Third lecture

New tools for finance

F) Error structures on the Wiener space

G) error structures on the Poisson space

H) sensitivity analysis of an SDE, application to finance

I) A non classical approach to finance

.1 Error structures on the Wiener space

Let's first recall the classical approach of the so-called Wiener integral.

2.1. The Wiener stochastic integral. Let $(T, \mathcal{T}, \mu)$ be a σ-finite measured space, $\big(\chi_n\big)_{n\in\mathbf{N}}$ an orthonormal basis of $L^2(T, \mathcal{T}, \mu)$, and $\big(g_n\big)_{n\in\mathbf{N}}$ a sequence of i.i.d. reduced Gaussian variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$.

If $f \in L^2(T, \mathcal{T}, \mu)$ were associated with $I(f) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ defined via

$$I(f) = \sum_n \langle f, \chi_n\rangle g_n,$$

I would be a homomorphism from $L^2(T, \mu)$ into $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

If $f, g \in L^2(T, \mathcal{T}, \mu)$ are such that $\langle f, g\rangle = 0$, then $I(f)$ and $I(g)$ are two independent Gaussian variables.

From now on, we will take either $(T, \mathcal{T}, \mu) = \big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx\big)$ or $\big([0,1], \mathcal{B}\big([0,1]\big), dx\big)$.

If we set

$$\boxed{B(t) = \sum_n \big\langle 1_{[0,t]}, \chi_n\big\rangle g_n = \sum_n \int_0^t \chi_n(y)\,dy \cdot g_n} \tag{1}$$

then $B(t)$ is a centered Gaussian process with covariance

$$\mathbb{E}\big[B(t)B(s)\big] = t \wedge s$$

i.e., a standard Brownian motion.

It can be shown that series (1) converges in both $\mathcal{C}_K\big(\mathbb{R}_+\big)$ a.s. and $L^p\big((\Omega, \mathcal{A}, \mathbb{P}), \mathcal{C}_K\big)$ for $p \in [1, \infty[$ (where K denotes a compact set in $\mathbb{R}_+$).

Due to the case where f is a step-function, the random variable $I(f)$ is denoted

$$I(f) = \int_0^\infty f(s)\,dB_s \quad \left(\text{resp. } \int_0^1 f(s)\,dB_s\right)$$

and called the Wiener integral of f.

2.2. Product error structures. The preceding construction actually involves the product probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = \big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0,1)\big)^{\mathbf{N}},$$

with the g_n's being the coordinate mappings. If we place on each factor an error structure

$$\big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0,1), \mathbf{d}_n, \gamma_n\big),$$

we obtain an error structure on $(\Omega, \mathcal{A}, \mathbb{P})$ as follows

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty} \big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0,1), \mathbf{d}_n, \gamma_n\big)$$

such that a random variable

$$F\left(g_0, g_1, \dots, g_n, \dots\right)$$

belongs to $\mathbb{D}$ iff $\forall n\ x \to F\left(g_0, \dots, g_{n-1}, x, g_n, \dots\right)$ belongs to $\mathbf{d}_n$ $\mathbb{P}$-a.s. and

$$\Gamma[F] = \sum_n \gamma_n[F],$$

γ_n acting on the n-th variable of F, belongs to $L^1(\mathbb{P})$.

the domain D is explicitely given by the theorem on products

2.3. The Ornstein–Uhlenbeck structure. On each factor, we consider the one-dimensional Ornstein–Uhlenbeck structure (see Chapters II and III Example 1). Hence, we obtain

$$\Gamma\left[g_n\right] = 1$$
$$\Gamma\left[g_m, g_n\right] = 0 \quad \text{if } m \neq n.$$

For $f \in L^2(\mathbb{R}_+)$, by $\int_0^\infty f(s)\,dB_s = \sum\limits_n \langle f, \chi_n\rangle g_n$ we obtain

$$\Gamma\left[\int_0^\infty f(s)\,dB_s\right] = \sum_n \left\langle f, \chi_n\right\rangle^2 = \|f\|^2_{L^2(\mathbb{R}_+)},$$

From the relation

$$\Gamma\left[\int_0^\infty f(s)\,dB_s\right] = \|f\|^2_{L^2(\mathbb{R}_+)} \tag{2}$$

we derive $\forall F \in \mathcal{C}^1 \cap \mathrm{Lip}(\mathbb{R}^m)$

$$\Gamma\left[F\left(\int f_1(s)\,dB_s, \dots, \int f_n(s)\,dB_s\right)\right] = \sum_{i,j} \frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_j} \int f_i(s) f_j(s)\,ds.$$

This relation defines Γ on a dense subspace of $L^2(\mathbb{P})$ since it contains the $\mathcal{C}^1 \cap \mathrm{Lip}$ functions of a finite number of g_n's, which prove to be dense by virtue of the construction of the product measure. In other words, any error structure on the Wiener space such that $\mathbb{D}$ contains $\int f dB$ fot $f \in \mathcal{C}_K^\infty(\mathbb{R}_+)$ and satisfies (2) is an extension of the Ornstein-Uhlenbeck structure, in fact coincides with it : it can be proved that (2) characterizes the Ornstein–Uhlenbeck structure on the Wiener space among the structures such that $\mathbb{D}$ contains $\int f\,dB$ for $f \in \mathcal{C}_K^\infty(\mathbb{R}_+)$.

Gradient. We can easily define a gradient operator with $\mathcal{H} = L^2(\mathbb{R}_+)$: for $G \in \mathbb{D}$ let's set

$$D[G] = \sum_n \frac{\partial G}{\partial g_n} \cdot \chi_n(t). \tag{3}$$

This approach makes sense according to the theorem on products and satisfies

$$\langle D[G], D[G] \rangle = \sum_n \left(\frac{\partial G}{\partial g_n} \right)^2 = \Gamma[G],$$

therefore D is a gradient.

