Second lecture

Error structures and sensitivity analysis

C) Properties of error structures
- recalling the definition
- Lipschitzian calculus
- images
- densities and Dloc
- finite and infinite products
- the gradient and the sharp
- Integration by part formulae

D) Application in simulation : error calculus on the Monte Carlo space
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E) Application in numerical analysis
- sensitivity of an ODE to a functional coefficient
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ERROR STRUCTURES

Main definition . *An error structure is a term*

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$$

where $(\Omega, \mathcal{A}, \mathbb{P})$ *is a probability space, and:*

(1) $\mathbb{D}$ *is a dense subvector space of* $L^2(\Omega, \mathcal{A}, \mathbb{P})$ *(also denoted* $L^2(\mathbb{P})$*);*

(2) Γ *is a positive symmetric bilinear application from* $\mathbb{D} \times \mathbb{D}$ *into* $L^1(\mathbb{P})$ *satisfying "the functional calculus of class* $\mathcal{C}^1 \cap \mathrm{Lip}$*". This expression means*

$$\forall u \in \mathbb{D}^m, \quad \forall v \in \mathbb{D}^n, \quad \forall F : \mathbb{R}^m \to \mathbb{R}, \quad \forall G : \mathbb{R}^n \to \mathbb{R}$$

with F, G *being of class* $\mathcal{C}^1$ *and Lipschitzian, we have* $F(u) \in \mathbb{D}$, $G(v) \in \mathbb{D}$ *and*

$$\Gamma[F(u), G(v)] = \sum_{i,j} \frac{\partial F}{\partial x_i}(u) \frac{\partial G}{\partial x_j}(v) \Gamma\big[u_i, v_j\big] \quad \mathbb{P}\text{-a.s.};$$

(3) *the bilinear form* $\mathcal{E}[u, v] = \frac{1}{2}\mathbb{E}\big[\Gamma[u, v]\big]$ *is "closed". This means that the space* $\mathbb{D}$ *equipped with the norm*

$$\|u\|_{\mathbb{D}} = \Big(\|u\|^2_{L^2(\mathbb{P})} + \mathcal{E}[u, u]\Big)^{1/2}$$

is complete.

If, in addition

(4) *the constant function* 1 *belongs to* $\mathbb{D}$ *(which implies* $\Gamma[1] = 0$ *by property 2), we say that the error structure is Markovian.*

We will always write $\mathcal{E}[u]$ for $\mathcal{E}[u, u]$ and $\Gamma[u]$ for $\Gamma[u, u]$.
With this definition, the form $\mathcal{E}$ is a *Dirichlet form*.

To this Dirichlet form corresponds a *Dirichlet operator* A (generator of the associated symmetric semi-group) which satisfies (with some hypotheses) :

$$A[F \circ u] = \sum_i F'_i \circ u \;\; A[u_i] + \frac{1}{2} \sum_{i,j} F''_{ij} \circ u \;\; \Gamma[u_i, u_j] \quad \mathbb{P}\text{-p.s.}.$$

D is stable by contractions

Proposition *If F is a contraction, i.e.*

$$|F(x) - F(y)| \leq \sum_{i=1}^{m} \left| x_i - y_i \right|$$

then for $u \in \mathbb{D}^m$, we have

$$\left(\Gamma[F \circ u]\right)^{1/2} \leq \sum_i \left(\Gamma[u_i]\right)^{1/2}$$

and

$$\left(\mathcal{E}[F \circ u]\right)^{1/2} \leq \sum_i \left(\mathcal{E}[u_i]\right)^{1/2}.$$

Theorem *For all $u \in \mathbb{D}$, the image by u of the (positive bounded) measure $\Gamma[u] \cdot \mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$:*

$$u_*\left(\Gamma[u] \cdot \mathbb{P}\right) \ll dx.$$

If $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz

$$\Gamma[F \circ u] = F'^2 \circ u \cdot \Gamma[u]$$

where F' is any version of the derivative (defined Lebesgue-a.e.) of F.

