n(Y)$. *If for* $\alpha_n \to +\infty$ *and an algebra* $\mathcal{D}$,

$$\alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)] \to < \overline{A}[\varphi], \chi >_{\mathbb{P}_Y} \qquad \forall \varphi \in \mathcal{D}, \quad \forall \chi \in L^2(\mathbb{P}_Y),$$

then (H1) *to* (H4) *hold,* $\overline{A} = -\underline{A} = \not{A}$ *are first order operators and* $\widetilde{A} = 0$.

Proof. The sequence $\alpha_n(\mathbb{E}[\varphi(Y_n)|Y = y] - \varphi(y))$ is weakly bounded in $L^2(\mathbb{P}_Y)$ hence strongly bounded, i.e.

$$\alpha_n^2 \int (\mathbb{E}[\varphi(Y_n)|Y = y] - \varphi(y))^2 \, \mathbb{P}_Y(dy) \leq K.$$

Now $\mathbb{E}[\varphi(Y_n)|Y = y] = \varphi(\eta_n(y))$, hence

$$\alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2] = \alpha_n \int (\varphi(\eta_n(y)) - \varphi(y))^2 \, \mathbb{P}_Y(dy) \leq \frac{K}{\alpha_n} \to 0.$$

The Dirichlet form is zero, hence it is local and $\not{A}$ is a first order operator. $\diamond$

For example let us consider the ordinary differential equation

$$x_t = x_0 + \int_0^t f(x_s) y_s ds$$

approximated by the Euler scheme

$$x_t^n = x_0 + \int_0^t f(x^n_{[ns]/n}) y_s ds$$

even if we suppose x_0 to be random, errors are of deterministic nature and as soon as f is $\mathcal{C}^1$ with at most linear growth and $\int_0^1 y_s^2 ds < +\infty$ we have for $\varphi, \chi \in \mathcal{C}_b^1$ (bounded with bounded derivative)

$$\begin{aligned} n\mathbb{E}[(\varphi(x_t^n) - \varphi(x_t))\chi(x_t)] &\to \mathbb{E}[u_t \varphi'(x_t)\chi(x_t)] \\ n\mathbb{E}[(\varphi(x_t) - \varphi(x_t^n))\chi(x_t^n)] &\to -\mathbb{E}[u_t \varphi'(x_t)\chi(x_t)] \\ \text{and} \quad \mathbb{E}[(\varphi(x_t^n) - \varphi(x_t)^2] &\to 0 \end{aligned}$$

where u_t is given by $u_t = -\frac{1}{2}\int_0^t f'(x_s) f(x_s) y_s^2 e^{\int_s^t f'(x_\alpha) y_\alpha d\alpha} ds$. (cf. [Ja-Pro-98] theorem 1.1). Thus

$$\overline{A}[\varphi](x) = \mathbb{E}[u_t \varphi'(x_t) | x_t = x] = -\underline{A}[\varphi](x)$$

and we have $\not{A} = \overline{A}$ and $\widetilde{A} = 0$. $\diamond$

Nevertheless, there are important cases where the conditional law of Y_n given Y is a Dirac mass, i.e. Y_n is a deterministic function of Y, and where the approximation of Y by Y_n yields even so a non zero Dirichlet form on $L^2(\mathbb{P}_Y)$.

This phenomenon is interesting, insofar as randomness (here the Dirichlet form) is generated by a deterministic device.

Example. We give here a simple example where both the conditional law of Y_n given $Y = y$ and the conditional law of Y given $Y_n = y$ are Dirac measures and where nevertheless the approximation gives rise to a non-zero Dirichlet form.

Let us consider the unit interval and the dyadic representation of real numbers. If Y is uniformly distributed we may write $Y = \sum_{k=0}^{\infty} \frac{a_k}{2^{k+1}}$ where the a_k are i.i.d. with law $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

Let us approximate Y by $Y_n = \sum_{k=0}^{n-1} \frac{a_k}{2^{k+1}} + \frac{1}{2}\sum_{k=n}^{\infty} \frac{a_k}{2^{k+1}}$. We see that Y and Y_n are deterministically linked :

$$Y_n = Y - \frac{1}{2^{n+1}}\{2^n Y\} \qquad Y = Y_n + \frac{1}{2^n}\{2^n Y_n\}.$$

Now, it is easily seen that on the algebra $\mathcal{D} = \mathcal{L}\{e^{2i\pi kx}, k \in \mathbb{Z}\}$ we have

$$3.4^n\mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\psi(Y_n) - \psi(Y))] \rightarrow \mathbb{E}[\varphi'\overline{\psi'}],$$

what gives in the real domain the Dirichlet form $\mathcal{E}[\varphi] = \frac{1}{2}\mathbb{E}[\varphi'^2]$. ◇

In its simplest form, the phenomenon appears precisely when a quantity is measured by a graduated instrument when looking for the asymptotic limits as the graduation fines down. Hence the phenomenon is surprisingly rather usual.