For $h \in L^2(\mathbb{R}_+)$, we obtain:

$$D\left[\int_0^\infty h(s)\, dB_s \right] = h \tag{4}$$

(since $\int_0^\infty h(s)\, dB_s = \sum\limits_n \langle h, \chi_n \rangle g_n$ and $D[g_n] = \chi_n$).

Proposition VI.6. *If* $h \in L^2(\mathbb{R}_+)$ *and* $F \in \mathbb{D}$,

$$\mathbb{E}\langle DF, h \rangle_{\mathcal{H}} = \mathbb{E}\left[F \int_0^\infty h\, dB \right].$$

Corollary VI.7. $\forall F, G \in \mathbb{D} \cap L^\infty$

$$\mathbb{E}\big[G \langle DF, h \rangle_{\mathcal{H}}\big] = -\mathbb{E}\big[F \langle DG, h \rangle\big] + \mathbb{E}\left[FG \int h\, dB \right].$$

Let $\mathcal{F}_t = \sigma\big(B_s, s \le t\big)$ be the natural filtration of the Brownian motion, we have

Lemma VI.8. *The operators* $\mathbb{E}\big[\cdot \mid \mathcal{F}_s\big]$ *are orthogonal projectors in* $\mathbb{D}$*, and for* $X \in \mathbb{D}$

$$D\big[\mathbb{E}[X \mid \mathcal{F}_s]\big] = \mathbb{E}\big[(DX)(t) 1_{t \le s} \mid \mathcal{F}_s\big].$$

We often write $D_t X$ *for* $DX(t)$.

We are now able to study the adjoint operator of the gradient: operator δ.

Proposition VI.9. *Let* $u_t \in L^2\big(\mathbb{R}_+ \times \Omega, dt \times d\mathbb{P}\big)$ *be an adapted process (*u_t *is* $\mathcal{F}_t$*-measurable up to* $\mathbb{P}$*-negligible sets,* $\forall t$*), then* $u_t \in \operatorname{dom} \delta$ *and*

$$\delta\big[u_t\big] = \int_0^\infty u_t\, dB_t.$$

so-called
Skorohod integral

Thus δ extends the Itô stochastic integral and coincides with it on adapted processes.

The sharp. The general definition lends the following relations:

$$\left(F\left(g_0,\ldots,g_n,\ldots\right)\right)^{\#}=\sum_n\frac{\partial F}{\partial g_n}\left(g_0,\ldots,g_n,\ldots\right)\hat{g}_n$$
$$\forall X\in\mathbb{D}\quad X^{\#}(\omega,\hat{\omega})=\int_0^\infty DX(t)\,d\hat{B}_t$$
$$\Gamma[X]=\hat{\mathbb{E}}\left[X^{\#2}\right].$$

From (4) we obtain:

$$\left(\int_0^\infty h(s)\,dB_s\right)^{\#}=\int_0^\infty h(s)\,d\hat{B}_s.$$

Proposition VI.10. *Let u be an adapted process in the closure of the space*

$$\left\{\sum_{i=1}^n F_i\,1_{]t_i,t_{i+1}]},\,F_i\in\mathcal{F}_{t_i},\,F_i\in\mathbb{D}\right\}$$

for the norm $\left(\mathbb{E}\int_0^\infty u^2(s)ds+\mathbb{E}\int_0^\infty\int_0^\infty(D_t[u(s)])^2\,dsdt\right)^{1/2}$. *Then*

$$\left(\int_0^\infty u_s\,dB_s\right)^{\#}=\int_0^\infty(u_s)^{\#}\,dB_s+\int_0^\infty u_s\,d\hat{B}_s.$$

The proof proceeds by approximation

the sharp is particularly usefull to compute errors for stochastic calculus

Ito's formula may be applied for B and for B^

As an application of the sharp, we propose the following exercises.

Exercise VI.11. Let $f_1(s,t)$ and $f_2(s,t)$ belong to $L^2\big(\mathbb{R}_+^2, ds\,dt\big)$ and be symmetric.

Let $U = \big(U_1, U_2\big)$ with

$$U_i = \int_0^\infty \int_0^t f_i(s,t)\,dB_s\,dB_t, \qquad i = 1, 2.$$

If $\det\big(\Gamma[U, U^t]\big) = 0$ a.e. then the law of U is carried by a straight line.

Hint. Show that

$$U_i^\# = \int_0^\infty \int_0^\infty f_i(s,t)\,dB_s\,d\hat{B}_t.$$

From $\big(\hat{\mathbb{E}}[U_1^\# U_2^\#]\big)^2 = \hat{\mathbb{E}}[U_1^{\#2}]\hat{\mathbb{E}}[U_2^{\#2}]$ deduce that a random variable $A(\omega)$ exists whereby

$$U_1^\#(\omega,\hat{\omega}) = A(\omega)U_2^\#(\omega,\hat{\omega}).$$

Use the symmetry of $U_1^\#$ and $U_2^\#$ in $(\omega,\hat{\omega})$ in order to deduce that A is constant.

Exercise VI.12. Let $f(s,t)$ be as in the preceding exercise, and g belong to $L^2(\mathbb{R}_+)$. If

$$\boxed{\begin{aligned} X &= \int_0^\infty g(s)\,dB_s \\ Y &= \int_0^\infty \int_0^t f(s,t)\,dB_s\,dB_t \end{aligned}}$$

show that

$$\boxed{\begin{aligned} \Gamma[X] &= \|g\|_{L^2}^2 \\ \Gamma[Y] &= \int_0^\infty \left(\int_0^\infty f(s,t)\,dB_s\right)^2 dt \\ \Gamma[X,Y] &= \int_0^\infty g(s)\left(\int_0^\infty f(s,t)\,dB_t\right) ds. \end{aligned}}$$

Show that if $\big(\Gamma[X,Y]\big)^2 = \Gamma[X]\Gamma[Y]$, the law of (X,Y) is carried by a parabola.