Proposition (proved for special error structures)
If $u = \left(u_1, \ldots, u_m\right) \in \mathbb{D}^m$, then the image by u of the measure $\det \Gamma\left[u_i, u_j\right]. \mathbb{P}$ *is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$.*

On the other hand, no error structure is known at present that does not satisfy this proposition the operator Γ can be extended to a larger space than $\mathbb{D}$:

Definition III.13. *A function $u\colon \Omega \to \mathbb{R}$ is said to be locally in $\mathbb{D}$, and we write $u \in \mathbb{D}_{\text{loc}}$ if a sequence of sets $\Omega_n \in \mathcal{A}$ exists such that*

- $\bigcup_n \Omega_n = \Omega$
- $\forall n\ \exists u_n \in \mathbb{D}\colon u_n = u \text{ on } \Omega_n.$

$\mathbb{D}_{\text{loc}}$ is preserved by locally Lipschitz functions.

Proposition III.14. *Let u be in $\mathbb{D}_{\text{loc}}$.*

1) *There exists a unique positive class $\Gamma[u]$ (defined $\mathbb{P}$-a.e.) such that*

$$\forall v \in \mathbb{D},\quad \forall B \in \mathcal{A},\quad u = v \text{ on } B \Rightarrow \Gamma[u] = \Gamma[v] \ \text{ on } B.$$

2) *The image by u of the σ-finite measure $\Gamma[u]\cdot\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure.*

3) *If $F\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $F \circ u \in \mathbb{D}_{\text{loc}}$ and*

$$\Gamma[F \circ u] = F'^2 \circ u \cdot \Gamma[u].$$

Images

Let $S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ be an error structure; consider an $\mathbf{R}^d$-valued random variable $X: \Omega \to \mathbf{R}^d$ such that $X \in \mathbb{D}^d$, i.e. $X = (X_1, \ldots, X_d)$, $X_i \in \mathbb{D}$ for $i = 1, \ldots, d$. We will define the *error structure image of S by X*.

First of all, the probability space on which this error structure will be defined is the image of $(\Omega, \mathcal{A}, \mathbb{P})$ by X:

$$(\Omega, \mathcal{A}, \mathbf{P}) \xrightarrow{X} (\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d), X_*\mathbb{P})$$

where $X_*\mathbb{P}$ is the law of X, i.e. the measure such that $(X_*\mathbf{P})(E) = \mathbb{P}(X^{-1}(E))\ \forall E \in \mathcal{B}(\mathbb{R}^d)$.

We may then set

$$\mathbb{D}_X = \{u \in L^2(X_*\mathbf{P}) \colon u \circ X \in \mathbb{D}\}$$

and

$$\Gamma_X[u](x) = \mathbb{E}[\Gamma[u \circ X] \mid X = x].$$

(Unless explicitly mentioned otherwise, the symbol $\mathbf{E}$ denotes the expectation or conditional expectation with respect to $\mathbb{P}$.)

Proposition $X_*S = (\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d), X_*\mathbf{P}, \mathbb{D}_X, \Gamma_X)$ *is an error structure, the coordinate maps of* $\mathbf{R}^d$ *are in* $\mathbb{D}_X$*, and* X_*S *is Markovian if* S *is Markovian.*

Let $S_1 = \big(\Omega_1, \mathcal{A}_1, \mathbb{P}_1, \mathbb{D}_1, \Gamma_1\big)$ and $S_2 = \big(\Omega_2, \mathcal{A}_2, \mathbb{P}_2, \mathbb{D}_2, \Gamma_2\big)$ be two error structures.

The aim is to define on the product probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = \big(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \times \mathbb{P}_2\big)$$

an operator Γ and its domain $\mathbb{D}$ in such a way that $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ is an error structure expressing the condition that the two coordinate mappings and their errors are independent.

Proposition. *Let's define*

$$\begin{aligned}(\Omega, \mathcal{A}, \mathbb{P}) &= \big(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \times \mathbb{P}_2\big)\\ \mathbb{D} &= \Big\{ f \in L^2(\mathbb{P}) : \textit{for } \mathbb{P}_1\text{-a.e. } x \;\; f(x,\cdot) \in \mathbb{D}_2 \\ &\qquad \textit{for } \mathbb{P}_2\text{-a.e. } y \;\; f(\cdot, y) \in \mathbb{D}_1 \quad \textit{and} \\ &\qquad \int \big(\Gamma_1[f(\cdot, y)](x) + \Gamma_2\big[f(x,\cdot)\big](y) d\mathbb{P}_1(x)\, d\mathbb{P}_2(y) < +\infty \Big\}\end{aligned}$$

and for $f \in \mathbb{D}$

$$\Gamma[f](x, y) = \Gamma_1[f(\cdot, y)](x) + \Gamma_2[f(x, \cdot)](y),$$

then $S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ *is an error structure denoted* $S = S_1 \times S_2$ *and called the product of* S_1 *and* S_2*, whereby* S *is Markovian if* S_1 *and* S_2 *are both Markovian.*