III. The Arbitrary Functions Principle, extensions to the Wiener space.

Let us denote $\{x\}$ the fractional part of the real number x and $\stackrel{d}{\Longrightarrow}$ the weak convergence of random variables. Let (X, Y) be a pair of random variables with values in $\mathbb{R} \times \mathbb{R}^r$, we refer to the following property or its extensions as the arbitrary functions principle:

$$(\{nX\}, Y) \quad \stackrel{d}{\Longrightarrow} \quad (U, Y)$$

where U is uniformly distributed on $[0,1]$ independent of Y.

This property is satisfied when X has a density or more generally a characteristic function vanishing at infinity. (cf. Poincaré (1912) Chap. VIII §92 and §93; Hopf (1936)). It yields an approximation property of X by the random variable $X_n = X - \frac{1}{n}\{nX\} = \frac{[nX]}{n}$ where $[x]$ denotes the entire part of x:

Proposition. *Let X be a real random variable with density and Y a random variable with values in $\mathbb{R}^r$. Let $X_n = \frac{[nX]}{n}$*

a) For all $\varphi \in \mathcal{C}^1 \bigcap \mathrm{Lip}(\mathbb{R})$ and for all integrable random variable Z,

$$(n(\varphi(X_n) - \varphi(X)), Y) \quad \stackrel{d}{\Longrightarrow} \quad (-U\varphi'(X), Y)$$

$$n^2\mathbb{E}[(\varphi(X_n) - \varphi(X))^2 Z] \quad \to \quad \frac{1}{3}\mathbb{E}[\varphi'^2(X)Z]$$

where U is uniformly distributed on $[0,1]$ independent of (X, Y).

b) $\forall \psi \in L^1([0,1])$ one has $(\psi(n(X_n - X)), Y) \quad \stackrel{d}{\Longrightarrow} \quad (\psi(-U), Y)$ under any probability measure $\tilde{\mathbb{P}} \ll \mathbb{P}$.

We extend such results to random variables defined on the Wiener space.

Periodic isometries.

Let (B_t) be a standard d-dimensional Brownian motion and let m be the Wiener measure, law of B. Let $t \mapsto M_t$ be a bounded deterministic measurable map, periodic with unit period, into the space of orthogonal $d \times d$-matrices such that $\int_0^1 M_s ds = 0$ (e.g. a rotation in $\mathbb{R}^d$ of angle $2\pi t$). The transform $B_t \mapsto \int_0^t M_s dB_s$ defines an isometric endomorphism in $L^p(m), 1 \leq p \leq \infty$. Let be $M_n(s) = M(ns)$ and $T_n = T_{M_n}$.

Proposition 2. *Under m we have*

$$(T_n(X), B) \quad \stackrel{d}{\Longrightarrow} \quad (X(w), B).$$

The weak convergence acts on $\mathbb{R} \times \mathcal{C}([0,1])$ and $X(w)$ denotes a random variable with the same law as X had under m function of a Brownian motion W independent of B.

Approximation of the Ornstein-Uhlenbeck structure.

From now on, we assume for simplicity that (B) is one-dimensional. Let θ be a periodic real function with unit period such that $\int_0^1 \theta(s)ds = 0$ and $\int_0^1 \theta^2(s)ds = 1$. We consider the transform R_n of the space $L^2_{\mathbb{C}}(m)$ defined by its action on the Wiener chaos:

If $X = \int_{s_1<\cdots<s_k} \hat{f}(s_1,\ldots,s_k)dB_{s_1}\ldots dB_{s_k}$ for $\hat{f} \in L^2_{sym}([0,1]^k,\mathbb{C})$,

$$R_n(X) = \int_{s_1<\cdots<s_k} \hat{f}(s_1,\ldots,s_k)e^{i\frac{1}{n}\theta(ns_1)}dB_{s_1}\ldots e^{i\frac{1}{n}\theta(ns_k)}dB_{s_k}.$$

R_n is an isometry from $L^2_{\mathbb{C}}(m)$ into itself.