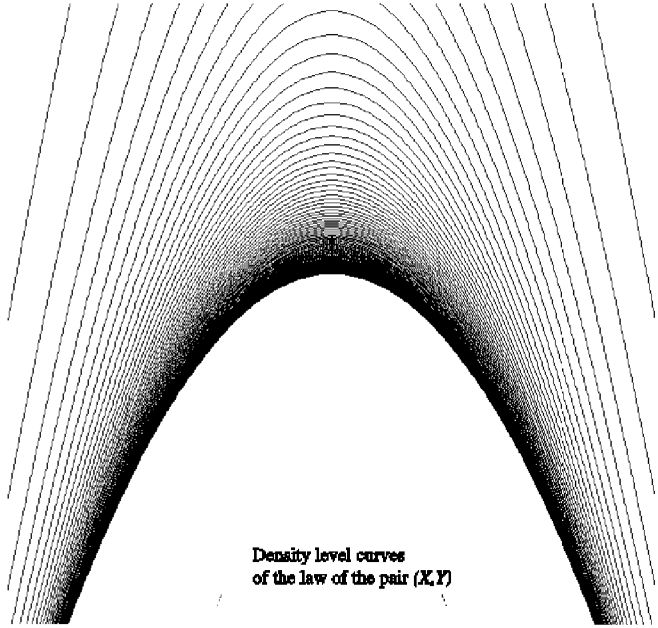
Numerical application. Let's consider the case

$$f(s,t) = 2h(s)h(t) - 2g(s)g(t)$$

for $g, h \in L^2(\mathbb{R}_+)$ with $\|h\|_{L^2} = \|g\|_{L^2} = 1$ and $\langle g, h\rangle = 0$.

The pair (X,Y) then possesses the density

$$\frac{1}{4\pi} e^{-y/2} e^{-x^2} \frac{1}{\sqrt{y + x^2}} 1_{\{y > -x^2\}},$$



and the matrix of the error variances is

$$\begin{pmatrix} \Gamma[X] & \Gamma[X,Y] \\ \Gamma[X,Y] & \Gamma[Y] \end{pmatrix} = \begin{pmatrix} 1 & -2X \\ -2X & 8X^2+4Y \end{pmatrix}.$$

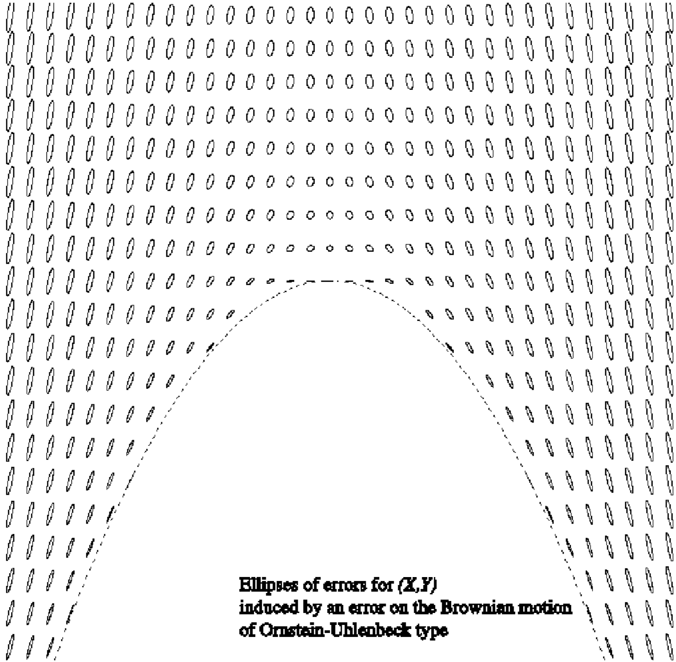
In other words, the image error structure by (X,Y) possesses a quadratic error operator $\Gamma_{(X,Y)}$ such that for $\mathcal{C}^1 \cap$ Lip-functions

$$\Gamma_{(X,Y)}[F](x,y) = F_1'^2(x,y) - 4x F_1'(x,y)F_2'(x,y) + (8x^2+4y)F_2'^2(x,y).$$

This can be graphically represented, as explained in Chapter I, by a field of ellipses of equations

$$(u\ \ v)\begin{pmatrix} 1 & -2x \\ -2x & 8x^2+4y \end{pmatrix}^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon^2$$

which are the level curves of small Gaussian densities.



2.4. Structures with erroneous time. Let's now choose $(T, \mathcal{T}, \mu) = \big([0,1], \mathcal{B}([0,1]), dx\big)$ for the sake of simplicity and let

$$\begin{cases} \chi_n = \sqrt{2}\cos 2\pi n t & \text{if } n > 0 \\ \chi_0 = 1 \\ \chi_n = \sqrt{2}\sin 2\pi n t & \text{if } n < 0 \end{cases}$$

be the trigonometric basis of $L^2\big([0,1]\big)$. We then follow the same construction as before

$$B_t = \sum_n \int_0^t \chi_n(s)ds \cdot g_n$$

$$\int_0^1 f(s)\,dB_s = \sum_n \hat{f}_n g_n$$

if

$$f(t) = \sum_n \hat{f}_n \chi_n$$

and

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_n \big(\mathbb{R}, \mathcal{B}(\mathbb{R}), m, H^1(m), \gamma_n\big),$$

where m is the reduced normal law and

$$\boxed{\gamma_n[u] = a_n u'^2}$$

with a_n constant and dependent on n.