• infinite products

Theorem on products *Let* $S_n = \big(\Omega_n, \mathcal{A}_n, \mathbb{P}_n, \mathbb{D}_n \Gamma_n\big)$, $n \geq 1$, *be error structures. The product structure*

$$S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_{n=1}^{\infty} S_n$$

is defined by

$$(\Omega, \mathcal{A}, \mathbb{P}) = \left(\prod_{n=1}^{\infty} \Omega_n, \otimes_{n=1}^{\infty} \mathcal{A}_n, \prod_{n=1}^{\infty} \mathbb{P}_n\right)$$

$$\mathbb{D} = \Big\{ f \in L^2(\mathbb{P}) : \forall n, \textit{ for almost every } w_1, w_2, \ldots, w_{n-1}, w_{n+1}, \ldots$$

$$\textit{for the product measure}$$

$$x \to f\big(w_1, \ldots, w_{n-1}, x, w_{n+1}, \ldots\big) \in \mathbb{D}_n \textit{ and}$$

$$\int \sum_n \Gamma_n[f]\, d\mathbb{P} < +\infty \Big\}$$

and for $f \in \mathbb{D}$

$$\Gamma[f] = \sum_{n=1}^{\infty} \Gamma_n[f].$$

S is an error structure, Markovian if each S_n is Markovian.

When we write $\Gamma_n[f]$, Γ_n acts on the n-th argument of f uniquely.

• The gradient and the sharp (#)

One of the features of the operator Γ is to be quadratic or bilinear like the variance or covariance that often makes computations awkward to perform. If we accept to consider random variables with values in Hilbert space, it is possible to overcome this problem by introducing a new operator, the gradient, which in some sense is a *linear version of the standard deviation of the error*.

The gradient. Let $S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ be an error structure. If $\mathcal{H}$ is a real Hilbert space, we denote either by $L^2\big((\Omega, \mathcal{A}, \mathbb{P}), \mathcal{H}\big)$ or $L^2(\mathbb{P}, \mathcal{H})$ the space of $\mathcal{H}$-valued random variables equipped with the scalar product $(U, V)_{L^2(\mathbb{P},\mathcal{H})} = \mathbb{E}\big[\langle U, V\rangle_{\mathcal{H}}\big]$.

Definition . *Let $\mathcal{H}$ be a Hilbert space. A linear operator D from $\mathbb{D}$ into $L^2(\mathbb{P}, \mathcal{H})$ is said to be a gradient (for S) if*

$$\forall u \in \mathbb{D} \quad \Gamma[u] = \langle Du, Du\rangle_{\mathcal{H}}.$$

In practice, a gradient always exists (once the space $\mathbb{D}$ is separable i.e. possesses a dense sequence).

Proposition (chain rule). *Let D be a gradient for S with values in $\mathcal{H}$. Then $\forall u \in \mathbb{D}^n$ $\forall F \in \mathcal{C}^1 \cap \mathrm{Lip}(\mathbb{R}^n)$,*

$$D[F \circ u] = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \circ u \; D[u_i] \quad \text{a.e.}$$

The sharp (#). It is a special case of gradient when $\mathcal{H}$ is taken to be

$$\mathcal{H} = L^2(\widehat{\Omega}, \widehat{\mathcal{A}}, \widehat{\mathbb{P}})$$

where $(\widehat{\Omega}, \widehat{\mathcal{A}}, \widehat{\mathbb{P}})$ is a copy of $(\Omega, \mathcal{A}, \mathbb{P})$. It satisfies $\forall u \in \mathbb{D}^n$, and $F \in \mathcal{C}^1 \cap \mathrm{Lip}\big(\mathbb{R}^n\big)$

$$\big(F(u_1, \ldots, u_n)\big)^{\#} = \sum_i \frac{\partial F}{\partial x_i} \circ u \cdot u_i^{\#}$$

It is particularly usefull in stochastic calculus.