From $n(e^{\frac{i}{n}\sum_{p=1}^k \theta(ns_p)} - 1) = i\sum_{p=1}^k \theta(ns_p)\int_0^1 e^{\alpha\frac{i}{n}\sum_p \theta(ns_p)}d\alpha$ it follows that if X belongs to the k-th chaos

$$\|n(R_n(X) - X)\|^2_{L^2} \le k^2\|X\|^2_{L^2}\|\theta\|^2_\infty.$$

In other words, denoting A the Ornstein-Uhlenbeck operator, $X \in \mathcal{D}(A)$ implies

$$\|n(R_n(X) - X)\|_{L^2} \le 2\|AX\|_{L^2}\|\theta\|_\infty.$$

Proposition. $\forall X \in \mathcal{D}(A)$

$$(-in(R_n(X) - X), B) \quad \stackrel{d}{\Longrightarrow} \quad (X^\#(\omega,w), B)$$

where W *is an Brownian motion independent of* B *and* $X^\# = \int_0^1 D_sX\,dW_s$.

$$n^2\mathbb{E}[|R_n(X) - X|^2] \to 2\mathcal{E}[X]$$

where $\mathcal{E}$ *is the Dirichlet form associated with the Ornstein-Uhlenbeck operator.*

Proof. If X belongs to the k-th chaos, expanding the exponential by its Taylor series gives

$$n(R_n(X)-X) = i\int_{s_1<\cdots<s_k} \hat{f}(s_1,\ldots,s_k)\sum_{p=1}^{k}\theta(ns_p)dB_{s_1}\ldots dB_{s_k} + Q_n$$

with $\|Q_n\|^2 \leq \frac{1}{4n}k^2\|\theta\|_\infty^2\|X\|^2$.

Then using that $\int_{s_1<\cdots<s_p<\cdots<s_k} h(s_1,\ldots,s_k)\theta(ns_p)dB_{s_1}\ldots dB_{s_p}\ldots dB_{s_k}$ converges stably to $\int_{s_1<\cdots<s_p<\cdots<s_k} h(s_1,\ldots,s_k)dB_{s_1}\ldots dW_{s_p}\ldots dB_{s_k}$ one gets

$$-in(R_n(X)-X) \quad \stackrel{s}{\Longrightarrow} \quad \begin{array}{l} \int_{t<s_2<\cdots<s_k}\hat{f}(t,s_2,\ldots,s_k)dW_t dB_{s_2}\ldots dB_{s_k} \\ +\int_{s_1<t<\cdots<s_k}\hat{f}(s_1,t,\ldots,s_k)dB_{s_1}dW_t\ldots dB_{s_k} \\ +\cdots \\ +\int_{s_1<\cdots<s_{k-1}<t}\hat{f}(s_1,\ldots,s_{k-1},t)dB_{s_1}\ldots dB_{s_{k-1}}dW_t \end{array}$$

which equals $\int D_s(X)dW_s = X^{\#}$.

The general case in obtained by approximation of X by X_k for the $\mathbb{D}^{2,2}$ norm and the same argument as in the proof of proposition 2 by the caracteristic functions gives the result. $\diamond$

Following the same lines, it is possible to show that the theoretical $\overline{A}$ and practical $\underline{A}$ bias operators defined on the algebra $\mathcal{L}\{e^{\int \xi dB}\ ;\ \xi\in\mathcal{C}^1\}$ by

$$\begin{array}{c} n^2\mathbb{E}[(R_n(X)-X)Y] = <\overline{A}X, Y>_{L^2(m)} \\ n^2\mathbb{E}[(X-R_n(X))R_n(Y)] = <\underline{A}X, Y>_{L^2(m)} \end{array}$$

are defined and equal to A.

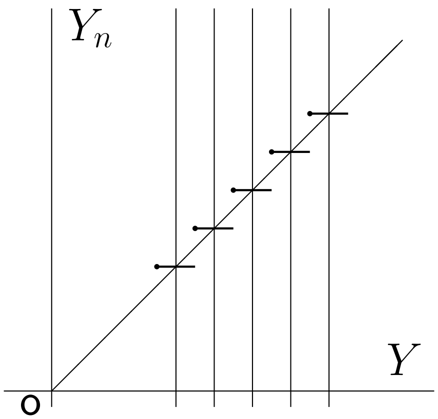
IV. Graduations of measuring instruments and Rajchman martingales

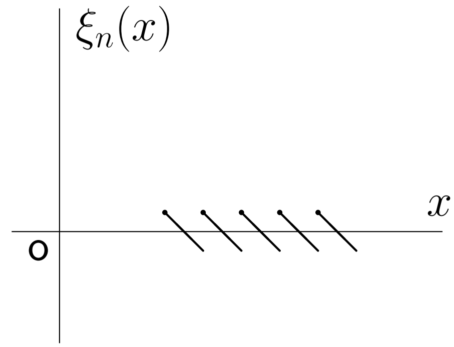
The basic example.

Let Y be a real random variable. It is approximated by Y_n to the nearest graduation, i.e.

$$Y_n = \frac{[nY]}{n} + \frac{1}{2n}.$$

($[x]$ denotes the entire part of x, and $\{x\} = x - [x]$ the fractional part).





We put $Y_n = Y + \xi_n(Y)$ where the function $\xi_n(x) = \frac{[nx]}{n} - \frac{1}{2n} - x$ is periodic with period $\frac{1}{n}$ and may be written $\xi_n(x) = \frac{1}{n}\theta(nx)$ with $\theta(x) = \frac{1}{2} - \{x\}$.