Example. $a_n = (2\pi n)^{2q}$, $q \in \mathbb{N}$. In this case

$$\Gamma\left[\int_0^1 f(s)\,dB_s\right] = \Gamma\left[\sum_n \hat{f}_n g_n\right] = \sum_n \hat{f}_n^2 (2\pi n)^{2q}$$

from the theorem on products, we know that

$$\int_0^1 f(s)\,dB_s \in \mathbb{D} \quad \text{if and only if} \quad \sum_n \hat{f}_n^2 (2\pi n)^{2q} < +\infty.$$

Proposition VI.13. $\int_0^1 f(s)\,dB_s \in \mathbb{D}$ *if and only if the q-th derivative $f^{(q)}$ of f in the sense of distribution belongs to* $L^2\big([0,1]\big)$*; then*

$$\boxed{\Gamma\left[\int_0^1 f(s)\,dB_s\right] = \int_0^1 f^{(q)2}(s)\,ds.}$$

Proof. This result stems from the fact that $\left(f^{(q)} \in L^2([0,1]) \text{ in the sense of } \mathcal{D}'\right)$ is equivalent with $\sum_n \hat{f}_n^2 (2\pi n)^{2q} < +\infty$, as easily seen using Fourier expansion. ◇

We can observe that the structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ is *white* in the strong sense of error structures.

Proposition VI.14.

a) *Let* $f \in L^2([0,1])$ *with* $f^{(q)} \in L^2([0,1])$ *with such support that*

$$g = \tau_\alpha f = \left(t \to f(t-\alpha)\right)$$

also lies in $L^2([0,1])$*. Then for* $U = \int_0^1 f(s)\,dB_s$ *and* $V = \int_0^1 g(s)\,dB_s$*, the image structures by both* U *and* V *are equal.*

b) *Let* $f, g \in L^2([0,1])$ *and* $f^{(q)}, g^{(q)} \in L^2([0,1])$ *such that* $fg = 0$*, then for* $U = \int_0^1 f(s)\,dB_s$ *and* $V = \int_0^1 g(s)\,dB_s$*, the image structure by the pair* (U, V) *is the product of the image structures by* U *and by* V*.*

This result is also valid for the Ornstein–Uhlenbeck structure obtained for $q = 0$.

the obtained structures in this example are
"white"
in the strong sense of independence (products) of error structures for what concerns disjoint time intervals and invariance by time translations

This may be put in relation with white noise theory

2.5. Structures of the generalized Mehler type. The error structures on the Wiener space constructed in the preceding Section 2.4 can be proved to belong to a more general family which will now be introduced.

Let $m = \mathcal{N}(0, 1)$ as usual. Let's consider the probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = \big(\mathbb{R}, \mathcal{B}(\mathbb{R}), m\big)^{\mathbb{N}}$$

with g_n as coordinate mappings.

Let $X = F\big(g_0, \dots, g_n, \dots\big)$ be a bounded random variable. Consider the transform P_t:

$$P_t X = \hat{\mathbb{E}}\Big[F\Big(g_0\sqrt{e^{-a_0 t}} + \hat{g}_0\sqrt{1 - e^{-a_0 t}}, \dots, g_n\sqrt{e^{-a_n t}} + \hat{g}_n\sqrt{1 - e^{-a_n t}}, \dots\Big)\Big]$$

where the $\hat{g}_n$'s are copies of the g_n's, $\hat{\mathbb{E}}$ is the corresponding expectation and the a_n are positive numbers: $a_n \geq 0\ \forall n$.

The following properties are easily proved along the same lines as in dimension one (see Chapter II).

2.5.1. P_t is well-defined and preserves the probability measure $\mathbb{P}$.

2.5.2. P_t is continuous from $L^2(\mathbb{P})$ into itself with norm ≤ 1

$$\mathbb{E}\big(P_t X\big)^2 \leq \mathbb{E}P_t\big(X^2\big) = \mathbb{E}X^2.$$

2.5.3. P_t is a Markovian semigroup

$$\begin{aligned} P_{t+s}(X) &= P_t\big(P_s(X)\big) \\ P_t(X) &\geq 0 \quad \text{if } X \geq 0 \\ P_t(1) &= 1. \end{aligned}$$

2.5.4. P_t is symmetric with respect to $\mathbb{P}$.

Let $Y = G\big(g_0, \dots, g_n, \dots\big)$, we then obtain:

$$\mathbb{E}\big[P_t X \cdot Y\big] = \mathbb{E}\big[F\big(\xi_0, \dots, \xi_n, \dots\big)G\big(y_0, \dots, y_n, \dots\big)\big]$$

where $\xi_0, \dots, \xi_n, \dots$ are i.i.d. reduced Gaussian variables and $y_0, \dots, y_n, \dots$ are also i.i.d. reduced Gaussian variables, such that $\operatorname{cov}\big(\xi_n, y_n\big) = \sqrt{e^{-a_n t}}$, i.e.

$$\mathbb{E}\big[P_t X \cdot Y\big] = \mathbb{E}\big[X \cdot P_t Y\big].$$

2.5.5. P_t is strongly continuous on $L^2(\mathbb{P})$. Indeed if X is bounded and cylindrical

$$\lim_{t\to 0} P_t X = X \quad \text{a.e.}$$

by virtue of dominated convergence, hence

$$\lim_{t\to 0} \mathbb{E}\big[\big(P_t X - X\big)^2\big] = 0$$

again by dominated convergence. From the density of bounded cylindrical random variables in $L^2(\mathbb{P})$, the result therefore follows.