• Integration by parts formulae

Let $S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ be an error structure. If $v \in \mathcal{D}A$, for $\forall u \in \mathbb{D}$ we have

$$\frac{1}{2}\mathbb{E}\big[\Gamma[u, v]\big] = -\mathbb{E}\big[uA[v]\big]. \tag{1}$$

This relation is already an integration by parts formula since Γ follows first-order differential calculus, in particular if $F \in$ Lip with Lebesgue derivative F'

$$\frac{1}{2}\mathbb{E}\big[F'(u)\Gamma[u, v]\big] = -\mathbb{E}\big[F(u)A[v]\big]. \tag{2}$$

We know that $\mathbb{D} \cap L^\infty$ is an algebra, hence if $u_1, u_2 \in \mathbb{D} \cap L^\infty$ we can apply (2) to u_1u_2 as follows

$$\frac{1}{2}\mathbb{E}\big[u_2\Gamma[u_1, v]\big] = -\mathbb{E}\big[u_1u_2A[v]\big] - \frac{1}{2}\mathbb{E}\big[u_1\Gamma[u_2, v]\big] \tag{3}$$

which yields for φ Lipschitz

$$\frac{1}{2}\mathbb{E}\big[u_2\varphi'(u_1)\Gamma[u_1, v]\big] = -\mathbb{E}\big[\varphi(u_1)u_2A[v]\big] - \frac{1}{2}\mathbb{E}\big[\varphi(u_1)\Gamma[u_2, v]\big]. \tag{4}$$

Let's now introduce a gradient D with values in $\mathcal{H}$ along with its adjoint operator δ. The preceding formula (4) with $u \in \mathbb{D}$, $U \in \text{dom}\ \delta$

$$\mathbb{E}[u\delta U] = \mathbb{E}\big[\langle D[u], U\rangle_{\mathcal{H}}\big] \tag{5}$$

provides, as above, for φ Lipschitz

$$\mathbb{E}\big[\varphi'(u)\langle D[u], U\rangle_{\mathcal{H}}\big] = \mathbb{E}[\varphi(u)\delta U]. \tag{6}$$

Moreover if $u_1, u_2 \in \mathbb{D} \cap L^\infty$ and $U \in \text{dom}\ \delta$

$$\mathbb{E}\big[u_2\langle Du_1, U\rangle_{\mathcal{H}}\big] = \mathbb{E}\big[u_1u_2\delta U\big] - \mathbb{E}\big[u_1\langle Du_2, U\rangle_{\mathcal{H}}\big]. \tag{7}$$

We refer to Monte Carlo space as the probability space used in simulation:

$$(\Omega, \mathcal{A}, \mathbb{P}) = \big([0,1], \mathcal{B}([0,1]), dx\big)^{\mathbb{N}}.$$

We denote the coordinate mappings $(U_n)_{n\geq 0}$. They are i.i.d. random variables uniformly distributed on the unit interval.

To obtain an error structure on this space, using the theorem on products, it suffices to choose an error structure on each factor; many (uncountably many) solutions exist.

We will focus on very simple (shift-invariant) structures useful in applications.

• The H^1-type product structure

It is the structure

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \big([0,1], \mathcal{B}([0,1]), dx, H^1([0,1], u \to u'^2)\big)^{\mathbb{N}}.$$

It is the simplest one, already very useful for sensitivity studies on Monte Carlo simulations.

• Structure without border terms

For some applications it is interesting to have integration by parts formulae without border terms. Consider the structure

$$\big([0,1], \mathcal{B}([0,1]), dx, \mathbf{d}, \gamma\big)$$

with $\gamma[u](x) = x^2(1-x)^2 u'^2(x)$ on the smallest closed domain $\mathbf{d}$ containing $\mathcal{C}^1[0,1]$. And consider the product structure

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \big([0,1], \mathcal{B}([0,1]), dx, \mathbf{d}, \gamma\big)^{\mathbb{N}}$$

The coordinate mappings U_n verify

$$\begin{aligned} U_n &\in \mathcal{D}A \subset \mathbb{D} \\ \Gamma\big[U_n\big] &= U_n^2(1-U_n)^2 \\ \Gamma\big[U_m, U_n\big] &= 0 \quad \forall m \neq n \\ A\big[U_n\big] &= U_n(1-U_n)(1-2U_n). \end{aligned}$$

Set $\varphi(x) = x^2(1-x)^2$.