Let $\mathbb{P}_Y$ the law of Y, we approximate Y by Y_n on the algebra $\mathcal{D} = \mathcal{C}^1 \cap Lip$ with the sequence $\alpha_n = n^2$.

The Rajchman class $\mathcal{R}$ is the set of bounded measures on $\mathbb{R}$ whose Fourier transform vanishes at infinity. These measures are continuous (do not charge points) and are a band in the space of bounded measures on $\mathbb{R}$.

Theorem. *If $\mathbb{P}_Y$ is a Rajchman measure,*

$$(n(Y_n - Y), Y) \quad \stackrel{d}{\Longrightarrow} \quad (V, Y)$$

where V is uniform on $(-\frac{1}{2}, \frac{1}{2})$ independent of Y, and for $\varphi \in \mathcal{C}^1 \cap Lip$

$$n^2 \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2] \quad \to \quad \frac{1}{12}\mathbb{E}_Y[\varphi'^2].$$

Here $\stackrel{d}{\Longrightarrow}$ denotes the weak convergence, $\mathbb{E}_Y$ is the expectation under $\mathbb{P}_Y$.

If $\mathbb{P}_Y$ is absolutely continuous and satisfies the Hamza condition (cf. Fukushima-Oshima-Takeda thm 3.1.6 p.105), e.g. as soon as $\mathbb{P}_Y$ has a continuous density, the form $\mathcal{E}[\varphi] = \frac{1}{24}\mathbb{E}_Y[\varphi'^2]$ is Dirichlet and admits the square field operator $\Gamma[\varphi] = \frac{1}{12}\varphi'^2$. The graduation yields therefore an error structure $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_Y, \mathbb{D}, \Gamma)$ whose operator Γ does not depend on $\mathbb{P}_Y$ provided that Y has a regular density. This translates in terms of errors the arbitrary functions principle.

I.3. **Historical comment.**

In his intuitive version, the idea underlying the arbitrary functions method is ancient. The historian J. von Plato dates it back to a book of J. von Kries published in 1886. We find indeed in this philosophical treatise the idea that if a roulette had equal and infinitely small black and white cases, then there would be an equal probability to fall on a case or on the neighbor one, hence by addition an equal probability to fall either on black or on white. But no precise proof was given. The idea remains at the common sense level.

A mathematical argument for the fairness of the roulette and for the equidistribution of other mechanical systems (little planets on the zodiac) was proposed by H. Poincaré in his course on probability published in 1912 (Chap. VIII §92 and especially §93). In present language, Poincaré shows the weak convergence of $tX+Y$ mod 2π when $t \uparrow \infty$ to the uniform law on $(0, 2\pi)$ when the pair (X, Y) has a density. He uses the characteristic functions. His proof supposes the density be $\mathcal{C}^1$ with bounded derivative in order to perform an integration by parts, but the proof would extend to the general absolutely case if we were using instead the Riemann-Lebesgue lemma.

The question is then developed without major changes by several authors, E. Borel (case of continuous density), M. Fréchet (case of Riemann-integrable density), B. Hostinski (bidimensional case) and is tackled anew by E. Hopf with the more general point of view of asymptotic behaviour of dissipative dynamical systems. Hopf has shown that these phenomena must be mathematically understood in the framework of ergodic theory and are related to mixing. Today the connection is close to Rajchman (or mixing) measures very interesting objects related to deep properties of descriptive set theory.

Girsanov theorem for Dirichlet forms.

Let us recall that an error structure is a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a local Dirichlet form with domain $\mathbb{D}$ (dense in $L^2(\Omega, \mathcal{A}, \mathbb{P})$) admitting a square field operator Γ. We denote $\mathcal{D}A$ the domain of the associated generator.

Theorem. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ be an error structure. Let be $f \in \mathbb{D} \cap L^\infty$ such that $f > 0$, $\mathbb{E}f = 1$, We put $\mathbb{P}_1 = f.\mathbb{P}$.*

a) The bilinear form $\mathcal{E}_1$ defined on $\mathcal{D}A \cap L^\infty$ by

$$\mathcal{E}_1[u,v] = -\mathbb{E}\left[fvA[u] + \frac{1}{2}v\Gamma[u,f]\right]$$

is closable in $L^2(\mathbb{P}_1)$ and satisfies for $u, v \in \mathcal{D}A \cap L^\infty$

$$\mathcal{E}_1[u,v] = - < A_1u, v > = - < u, A_1v > = \frac{1}{2}\mathbb{E}[f\Gamma[u,v]]$$

where $A_1[u] = A[u] + \frac{1}{2f}\Gamma[u,f]$.

b) Let $(\mathbb{D}_1, \mathcal{E}_1)$ be the smallest closed extension of $(\mathcal{D}A \cap L^\infty, \mathcal{E}_1)$. Then $\mathbb{D} \subset \mathbb{D}_1$, $\mathcal{E}_1$ is local and admits a square field operator Γ_1, and

$$\Gamma_1 = \Gamma \quad \text{on} \quad \mathbb{D}$$

in addition $\mathcal{D}A \subset \mathcal{D}A_1$ and $A_1[u] = A[u] + \frac{1}{2f}\Gamma[u,f]$ for all $u \in \mathcal{D}A$.