2.5.6. Let's define

$$\mathbb{D} = \left\{ X \in L^2(\mathbb{P}) \colon \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}\big[\big(X - P_t X\big) X\big] < +\infty \right\}$$

and for $X \in \mathbb{D}$

$$\mathcal{E}[X] = \lim_{t \downarrow 0} \uparrow \frac{1}{t} \mathbb{E}\big[\big(X - P_t X\big) X\big].$$

By approximation on cylindrical functions, it can be shown that this construction provides the product error structure

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty} \big(\mathbb{R}, \mathcal{B}(\mathbb{R}), m, H^1(m), u \to a_n u'^2\big)$$

and

$$\mathbb{D} = \left\{ X = F\big(g_0, \ldots, g_n, \ldots\big) \colon \forall n \frac{\partial F}{\partial g_n} \in H^1(m) \right.$$
$$\left. \sum_n a_n \left(\frac{\partial F}{\partial g_n}\right)^2 \in L^1(\mathbb{P}) \right\}$$

$$\Gamma[X] = \sum_n a_n \left(\frac{\partial F}{\partial g_n}\right)^2.$$

Let's now introduce the semigroup p_t on $L^2(\mathbb{R}_+)$ defined for

$$f = \sum_n \langle f, \chi_n \rangle \chi_n$$

by

$$p_t f = \sum_n \langle f, \chi_n \rangle e^{-a_n t} \chi_n.$$

(p_t) is a symmetric strongly continuous contraction semigroup on $L^2(\mathbb{R}_+)$ with eigenvectors χ_n. Let $(B, \mathcal{D}B)$ be its generator. Since

$$\| p_t f - f \|_{L^2}^2 = \sum_n \langle f, \chi_n \rangle^2 \big(1 - e^{-a_n t}\big)^2$$

we can observe that if

$$\sum_n \langle f, \chi_n \rangle^2 a_n^2 < +\infty$$

then $f \in \mathcal{D}B$ and $Bf = -\sum_n \langle f, \chi_n \rangle a_n \chi_n$ which leads to

Proposition VI.16. *$\int_0^\infty f(s)\,dB_s \in \mathbb{D}$ if and only if*

$$\sum_n \langle f, \chi_n\rangle^2 a_n < +\infty,$$

i.e. using, in this case, the symbolic calculus notation

$$f \in \mathcal{D}(\sqrt{-B}), \quad \sqrt{-B} f = \sum_n \langle f, \chi_n\rangle \sqrt{a_n}\chi_n,$$

we then have:

$$\boxed{\Gamma\left[\int_0^\infty f(s)\,dB_s\right] = \langle \sqrt{-B} f, \sqrt{-B} f\rangle_{L^2(\mathbb{R}_+)}.}$$

Let's emphasize that the semigroup p_t on $L^2(\mathbb{R}_+)$ is not necessarily positive on positive functions. As a matter of fact, we obtained *any* symmetric, strongly continuous contraction semigroup on $L^2(\mathbb{R}_+)$, and we can start the construction with such a semigroup as input data.

Exercise VI.17. Show that for $f \in L^2(\mathbb{R}_+)$

$$P_t\left(\int_0^\infty f\,dB\right) = \int_0^\infty (p_{\frac{t}{2}} f)\,dB$$

$$P_t\left(\exp\left\{\int f\,dB - \frac{1}{2}\|f\|_{L^2}^2\right\}\right) = \exp\left\{\int p_{\frac{t}{2}} f\,dB - \frac{1}{2}\|p_{\frac{t}{2}} f\|_{L^2}^2\right\}$$

$$P_t\left(\left(\sin\int f\,dB\right) e^{\frac{1}{2}\|f\|_{L^2}^2}\right) = \left(\sin\int p_{\frac{t}{2}} f\,dB\right) e^{\frac{1}{2}\|p_{\frac{t}{2}} f\|_{L^2}^2}.$$

2.5.9. Considering the Wiener measure as carried by $\mathcal{C}_0(\mathbb{R}_+)$ and using the symbolic calculus for operators in $L^2(\mathbb{R}_+)$ the generalized Mehler formula can be demonstrated: $\forall F \in L^2(\Omega, \mathcal{A}, \mathbb{P})$

$$\boxed{P_t F = \hat{\mathbb{E}}\left[F\left(\int_0^\infty (p_{\frac{t}{2}} 1_{[0,\cdot]})(u)\,dB_u + \int_0^\infty (\sqrt{1-p_t}\,1_{[0,\cdot]})(v)\,d\hat{B}_v\right)\right].}$$

This Mehler formula provides an intuitive interpretation of the error on the Brownian path modeled by this error structure. For example, in the Ornstein–Uhlenbeck case where $p_t u = e^{-t} u$, we can see that the path ω is perturbed in the following way

$$\omega \longrightarrow e^{-\frac{\varepsilon}{2}}\omega + \sqrt{1-e^{\varepsilon}}\,\hat{\omega}$$

where $\hat{\omega}$ is an independent standard Brownian motion and ε a small parameter.

In the case of the weighted Ornstein–Uhlenbeck case (see Exercise VI.20 below)

$$\omega(s) = \int_0^s dB_u \longrightarrow \int_0^s e^{-\alpha(u)\varepsilon/2}\,dB_u + \int_0^s \sqrt{1-e^{-\alpha(u)\varepsilon}}\,d\hat{B}_u$$

(where α is a positive function in $L^1_{\text{loc}}(\mathbb{R}_+)$).

.2 Error structures on the Poisson space

Several error structures can easily be constructed either on the Poisson process on $\mathbb{R}_+$ or on the general Poisson space. As in the case of Brownian motion, these structures allow studying more sophisticated objects, such as marked processes and processes with independent increments, which can be defined in terms of a general Poisson point process.