• According to the theorem on products, if $F = f\big(U_0, U_1, \ldots, U_n, \ldots\big)$ is a real random variable, then $F \in \mathbb{D}$ iff

$$\forall n \;\; x \to f\big(U_0, \ldots, U_{n-1}, x, U_{n+1}, \ldots\big) \in \mathbf{d}$$

and

$$\mathbb{E}\left[\sum_n f_n'^2(U_0, U_1, \ldots)\varphi(U_n)\right] < +\infty.$$

• We can define a gradient with $\mathcal{H} = \ell^2$ and

$$DF = \left(f_n'(U_0, U_1, \ldots)\sqrt{\varphi(U_n)}\right)_{n\in\mathbb{N}}.$$

• If $a \in \ell^2$, $a = (a_n)$, we observe that $a \in \text{dom } \delta$ and $\delta[a] = \sum_n a_n\big(2U_n - 1\big)$ (which is a square integrable martingale) such that, $\forall F \in \mathbb{D}$

$$\mathbb{E}\big[\langle DF, a\rangle\big] = \mathbb{E}\left[F\sum_n a_n\big(2U_n - 1\big)\right].$$

• Applying this relation to FG for $F, G \in \mathbb{D} \cap L^\infty$ yields

$$\mathbb{E}\big[G\langle DF, a\rangle_{\ell^2}\big] = \mathbb{E}\left[F\left(G\sum_n a_n\big(2U_n - 1\big) - \langle DG, a\rangle\right)\right].$$

• We can similarly obtain an explicit formula for the adjoint operator δ acting on $Y = \big(Y_n(U_0, U_1, \ldots)\big)_{n\geq 0}$ and an explicit formula for the generator A acting on $F = f\big(U_0, U_1, \ldots\big)$.

• Sensitivity of a simulated Markov chain

Suppose by discretization of an SDE with respect to a martingale or a process with independent increments, we obtain a Markov chain

$$\begin{cases} S_0 = x \\ S_{n+1} = S_n + \sigma(S_n)(Y_{n+1} - Y_n) \end{cases}$$

where Y_n is a martingale simulated by

$$Y_n - Y_{n-1} = \xi(n, U_n) \quad \text{with} \int_0^1 \xi(n,x)\,dx = 0$$

and where σ is a Lipschitz function.

• *Sensitivity to the starting point*

The improvement of using error structures is here only to allow calculations with *Lipschitz* functions.

• *Internalization for* $\frac{d}{dx}\mathbb{E}[\Psi(S_N)]$

We are seeking $\psi(\omega, x)$ such that

$$\frac{d}{dx}\mathbb{E}[\Psi(S_N)] = \mathbb{E}[\Psi(S_N)\psi(\omega,x)].$$

We place an error only on U_1 and choose the preceding error structure without border terms. Assuming $\xi(1,x)$ to be $\mathcal{C}^2$ in x and using the integration by parts formula

$$\mathbb{E}[GDF] = -\mathbb{E}\left[F\frac{d}{dU_1}(G\sqrt{\varphi(U_1)})\right]$$

leads to

$$\frac{d}{dx}\mathbb{E}[\Psi(S_N)] = \mathbb{E}\left[\Psi(S_N)\left(\frac{\xi''_{x^2}(1,U_1)(1+\sigma'(x)\xi(1,U_1))}{\sigma(x)\xi'^2_x(1,U_1)} - \frac{\sigma'(x)}{\sigma(x)}\right)\right].$$

• *Deriving information on the law of S_N.*

We introduce an error on each U_n and then work with the Monte Carlo space using the above defined structure. If $F = f\big(U_1, \dots, U_n, \dots\big)$ we obtain:

$$DF = \left(f_i'\sqrt{\varphi(U_i)}\right)_{i\geq 1} \qquad \text{(remember that } \varphi(x) = x^2(1-x)^2\text{).}$$

With $F = \Psi\big(S_N\big)$, we get $DF = \Psi'(S_N)DS_N$.