Rajchman measures.

Definition. *A measure μ on the torus $\mathbb{T}^1$ is said to be Rajchman if*

$$\hat{\mu} = \int_{\mathbb{T}^1} e^{2i\pi nx}\, d\mu(x) \to 0 \qquad \textit{when } |n| \uparrow \infty.$$

The set of Rajchman measures $\mathcal{R}$ is a band : if $\mu \in \mathcal{R}$ and if $\nu \ll |\mu|$ then $\nu \in \mathcal{R}$.

Lemma. *Let X be a real random variable and let $\Psi_X(u) = \mathbb{E}e^{iuX}$ be its characteristic function. Then*

$$\lim_{|u|\to\infty} \Psi_X(u) = 0 \quad \Longleftrightarrow \quad \mathbb{P}_{\{X\}} \in \mathcal{R}.$$

A probability measure on $\mathbb{R}$ satisfying the conditions of the lemma will be called Rajchman.

Examples. Thanks to the Riemann-Lebesgue lemma, absolutely continuous measures are in $\mathcal{R}$. It follows from the lemma that if a measure ν satisfies $\nu \star \cdots \star \nu \in \mathcal{R}$ then $\nu \in \mathcal{R}$.

Some Cantor type measures on perfect sets on $[0,1]$ are Rajchman, and some are not depending on number theoretical properties of the similarity ratio.

The preceding definitions and properties extend to $\mathbb{T}^d$: a measure μ on $\mathbb{T}^d$ is said to be in $\mathcal{R}$ if $\hat{\mu}(k) \to 0$ as $k \to \infty$ in $\mathbb{Z}^d$. The set of measures in $\mathcal{R}$ is a band. If X is $\mathbb{R}^d$-valued, $\lim_{|u|\to\infty} \mathbb{E}e^{i<u,X>} = 0$ is equivalent to $\mathbb{P}_{\{X\}} \in \mathcal{R}$ where $\{x\} = (\{x_1\}, \ldots, \{x_d\})$.

Finite dimensional case.

We suppose Y is $\mathbb{R}^d$-valued, measured with an equidistant graduation corresponding to an orthonormal rectilinear coordinate system, and estimated to the nearest graduation component by component. Thus we put

$$Y_n = Y + \frac{1}{n}\theta(nY)$$

with $\theta(y) = (\frac{1}{2} - \{y_1\}, \cdots, \frac{1}{2} - \{y_d\})$.

Theorem. *a) If $\mathbb{P}_Y$ is Rajchman and if X is $\mathbb{R}^m$-valued*

$$(X, n(Y_n - Y)) \quad \stackrel{d}{\Longrightarrow} \quad (X, (V_1, \ldots, V_d))$$

where the V_i's are i.i.d. uniformly distributed on $(-\frac{1}{2}, \frac{1}{2})$ independent of X.
For all $\varphi \in \mathcal{C}^1 \cap lip(\mathbb{R}^d)$

$$(X, n(\varphi(Y_n) - \varphi(Y))) \quad \stackrel{d}{\Longrightarrow} \quad (X, \sum_{i=1}^{d} V_i \varphi'_i(Y))$$

$$n^2 \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 | Y = y] \to \frac{1}{12} \sum_{i=1}^{d} \varphi'^2_i(y) \qquad \textit{in } L^1(\mathbb{P}_Y)$$

in particular

$$n^2 \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2] \to \mathbb{E}_Y[\frac{1}{12} \sum_{i=1}^{d} \varphi'^2_i(y)].$$

b) If φ is of class $\mathcal{C}^2$, the conditional expectation $n^2\mathbb{E}[\varphi(Y_n)-\varphi(Y)|Y=y]$ possesses a version $n^2(\varphi(y+\frac{1}{n}\theta(ny))-\varphi(y))$ independent of the probability measure $\mathbb{P}$ which converges in the sense of distributions to the function $\frac{1}{24}\triangle\varphi$.

