B) Error structures

- Languages with or without extension tool :

  the example of probability theory

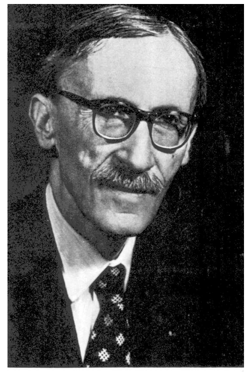
the role of σsigma -additivity in probability theory

measure theory (Lebesgue) +

Borel

Steinhaus

Fréchet

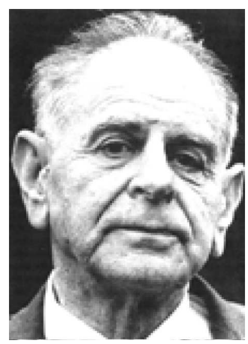


PAUL LÉVY 1886–1971

Paul Lévy

Kolmogorov (1933) *Grungbegriffe der Wahrscheinlichkeitsrechnung*

Let us mention that Karl Popper, who has an education of psychologist, is still in 1955 not convinced of the interest of embedding probability into measure theory : "*Kolmogorov's system can be taken, however, as one of the interpretations of mine*"



Popper

The error calculus of Gauss has the limitation that it has no mean of extension. If the error on (V_1, V_2, V_3) is known it gives the error on any differentiable function of (V_1, V_2, V_3) but that's all.

Now, in the usual probabilistic situations where a sequence of quantities $X_1, X_2, \ldots, X_n, \ldots$ is given and where the errors are known on the regular functions of a finite number of them, we would like to deduce the error on a function of an infinite number of the X_i's or at least on some such functions.

It is actually possible to reinforce this error calculus giving it a powerful extension tool and preserving the coherence property. In addition, it will give us the comfortable possibility to handle Lipschitz functions as well.

Adding an extension tool to the language of Gauss : the idea

For this we come back to the idea that the erroneous quantities are themselves random, as Gauss had supposed for his proof of the 'law of errors', say defined on $(\Omega, \mathcal{A}, \mathbb{P})$. The quadratic error on a random variable X is then itself a random variable that we will denote by $\Gamma[X]$. Intuitively we still suppose the errors are infinitely small although this doesn't appear in the notation. It is as we had an infinitely small unit to measure errors fixed in the whole problem. The extension tool is the following, we assume that if $X_n \to X$ in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ and if the error $\Gamma[X_m - X_n]$ on $X_m - X_n$ can be made as small as we want in $L^1(\Omega, \mathcal{A}, \mathbb{P})$ for m, n large enough, then the error $\Gamma[X_n - X]$ on $X_n - X$ goes to zero in L^1.

It is a *reinforced coherence principle* since this means that the error on a random variable X is attached to X and that furthermore if the sequence of pairs $(X_n,\ \text{error on } X_n)$ converges in a suitable sense, it converges necessarily to $(X,\ \text{error on } X)$.

ERROR STRUCTURES

Main definition . *An error structure is a term*

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$$

where $(\Omega, \mathcal{A}, \mathbb{P})$ *is a probability space, and:*

(1) $\mathbb{D}$ *is a dense subvector space of* $L^2(\Omega, \mathcal{A}, \mathbb{P})$ *(also denoted* $L^2(\mathbb{P})$*);*

(2) Γ *is a positive symmetric bilinear application from* $\mathbb{D} \times \mathbb{D}$ *into* $L^1(\mathbb{P})$ *satisfying "the functional calculus of class* $\mathcal{C}^1 \cap \mathrm{Lip}$*". This expression means*

$$\forall u \in \mathbb{D}^m, \quad \forall v \in \mathbb{D}^n, \quad \forall F : \mathbb{R}^m \to \mathbb{R}, \quad \forall G : \mathbb{R}^n \to \mathbb{R}$$

with F, G *being of class* $\mathcal{C}^1$ *and Lipschitzian, we have* $F(u) \in \mathbb{D}$, $G(v) \in \mathbb{D}$ *and*

$$\Gamma[F(u), G(v)] = \sum_{i,j} \frac{\partial F}{\partial x_i}(u) \frac{\partial G}{\partial x_j}(v) \Gamma\big[u_i, v_j\big] \quad \mathbb{P}\text{-a.s.};$$

(3) *the bilinear form* $\mathcal{E}[u, v] = \frac{1}{2}\mathbb{E}\big[\Gamma[u, v]\big]$ *is "closed". This means that the space* $\mathbb{D}$ *equipped with the norm*

$$\|u\|_{\mathbb{D}} = \Big(\|u\|^2_{L^2(\mathbb{P})} + \mathcal{E}[u, u]\Big)^{1/2}$$

is complete.