For $a \in \ell^2$, using the IPF (integration by parts formula)

$$\mathbb{E}\big[G\langle DF, a\rangle\big] = -\mathbb{E}\left[F\left(\langle DG, a\rangle - G\sum_n a_n\big(2U_n - 1\big)\right)\right]$$

with $G = \frac{1}{\langle DS_N, a\rangle}$ yields the relation:

$$\mathbb{E}\big[\Psi'(S_N)\big] = \mathbb{E}\left[\Psi\big(S_N\big)\left(\frac{\sum_n a_n(2U_n - 1)}{\langle DS_N, a\rangle} - \left\langle D\frac{1}{\langle DS_N, a\rangle}, a\right\rangle\right)\right]$$

which can, with a suitable assumption, yield the regularity of the law of S_N, this is Malliavin's method.

Now according to the density criterion in error structures we have with the Lipschitz hypotheses

$$\Gamma\big[S_N\big] = \sigma^2\big(S_{N-1}\big)\xi'^2\big(N, U_N\big)\varphi\big(U_N\big) + \cdots + \left(\prod_{k=1}^{N}\big(1 + \sigma'\big(S_{k-1}\big)\xi\big(k, U_k\big)\big)\right)^2 \big(\sigma(x)\xi'\big(1, U_1\big)\big)^2\varphi\big(U_1\big).$$

We observe that if $\sigma(x) \neq 0$ and $\xi'(k, x) \neq 0\ \forall k$, then S_N has density.

Moreover, observing from the above calculation of DS_N that

$$\det\begin{pmatrix}\Gamma\big[S_N\big] & \Gamma\big[S_N, S_{N-1}\big]\\ \Gamma\big[S_N, S_{N-1}\big] & \Gamma\big[S_{N-1}\big]\end{pmatrix} = \Gamma\big[S_{N-1}\big]\big(\sigma^2(S_{N-1})\xi'^2\varphi\big(U_N\big)\big) > 0,$$

we obtain that the pair $\big(S_N, S_{N-1}\big)$ also has a density.

• The pseudo-Gaussian structure on the Monte Carlo space

Consider the Ornstein–Uhlenbeck structure on $\mathbb{R}$

$$\left(\mathbb{R}, \mathcal{B}(\mathbb{R}), m, H^1(m), u \to u'^2\right)$$

with $m = \mathcal{N}(0, 1)$.

Let's denote $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$ the distribution function of the reduced normal law and

$$\varphi_1(x) = \frac{1}{2\pi} \exp -\left(N^{-1}(x)\right)^2,$$

The image by N of the Ornstein–Uhlenbeck structure then gives a structure on $\left([0, 1], \mathcal{B}\left([0, 1]\right), dx\right)$, i.e.

$$\left([0, 1], \mathcal{B}\left([0, 1]\right), dx, \mathbf{d}_1, \gamma_1\right)$$

with $\quad \mathbf{d}_1 = \left\{u \in L^2[0, 1] : u \circ N \in H^1(m)\right\}$

and $\quad \gamma_1[u](x) = \varphi_1(x) u'^2(x)$.

Although the function φ_1 is not as simple as φ, this structure still possesses an IPF without border terms like the preceding structure, and gives rise to an efficient IPF on the Monte Carlo space.

Another interesting property of this structure is that it satisfies a Poincaré-type inequality.

Proposition. *Let S be the product*

$$S = \left([0, 1], \mathcal{B}\left([0, 1]\right), dx, \mathbf{d}_1, \gamma_1\right)^{\mathbb{N}},$$

then $\forall F \in \mathbb{D}$ we the following inequalities hold:

$$\operatorname{var}[F] = \mathbb{E}\left[(F - \mathbb{E}F)^2\right] \leq 2\mathcal{E}[F].$$

Remarks on error calculus for Monte Carlo simulations

- Pseudorandom generators aren't perfect. A way of modelling this coarseness is to put an error structure on the Monte Carlo space $([0,1], dx)^N$

- Applying this idea to simulation encounters the difficulty that most of algorithms are discontinuous (rejection method, etc.).
Fortunately, concerning the variances, we may work in D_{loc} .