Among the works on the variational calculus on the Poisson space let us first cite Bichteler–Gravereau–Jacod [1987] and Wu [1987]. The construction produced by these authors yields the same objects as our approach in Section 3.2. Carlen and Pardoux, in 1990, introduced a different structure on the Poisson process on $\mathbb{R}_+$ and displayed some interesting properties. This domain represents still an active field of research (Nualart and Vives [1990], Privault [1993], Decreusefond [1998], etc.).

Our initial approach will consist of following to the greatest extent possible the classical construction of a Poisson point process, which we will first recall:

3.1. Construction of a Poisson point process with Intensity measure μ. Let's begin with the case where μ is a finite measure.

3.1.1. Let $(G, \mathcal{G}, \mu)$ be a measurable space equipped with a finite positive measure μ. We set $\theta = \mu(G)$ and $\mu_0 = \frac{1}{\theta} \cdot \mu$. Considering the product probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = \big(G, \mathcal{G}, \mu_0\big)^{\mathbb{N}^*} \times \big(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_\theta\big),$$

where $\mathcal{P}(\mathbb{N})$ denotes the σ-field of all subsets of integers $\mathbb{N}$ and P_θ denotes the Poisson law on $\mathbb{N}$ with parameter θ defined by

$$P_\theta(\{n\}) = e^{-\theta} \frac{\theta^n}{n!}, \quad n \in \mathbb{N},$$

and if we denote the coordinate mappings of this product space by $\big(X_n\big)_{n>0}$ and Y, we obtain for the X_n's a sequence of random variables with values in $(G, \mathcal{G})$ which are i.i.d. with law μ_0 and for Y an integer-valued random variable with law P_θ independent of the X_n's.

The following formula

$$N(\omega) = \sum_{n=1}^{Y(\omega)} \delta_{X_n(\omega)},$$

where δ is the Dirac measure (using the convention $\sum_1^0 = 0$) defines a random variable with values in the space of "point measures", i.e. measures which are sum of Dirac measures. Such a random variable is usually called a "point process."

Proposition VI.22. *The point process N features the following properties:*

a) *If $A_1, \ldots, A_n$ are in $\mathcal{G}$ and pairwise disjoint then the random variables $N(A_1), \ldots, N(A_n)$ are independent.*

b) *For $A \in \mathcal{G}$, $N(A)$ follows a Poisson law with parameter $\mu(A)$.*

Proof. This result is classical (see Neveu [1977] or Bouleau [2000]). ◇

Since the expectation of a Poisson variable is equal to the parameter, we have $\forall A \in \mathcal{G}$

$$\mu(A) = \mathbb{E}\big[N(A)\big]$$

such that μ can be called the *intensity* of point process N.

3.1.2. Let's now assume that the space $(G, \mathcal{G}, \mu)$ is only σ-finite. A sequence $G_k \in \mathcal{G}$ then exists such that:

- the G_k are pairwise disjoint
- $\bigcup_k G_k = G$
- $\mu\big(G_k\big) < +\infty$.

Let's denote $\big(\Omega_k, \mathcal{A}_k, \mathbb{P}_k\big)$ and N_k the probability spaces and point processes obtained by the preceding procedure on G_k; moreover let's set

$$(\Omega, \mathcal{A}, \mathbb{P}) = \prod_k \big(\Omega_k, \mathcal{A}_k, \mathbb{P}_k\big)$$
$$N = \sum_k N_k.$$

We then obtain the same properties for N as in Proposition VI.23, once the parameters of the Poisson laws used are finite.

Such a random point measure is called a Poisson point process with intensity μ.

3.1.3.. Let's indicate the Laplace characteristic functional of N.

For $f \geq 0$ and $\mathcal{G}$-measurable

$$\mathbb{E}e^{-N(f)} = \exp\left\{-\int \big(1 - e^{-f}\big)\, d\mu\right\}.$$

3.2. The white error structure on the general Poisson space. The first error structure that we will consider on the Poisson space displays the property that each point thrown in space G is erroneous and modeled by the same error structure on $(G, \mathcal{G})$, moreover if we examine the points in A_1 and their errors along with the points in A_2 and their errors, there is independence if $A_1 \cap A_2 = \emptyset$. This property justifies the expression “white error structure”.

3.2.1. Let’s begin with the case where μ is finite. Suppose an error structure is given on $\big(G, \mathcal{G}, \mu_0\big)$ e.g. $\big(G, \mathcal{G}, \mu_0, \mathbf{d}, \gamma\big)$; using the theorem on products once more, if we set

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \big(G, \mathcal{G}, \mu_0, \mathbf{d}, \gamma\big)^{\mathbb{N}^*} \times \big(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_\theta, L^2(P_\theta), 0\big),$$

we obtain an error structure that is Markovian if $(G, \mathcal{G}, \mu_0, \mathbf{d}, \gamma)$ is Markovian.