c) If $\mathbb{P}_Y \ll dy$ on $\mathbb{R}^d$, $\forall\psi\in L^1([0,1])$

$$(X,\psi(n(Y_n-Y))) \quad \overset{d}{\Longrightarrow} (X,\psi(V)).$$

d) We consider the bias operators on the algebra $\mathcal{C}_b^2$ of bounded functions with bounded derivatives up to order 2 with the sequence $\alpha_n = n^2$. If $\mathbb{P}_Y\in\mathcal{R}$ and if one of the following condition is fulfilled

i) $\forall i=1,\ldots,d$ the partial derivative $\partial_i\mathbb{P}_Y$ in the sense of distributions is a measure $\ll\mathbb{P}_Y$ of the form $\rho_i\mathbb{P}_Y$ with $\rho_i\in L^2(\mathbb{P}_Y)$

ii) $\mathbb{P}_Y = h1_G\frac{dy}{|G|}$ with G open set, $h\in H^1\cap L^\infty(G)$, $h>0$

then hypotheses (H1) *to* (H4) *are satisfied and*

$$\begin{aligned}
\overline{A}[\varphi] &= \tfrac{1}{24}\triangle\varphi \\
\widetilde{A}[\varphi] &= \tfrac{1}{24}\triangle\varphi + \tfrac{1}{24}\textstyle\sum\varphi_i'\rho_i \qquad &\textit{case i)} \\
\widetilde{A}[\varphi] &= \tfrac{1}{24}\triangle\varphi + \tfrac{1}{24}\tfrac{1}{h}\textstyle\sum h_i'\varphi_i' \qquad &\textit{case ii)} \\
\Gamma[\varphi] &= \tfrac{1}{12}\textstyle\sum\varphi_i'^2.
\end{aligned}$$

Rajchman martingales.

Let $(\mathcal{F}_t)$ be a right continuous filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ and M be a continuous local $(\mathcal{F}_t, \mathbb{P})$-martingale nought at zero. M will be said to be Rajchman if the measure $d\langle M, M\rangle_s$ belongs to $\mathcal{R}$ almost surely.
We will show that the method followed by Rootzén extends to Rajchman martingales and provides the following

Theorem. *Let M be a continuous local martingale which is Rajchman and s.t. $\langle M, M\rangle_\infty = \infty$.*

Let f be a bounded Riemann-integrable periodic function with unit period on $\mathbb{R}$ s.t. $\int_0^1 f(s)ds = 0$. Then for any random variable X

$$(X, \int_0^{\cdot} f(ns)\, dM_s) \quad \overset{d}{\Longrightarrow} \quad (X, W_{\|f\|^2\langle M,M\rangle_\cdot}),$$

the weak convergence is understood on $\mathbb{R} \times \mathcal{C}([0,1])$ and W is an independent standard Brownian motion.

The theorem shows that the random measure dM_s behaves in some sense like a Rajchman measure. Indeed if $\mathbb{P}_Y \in \mathcal{R}$ we have

$$\int_{-\infty}^{y} g(nx)\mathbb{P}_Y(dx) \to \int_0^1 g(x)dx \int_{-\infty}^{y} \mathbb{P}_Y(dx)$$

as soon as g is periodic with unit period, Riemann-integrable and bounded. Now applying the theorem to the Brownian motion gives the similar relation

$$\int_0^t f(ns)\, dB_s \quad \overset{d}{\Longrightarrow} \quad (\int_0^1 f^2(ns)ds)^{1/2} \int_0^t dW_s.$$

Limit quadratic form for Rajchman martingales.

We study the induced limit quadratic form when the martingale M is approximated by the martingale $M_t^n = M_t + \int_0^t \frac{1}{n} f(ns) dM_s$. The notation is the same as in the preceding section and f satisfies the same hypotheses as in the preceding theorem:

f is a bounded Riemann-integrable periodic function with unit period on $\mathbb{R}$ s.t. $\int_0^1 f(s)ds = 0$.

Theorem. *Let M be a Rajchman martingale s.t. $M_1 \in L^2$ and η, ζ bounded adapted processes. Then*

$$n^2 \mathbb{E}\left[(\exp\{i \int_0^1 \eta_s dM_s^n\} - \exp\{i \int_0^1 \eta_s dM_s\})(\exp\{i \int_0^1 \zeta_s dM_s^n\} - \exp\{i \int_0^1 \zeta_s dM_s\}) \right]$$

$$\to -\mathbb{E}\left[\exp\{i \int_0^1 (\eta_s + \zeta_s) dM_s\} \int_0^1 \eta_s \zeta_s \, d\langle M, M\rangle_s \right] \int_0^1 f^2(s) ds.$$

Sufficient closability conditions on the Wiener space.

The closability problem of the limit quadratic forms obtained in the preceding section, may be tackled with the tools available on the Wiener space.

Let us approximate the Brownian motion $(B_t)_{t\in[0,1]}$ by the process $B_t^n = B_t + \int_0^t \frac{1}{n} f(ns)\, dB_s$ where f satisfies the same hypotheses as before. We consider here only deterministic integrands.