If, in addition

(4) *the constant function* 1 *belongs to* $\mathbb{D}$ *(which implies* $\Gamma[1] = 0$ *by property 2), we say that the error structure is Markovian.*

We will always write $\mathcal{E}[u]$ for $\mathcal{E}[u, u]$ and $\Gamma[u]$ for $\Gamma[u, u]$.
With this definition, the form $\mathcal{E}$ is a *Dirichlet form*.

To this Dirichlet form corresponds a *Dirichlet operator* A (generator of the associated symmetric semi-group) which satisfies (with some hypotheses) :

$$A[F \circ u] = \sum_i F'_i \circ u \;\; A[u_i] + \frac{1}{2} \sum_{i,j} F''_{ij} \circ u \;\; \Gamma[u_i, u_j] \quad \mathbb{P}\text{-p.s.}.$$

Example 1.

$$\begin{aligned}\Omega &= \mathbb{R}\\ \mathcal{A} &= \text{Borel } \sigma\text{-field } \mathcal{B}(\mathbb{R})\\ \mathbb{P} &= \mathcal{N}(0,1) \text{ reduced normal law}\end{aligned}$$

$$\mathbb{D} = H^1\big(\mathcal{N}(0,1)\big) = \big\{u \in L^2(\mathbb{P}),\ u' \text{ in the distribution sense belongs to } L^2(\mathbb{P})\big\}$$

$$\Gamma[u] = u'^2$$

then, $\big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0,1), H^1(\mathcal{N}(0,1)), \Gamma\big)$ is an error structure. We also obtained the generator:

$$\mathcal{D}A = \big\{f \in L^2(\mathbb{P}) :\ f'' - xf' \text{ in the distribution sense} \in L^2(\mathbb{P})\big\}$$

and

$$Af = \frac{1}{2}f'' - \frac{1}{2}I \cdot f'$$

where I is the identity map on $\mathbb{R}$.

Example 2.

$$\begin{aligned}
\Omega &= [0,1] \\
\mathcal{A} &= \text{Borel } \sigma\text{-field} \\
\mathbb{P} &= \text{Lebesgue measure} \\
\mathbb{D} &= \Big\{u \in L^2\big([0,1],dx\big) : \text{the derivative } u' \text{ in the distribution} \\
&\qquad\qquad \text{sense over }]0,1[\text{ belongs to } L^2([0,1],dx)\Big\} \\
\Gamma[u] &= u'^2.
\end{aligned}$$

The space $\mathbb{D}$ defined herein is denoted $H^1\big([0,1]\big)$.

Example 3. Let U be a domain (connected open set) in $\mathbb{R}^d$ with unit volume, $\mathcal{B}(U)$ be the Borel σ-field and $dx = dx_1, \ldots dx_d$ be the Lebesgue measure

$$\begin{aligned}
\mathbb{D} = \Big\{u \in L^2(U,dx) &: \text{the gradient } \nabla u \text{ in the distribution sense} \\
&\text{belongs to } L^2\big(U,dx;\mathbb{R}^d\big)\Big\}
\end{aligned}$$

$$\Gamma[u] = |\nabla u|^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_d}\right)^2.$$

Then $(U, \mathcal{B}(U), dx, \mathbb{D}, \Gamma)$ is an error structure.

COMPARISON OF APPROCHES

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COMPARISON OF APPROCHES

the interpretation
in terms of packs

Deterministic approaches	Deterministic sensitivity analysis : derivation with respect to the parameters of the model	Interval calculus
Probabilistic approaches	Error calculus using Dirichlet forms first order calculus only dealing with variances | second order calculus with variances and biases	Probability theory
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