Then any quantity depending on

$$N = \sum_{n=1}^{Y} \delta_{X_n}$$

and sufficiently regular will be equipped with a quadratic error:

Proposition VI.23. *Let* $U = F\big(Y, X_1, X_2, \ldots, X_n, \ldots\big)$ *be a random variable in* $L^2(\Omega, \mathcal{A}, \mathbb{P})$, *then*

a) $U \in \mathbb{D}$ *iff* $\forall m \in \mathbb{N}, \forall k \in \mathbb{N}^*$, *for* $\mu_0^{\otimes \mathbb{N}^*}$*-a.e.* $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots$

$$F\big(m, x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots\big) \in \mathbf{d}$$

and $\mathbb{E}\left[\sum_{k=1}^{\infty} \gamma_k[F]\right] < +\infty$ *(where, as usual,* γ_k *is* γ *acting upon the k-th variable).*

b) *for* $U \in \mathbb{D}$

$$\Gamma[U] = \sum_{k=1}^{\infty} \gamma_k\big[F\big(Y, X_1, \ldots, X_{k-1}, \cdot, X_{k+1}, \ldots\big)\big](X_k).$$

theorem

on

products

Proof. This is simply the theorem on products. $\diamond$

This setting leads to the following proposition:

Proposition VI.24. *Let* $f, g \in \mathbf{d}$, *then* $N(f)$ *and* $N(g)$ *are in* $\mathbb{D}$ *and*

$$\Gamma\big[N(f)\big] = N\big(\gamma[f]\big)$$
$$\Gamma\big[N(f), N(g)\big] = N\big(\gamma[f, g]\big).$$

Proof. By $\mathbb{E}|N(f) - N(g)| \le \mathbb{E}[N|f - g|] = \mu|f - g|$, the random variable $N(f)$ depends solely upon the μ-equivalence class of f.

From the Laplace characteristic functional, we obtain:

$$\mathbb{E}\big[N(f)^2\big] = \int f^2\,d\mu + \left(\int f\,d\mu\right)^2$$

thus proving that $N(f) \in L^2(\mathbb{P})$ if $f \in L^2(\mu)$. Then for $f \in \mathbf{d}$,

$$\begin{aligned}\Gamma\big[N(f)\big] &= \sum_{k=1}^{\infty} \gamma_k \left[\sum_{n=1}^{Y} f(X_n)\right] = \sum_{k=1}^{\infty} 1_{\{k\le Y\}}\gamma[f](X_k)\\ &= \sum_{k=1}^{Y} \gamma[f](X_k) = N\big(\gamma[f]\big).\end{aligned}$$

The required result follows. $\diamond$

By functional calculus, this proposition allows computing Γ on random variables of the form $F\big(N(f_1),\dots,N(f_k)\big)$ for $F \in \mathcal{C}^1 \cap \text{Lip}$ and $f_i \in \mathbf{d}$.

Let $(a, \mathcal{D}a)$ be the generator of the structure $\big(G, \mathcal{G}, \mu_0, \mathbf{d}, \gamma\big)$, we also have:

Proposition VI.25. *If $f \in \mathcal{D}a$, then $N(f) \in \mathcal{D}A$ and*

$$A\big[N(f)\big] = N\big(a[f]\big).$$

Proof. The proof is straightforward from the definition of N. $\diamond$

For example if $f \geq 0$, $f \in \mathcal{D}a$, then

$$A\big[e^{-\lambda N(f)}\big] = e^{-\lambda N(f)} N\left(\frac{\lambda^2}{2}\gamma[f] - \lambda a[f]\right).$$

3.2.2. Chaos. Let's provide some brief comments on the chaos decomposition of the Poisson space. Let's set $\tilde{N} = N - \mu$. If $A_1,\dots,A_k$ are pairwise disjoint sets in $\mathcal{G}$, we define

$$I_k\big(1_{A_1} \otimes \cdots \otimes 1_{A_k}\big) = \tilde{N}\big(A_1\big)\cdots\tilde{N}\big(A_k\big),$$

the operator I_k extends uniquely to a linear operator on $L^2\big(G^k, \mathcal{G}^{\otimes k}, \mu^k\big)$ such that:

- $I_k(f) = I_k(\tilde{f})$, where $\tilde{f}$ is the symmetrized function of f
- $\mathbb{E} I_k(f) = 0\ \forall k \geq 1$, $I_0(f) = \int f\,d\mu$
- $\mathbb{E}\big[I_p(f) I_q(g)\big] = 0$ if $p \neq q$

- $\mathbb{E}\big[(I_p(f))^2\big] = p!\langle \tilde{f}, \tilde{g}\rangle_{L^2(\mu^p)}$.

If C_n is the subvector space of $L^2(\Omega, \mathcal{A}, \mathbb{P})$ of $I_n(f)$, we then have the direct sum:

$$L^2(\Omega, \mathcal{A}, \mathbb{P}) = \bigoplus_{n=0}^{\infty} C_n.$$

The link of the white error structure on the Poisson space with the chaos decomposition is slightly analogous to the relation of generalized Mehler-type error structures with the chaos decomposition on the Wiener space. It can be shown that if (P_t) is the semigroup on $L^2(\mathbb{P})$ associated with error structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$, then $\forall f \in L^2\big(G^n, \mathcal{G}^{\otimes n}, \mu^n\big)$

$$P_t\big(I_n(f)\big) = I_n\big(p_t^{\otimes n} f\big),$$

where (p_t) is the semigroup on $L^2(\mu_0)$ associated with the error structure $\big(G, \mathcal{G}, \mu_0, \mathbf{d}, \gamma\big)$.

It must nevertheless be emphasized that p_t here is necessarily positive on positive functions whereas this condition was not compulsory in the case of the Wiener space.

Exercise VI.26. Let d be a gradient for $\big(G, \mathcal{G}, \mu_0, \mathbf{d}, \gamma\big)$ with values in the Hilbert space H. Let's define $\mathcal{H}$ by the direct sum

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} H_n,$$

where H_n are copies of H.

Show that for $U = F\big(Y, X_1, \ldots, X_n, \ldots\big) \in \mathbb{D}$

$$D[U] = \sum_{k=1}^{\infty} d_k\big[F\big(Y, X_1, \ldots, X_{k-1}, \cdot, X_{k+1}, \ldots\big)\big]\big(X_k\big)$$

defines a gradient for $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$.

3.2.3. σ-finite case. When μ is σ-finite, the construction may be performed in one of several manners which do not all yield the same domains.