Theorem. *a) Let $\xi \in L^2([0,1])$, and let X be a random variable defined on the Wiener space, i.e. a Wiener functional, then*

$$\left(X, n(\exp\{i\int_0^1 \xi dB^n\} - \exp\{i\int_0^1 \xi dB\})\right) \stackrel{d}{\Longrightarrow} \left(X, \|f\|_{L^2}(\exp\{i\int_0^1 \xi dB\})^{\#}\right)$$

here for any regular Wiener functional Z we put $Z^{\#}(\omega, w) = \int_0^1 D_s Z\, dW_s$, where W is an independent Brownian motion.

b)

$$n^2 \mathbb{E}\left[(e^{i\xi.B^n} - e^{i\xi.B})^2\right] \to -\mathbb{E}[e^{2i\xi.B}] \int_0^1 \xi^2 ds \|f\|_{L^2}^2$$

on the algebra $\mathcal{L}\{e^{i\xi.B}\}$ the quadratic form $-\frac{1}{2}\mathbb{E}[e^{2i\xi.B}]\int_0^1 \xi^2 ds$ is closable, its closure is the Ornstein-Uhlenbeck form.

Link with discretization of SDE's.

In the case $f(x) = \theta(x) = \frac{1}{2} - \{x\}$, the approximation used consists in approximating B_t by $B_t + \int_0^t \frac{1}{n}\theta(ns)dB_s$. It is the most natural approximation suggested by the Rajchman property and the arbitrary functions principle. It yields also other approximation operators on the Wiener space.

But it is different from the approximations usually encountered in the discretization of stochastic differential equations.

In order to draw a link between the preceding study and works concerning the discretization of SDE's by the Euler scheme as by the Milstein scheme, it is possible to display the Dirichlet form associated with these discretizations.

Our aim here is just a connection of ideas and we limit the question to deterministic integrands, what, of course, simplifies highly the problem, but without loosing some interesting considerations.

Let us denote as before $[x]$ the entire part of x and $\widehat{x}$ the nearest integer of x with the convention $(n+\frac{1}{2})\widehat{\ } = n$. If ξ is with compact support in $[0,1[$ and if B is a Brownian motion vanishing at zero and on $\mathbb{R}_-$ we have

$$
\begin{array}{llll}
(i) & \int \xi_s dB_{\frac{[ns]}{n}+\frac{1}{n}} & = \int \xi_{\frac{[ns]}{n}} dB_s & = \sum_{k=0}^{\infty} \xi_{\frac{k}{n}} (B_{\frac{k+1}{n}} - B_{\frac{k}{n}}) \\
(ii) & \int \xi_s dB_{\frac{[ns]}{n}} & = \int \xi_{\frac{[ns]}{n}+\frac{1}{n}} dB_s & = \sum_{k=0}^{\infty} \xi_{\frac{k+1}{n}} (B_{\frac{k+1}{n}} - B_{\frac{k}{n}}) \\
(iii) & \int \xi_s dB_{\frac{\widehat{ns}}{n}} & = \int \xi_{\frac{[ns]}{n}+\frac{1}{2n}} dB_s & = \sum_{k=0}^{\infty} \xi_{\frac{k}{n}+\frac{1}{2n}} (B_{\frac{k+1}{n}} - B_{\frac{k}{n}}) \\
(iv) & \int \xi_s dB_{\frac{[ns]}{n}+\frac{1}{2n}} & = \int \xi_{\frac{\widehat{ns}}{n}} dB_s & = \sum_{k=0}^{\infty} \xi_{\frac{k}{n}} (B_{\frac{k+1}{n}+\frac{1}{2n}} - B_{\frac{k}{n}+\frac{1}{2n}}).
\end{array}
$$

Approximation (i) corresponds to the Euler scheme and for ξ adapted process it would yield the Ito integral.

The schemes (iii) and (iv) lead under the hypotheses of stochastic calculus for semi-martingales to the Stratonowitch integral and the scheme (ii) under suitable hypotheses provides the backward integral.

We focuse on the hypothesis (H3) on the algebra $\mathcal{L}\{e^{i\int \xi dB};\ \xi \in \mathcal{C}^2_K(]0,1[)\}$.

Theorem. *a) Approximations (iii) and (iv) give rise to the same limit :*

$$\lim_n n^2\mathbb{E}\left[(e^{i\int \xi(\frac{[ns]}{n}+\frac{1}{2n})dB_s} - e^{i\int \xi dB})^2\right] =$$

$$= \lim_n n^2\mathbb{E}\left[(e^{i\int \xi(\frac{\widehat{\widehat{ns}}}{n})dB_s} - e^{i\int \xi dB})^2\right] = -\frac{1}{12}\mathbb{E}[e^{2i\int \xi dB}\int \xi^2 ds]$$

which, up to a multiplicative coefficient, is the Ornstein-Uhlenbeck structure on the Wiener space. The hypothesis (H3) is therefore satisfied.