- Concerning the biases,

  - by the fact that $$A[1/N \Sigma f(U_n)] = 1/N \Sigma A[f(U_n)]$$

  the biases do not vanish in general by averaging, they may increase by non linearity

  - I am not able at present to give a satisfactory definition of $(DA)_{loc}$

Application to numerical analysis

• Error on $f(X)$ when f is erroneous

Let Ω_1 be a space of functions (not necessarily a vector space), e.g. from $\mathbb{R}^d$ into $\mathbb{R}$, and let's consider an error structure $S_1 = \big(\Omega_1, \mathcal{A}_1, \mathbb{P}_1, \mathbb{D}_1, \Gamma_1\big)$ with the following properties:

one way for tackle this problem

other methods are possible

a) The measure $\mathbb{P}_1$ is carried by the $\mathcal{C}^1 \cap \mathrm{Lip}$ function in Ω_1.

b) Let $S_2 = \big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P}_2, \mathbb{D}_2, \Gamma_2\big)$ be an error structure on $\mathbb{R}^d$ such that $\mathcal{C}^1 \cap \mathrm{Lip} \subset \mathbb{D}_2$. Let's suppose the following: If we denote V_x the valuation at x, defined via

$$V_x(f) = f(x) \quad f \in \Omega_1,$$

V_x is a real functional on Ω_1 (a linear form if Ω_1 is a vector space). We now suppose that for $\mathbb{P}_2$ a.e. x $\;V_x \in \mathbb{D}_1$, and the random variable F defined on $\Omega_1 \times \mathbb{R}^d$ by

$$F(f, x) = V_x(f)$$

satisfies

$$\int \big(\Gamma_1[F] + \Gamma_2[F]\big)\, d\mathbb{P}_1\, d\mathbb{P}_2 < +\infty.$$

The theorem on products then applies and we can write:

$$\Gamma[F] = \Gamma_1[F] + \Gamma_2[F]. \qquad (1)$$

Let's consider that the two structures S_1 and S_2 have a sharp operator. This assumption gives rise to a sharp on the product structure.

Proposition. *With the above hypotheses, yet with $d = 1$ for the sake of simplicity, let $X \in \mathbb{D}_2$, then $f(X) \in \mathbb{D}$ and*

$$\big(f(X)\big)^{\#} = f^{\#}(X) + f'(X)X^{\#}. \qquad (2)$$

Example

Consider space of analytic functions in the unit disk with real coefficients

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with the a_n to be random with a product structure on them.

Choice of the a priori probability measure. If we choose the a_n to be i.i.d., the measure $\mathbb{P}_1$ is carried by a very small set and the scaling $f \to \lambda f$ gives from $\mathbb{P}_1$ a singular measure.

In order for $\mathbb{P}_1$ to weigh on a cone or a vector space, we use the following result.

Property. *Let μ be a probability measure on $\mathbb{R}$ with a density ($\mu \ll dx$). We set*

$$\mu_n = \alpha_n \mu + \left(1 - \alpha_n\right)\delta_0$$

with $\alpha_n \in]0, 1[$, $\sum_n \alpha_n < +\infty$. Let a_n be the coordinate maps from $\mathbb{R}^{\mathbb{N}}$ into $\mathbb{R}$, then under the probability measure $\otimes_n \mu_n$, only a finite number of a_n are non zero and the scaling

$$a = \left(a_0, a_1, \ldots, a_n, \ldots\right) \mapsto \lambda a = \left(\lambda a_0, \lambda a_1, \ldots, \lambda a_n, \ldots\right) \qquad (\lambda \neq 0)$$

transforms $\otimes_n \mu_n$ into an absolutely continuous measure [equivalent measure if $\frac{d\mu}{dx} > 0$].

Hereafter, we will suppose the measure $\mathbb{P}_1 = \otimes_n \mu_n$ with μ_n chosen as above. The a_n's are the coordinate mappings of the product space.