b) The approximations (i) and (ii) provide the same limit quadratic form

$$\lim_n n^2\mathbb{E}\left[(e^{i\int \xi(\frac{[ns]}{n})dB_s} - e^{i\int \xi dB})^2\right] = \lim_n n^2\mathbb{E}\left[(e^{i\int \xi(\frac{[ns]}{n}+\frac{1}{n})dB_s} - e^{i\int \xi dB})^2\right]$$

$$= -\mathbb{E}\left[e^{2i\int \xi dB}(\frac{1}{12}\int \xi'^2 ds + \frac{1}{4}(\int \xi' dB)^2)\right],$$

hypothesis (H3) is satisfied and the limit Dirichlet form is the sum of a generalized Mehler type form

$$\mathcal{E}_1[e^{\int \xi dB}, e^{\int \eta dB}] = \frac{1}{12}\mathbb{E}[e^{\xi+\eta)dB}\int \xi'\eta' ds]$$

(corresponding to the second quantization of the heat semi-group on $[0,1]$) and a form

$$\mathcal{E}_2[e^{\int \xi dB}, e^{\int \eta dB}] = \frac{1}{4}\mathbb{E}[e^{\xi+\eta)dB}\int \xi' dB\int \eta' dB]$$

which is a rather singular Dirichlet form on the Wiener space in the sense that its gradient is one-dimensional and its square field operator writes

$$\Gamma_2[e^{\int \xi dB}] = \left(\frac{1}{2}e^{\int \xi dB}\int \xi' dB\right)^2.$$

Concerning the Euler scheme for SDE's, we may remark that the preceding results which yield

$$\left(n\int_0^. (s-\frac{[ns]}{s})dB_s, n\int_0^. (B_s - B_{\frac{[ns]}{n}})ds, B_.\right) \stackrel{d}{\Longrightarrow} (\frac{1}{\sqrt{3}}W_., -\frac{1}{\sqrt{3}}W_., B_.)$$

are generally hidden by a dominating phenomenon

$$\left(\sqrt{n}\int_0^. (B_s - B_{\frac{[ns]}{n}})dB_s, B_.\right) \stackrel{d}{\Longrightarrow} (\frac{1}{\sqrt{2}}\widetilde{W}_., , B_.)$$

due to the fact that when a variable of the second chaos (or in further chaos) converges stably to a Gaussian limit, this one appears to be independent of the first chaos and therefore of B itself.

The convergence of arbitrary functions principle type acts even on the first chaos. It concerns, for example, SDE's of the form

$$\begin{cases} X^1_t = x^1_0 + \int_0^t f^{11}(X^2_s)dB_s + \int_0^t f^{12}(X^1_s, X^2_s)ds \\ X^2_t = x^2_0 + \int_0^t f^{22}(X^1_s, X^2_s)ds \end{cases}$$

where X^1 is with values in $\mathbb{R}^{k_1}$, X^2 in $\mathbb{R}^{k_2}$, B in $\mathbb{R}^d$ and f^{ij} are matrices with suitable dimensions. Such equations are encountered to describe the movement of mechanical systems under the action of forces with a random noise, when the noisy forces depend only on the position of the system and the time. Typically

$$\begin{cases} X_t = X_0 + \int_0^t V_s ds \\ V_t = V_0 + \int_0^t a(X_s, V_s, s)ds + \int_0^t b(X_s, s)dB_s \end{cases}$$

perturbation of the equation $\frac{d^2x}{dt^2} = a(x, \frac{dx}{dt}, t)$. In such equations the stochastic integral may be understood as Ito as well as Stratonovitch.

For the above SDE the method of Kurtz-Protter without major changes yields the following result that we state in the case $k_1 = k_2 = d = 1$ for simplicity.

Theorem. *If functions f^{ij} are $\mathcal{C}_b^1$, and if X^n is the solution of (22) by the Euler scheme,*

$$(n(X^n - X), X, B) \stackrel{d}{\Longrightarrow} (U, X, B)$$

where the process U is solution of the SDE

$$U(t) = \sum_{k,j} \int_0^t \frac{\partial f^{ij}}{\partial x_k}(X_s) U_s^k dY_s^j - \sum_{k,j} \int_0^t \frac{\partial f^{ij}}{\partial x_k}(X_s) \sum_m f^{km}(X_s) dZ_s^{mj}$$

where $Y_s = (B_s, s)^t$ and

$$\begin{aligned} dZ_s^{12} &= \tfrac{1}{\sqrt{3}} dW_s + \tfrac{1}{2} dB_s \\ dZ_s^{21} &= -\tfrac{1}{\sqrt{3}} dW_s + \tfrac{1}{2} dB_s \\ dZ_s^{22} &= \tfrac{ds}{2} \end{aligned}$$

and as ever W is an independent Brownian motion.

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