Choice of Γ. We consider here the simplest case where

$$\Gamma\left[a_n\right] = a_n^2, \quad \Gamma\left[a_m, a_n\right] = 0 \quad m \neq n$$

Calculation.

$$\Gamma\left[V_x\right](f) = \sum_n a_n^2 x^{2n}.$$

Since

$$f\left(ze^{2i\pi t}\right) = \sum_{n=0}^{\infty} a_n z^n e^{2i\pi nt},$$

using that $\left(e^{2i\pi nt}\right)_{n\in\mathbf{Z}}$ is a basis of $L^2_{\mathbf{C}}[0,1]$, we obtain

$$\Gamma\left[V_x\right](f) = \int_0^1 \left|f\left(xe^{2i\pi t}\right)\right|^2 dt \tag{3}$$

Let's now consider an erroneous random variable X defined on $\left(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_2, \mathbb{D}_2, \Gamma_2\right)$ as above and examine the error on

$$F(f, X) = f(X) = V_X(f).$$

From (2) and (3) we have

$$\Gamma\left[f(X)\right] = \int_0^1 \left|f\left(Xe^{2i\pi t}\right)\right|^2 dt + f'^2(X)\Gamma[X] \tag{4}$$

• Sensitivity of the solution of an ODE to a functional coefficient

To study the sensitivity of the solution of

$$y' = f(x, y)$$

to f, let's consider the case where f is approximated by polynomials in two variables

$$f(x, y) = \sum a_{pq} x^p y^q.$$

We choose the measure $\mathbb{P}_1$ and Γ_1, as explained above and in assuming measures μ_n to be centered for purpose of simplicity.

Then, if we take hypothesis (α)

$$\Gamma\left[a_{pq}\right] = a_{pq}^2$$

we obtain a sharp defined by

$$a_{pq}^{\#} = a_{pq} \frac{\widehat{a_{pq}}}{\beta_{pq}}$$

where

$$\beta_{pq} = \left\| a_n \right\|_{L^2(\mu_n)} = \left(\int_{\mathbb{R}} x^2 \, d\mu_n(x) \right)^{1/2}.$$

This sharp defines a sharp on the product space and if we consider the starting point y_0 and the value x to be random and erroneous, denoting

$$y = \varphi\left(x, y_0\right)$$

the solution to

$$\begin{cases} y' = f(x, y) \\ y(0) = y_0, \end{cases}$$

we then seek to compute $\Gamma[Y]$ for

$$Y = \varphi\left(X, Y_0\right).$$

This is the problem we attempt to solve :

sensitivity of Y with respect to all erroneous quantities

First, suppose f alone is erroneous.

Let's remark that by the representation

$$f(t,y)=\sum_{p,q} a_{pq} t^p y^q,$$

the formula

$$(f(t,Y))^{\#}=f^{\#}(t,Y)+f_y'(t,Y)Y^{\#}$$

is still valid even when Y is not independent of f.

Hence from

$$y_x=y_0+\int_0^x f\big(t,y_t\big)\,dt,$$

we have

$$y_x^{\#}=\int_0^x \big(f^{\#}\big(t,y_t\big)+f_2'\big(t,y_t\big)y_t^{\#}\big)\,dt.$$

Let

$$M_x=\exp\int_0^x f_2'\big(t,y_t\big)\,dt$$

by the usual method of variation of the constant. This yields

$$y_x^{\#}=M_x\int_0^x \frac{f^{\#}(t,y_t)}{M_t}\,dt.$$

Finally, when f, Y_0 and X are erroneous, we obtain

$$\begin{aligned}\Gamma[Y]=M_X^2\sum_{p,q}\left(\int_0^X \frac{t^p y_t^q}{M_t}\,dt\right)^2 a_{pq}^2 \\ +\left(\sum_{p,q} a_{pq} X^p Y_0^q\right)^2 \Gamma[X] \\ +M_X^2\Gamma\big[Y_0\big].\end{aligned}$$

error due to f

error due to X

error due to Yo

where $M_X=\exp\{\sum_{p,q} q a_{pq}\int_0^X t^p\varphi(t,Y_0)^{q-1}\,dt$. Let's recall herein that all of these sums are finite.

Remarks on sensitivity analysis for finite elements methods

(1) $$L(u) = f$$ with boundary conditions

- Errors are generally thought by numerical analysts as a global norm of $(u_n - u)$ in some functional space. This is a poor and non transitive information.

- In the case problem (1) is solved by a Galerkin method, data and the solution are represented in a finite dimensional space

$$f_n = \Sigma a_k e_k \qquad u_n = \Sigma b_k e_k$$

Putting an error structure on the a_k's gives errors and co-errors on the b_k's and their biases.

If u_n is used as data in a new boundary problem, we have the right information to go on.

- We may also represent the boundary itself with an error, by putting its graph in an error structure. That allows us to perform sensitivity analysis with respect to the boundary.