If we try to strictly follow the probabilistic construction (see sub-section 3.1.2) it can be assumed that we have error structures on each G_k

$$S_k = \left(G_k, \mathcal{G}\big|_{G_k}, \frac{1}{\mu(G_k)}\mu\Big|_{G_k}, \mathbf{d}_k, \gamma_k \right)$$

hence, as before, we have error structures on $(\Omega_k, \mathcal{A}_k, \mathbb{P}_k)$, e.g. $(\Omega_k, \mathcal{A}_k, \mathbb{P}_k, \mathbb{D}_k \Gamma_k)$, and Poisson point processes N_k.

We have noted that on

$$(\Omega, \mathcal{A}, \mathbb{P}) = \prod_k (\Omega_k, \mathcal{A}_k, \mathbb{P}_k)$$

$$N = \sum_k N_k$$

is a Poisson point process with intensity μ. Thus, it is natural to take

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_k (\Omega_k, \mathcal{A}_k, \mathbb{P}_k, \mathbb{D}_k, \Gamma_k).$$

Let's define

$$\underline{\mathbf{d}} = \left\{ f \in L^2(\mu) \colon \forall k \ f|_{G_k} \in \mathbf{d}_k \right\}$$

and for all $f \in \underline{\mathbf{d}}$, let's set

$$\gamma[f] = \sum_k \gamma_k \left[f|_{G_k} \right],$$

we then have:

Proposition VI.27. *Let $f \in \underline{\mathbf{d}}$ be such that $f \in L^1 \cap L^2(\mu)$ and $\gamma[f] \in L^1(\mu)$. Then $N(f) \in \mathbb{D}$ and*

$$\Gamma\big[N(f)\big] = N\big(\gamma[f]\big).$$

To clearly see what happens with the domains, let's proceed with the particular case where

$$(G, \mathcal{G}) = \big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big),$$

μ is the Lebesgue measure on $\mathbb{R}_+$, G_k are the intervals $[k, k+1[$, and the error structures S_k are

$$\Big([k, k+1[, \mathcal{B}\big([k, k+1[\big), dx, H^1\big([k, k+1[\big), u \to u'^2 \Big).$$

We then have in $\underline{\mathbf{d}}$ not only continuous functions with derivatives in $L^2_{\mathrm{loc}}(dx)$, but also discontinuous functions with jumps at the integers.

Practically, this is not troublesome. We thus have

Lemma. *The random σ-finite measure*

$$\tilde{N} = N - \mu$$

extends uniquely to $L^2(\mathbb{R}_+)$ and for $f \in H^1(\mathbb{R}_+, dx)$

$$\Gamma\big[\tilde{N}(f)\big] = N\big(f'^2\big).$$

3.2.4. Application to the Poisson process on $\mathbb{R}_+$. Let's recall herein our notation.

On $[k, k+1[$, we have an error structure:

$$S_k = \Big([k, k+1[, \, \mathcal{B}\big([k, k+1[\big), \, dx, \, H^1\big([k, k+1[\big), \, u \xrightarrow{\gamma_k} u'^2\Big).$$

With these error structures, we built Poisson point processes on $[k, k+1[$ and then placed error structures on them:

$$\big(\Omega_k, \mathcal{A}_k, \mathbb{P}_k, \mathbb{D}_k, \Gamma_k\big) = \big(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_1, L^2(P_1), 0\big) \times \big(S_k\big)^{\mathbb{N}^*}.$$

If $Y^k, X_1^k, X_2^k, \ldots, X_n^k, \ldots$ denote the coordinate maps, the point process is defined by

$$N^k = \sum_{n=1}^{Y^k} \delta_{X_n^k}.$$

We have proved that for $f \in H^1\big([k, k+1[\big)$

$$\Gamma_k\big[N^k(f)\big] = N^k\big(f'^2\big)$$

and for $f \in \mathcal{C}^2\big([k, k+1]\big)$ with $f'(k) = f'(k+1) = 0$,

$$A_k\big[N^k(f)\big] = \frac{1}{2} N^k(f'').$$

(cf. Example III.3 and Propositions VI.24 and VI.25).

We now take the product

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_{k=0}^{\infty} \big(\Omega_k, \mathcal{A}_k, \mathbb{P}_k, \mathbb{D}_k, \Gamma_k\big)$$

and set

$$N = \sum_{k=0}^{\infty} N^k.$$

Let's denote ξ_k the coordinate mappings of this last product, we then have from the theorem on products

Lemma VI.28.

- $\forall k \in \mathbb{N}, \forall n \in \mathbb{N}^*, X_n^k \circ \xi_k \in \mathbb{D}$
- $\Gamma\big[X_n^k \circ \xi_k\big] = 1$
- $\Gamma\big[X_m^k \circ \xi_k, X_n^\ell \circ \xi_\ell\big] = 0 \quad \textit{if} \quad k \neq \ell \quad \textit{or} \quad m \neq n.$

If we set $N_t = N\big([0,t]\big)$, N_t is a usual Poisson process with unit intensity on $\mathbb{R}_+$. Let $T_1, T_2, \dots, T_i, \dots$ be its jump times. We can prove

Proposition VI.29. *T_i belongs to $\mathbb{D}$.*

$$\Gamma\big[T_i\big] = 1, \quad \Gamma\big[T_i, T_j\big] = 0 \ \text{if}\, i \neq j.$$

Corollary VI.30.

a) *If F is $\mathcal{C}^1 \cap \mathrm{Lip}$*

$$\Gamma\big[F\big(T_1, \dots, T_p\big)\big] = \sum_{i=1}^{p} F_i'^2\big(T_1, \dots, T_p\big).$$

b) *For $f \in H^1(\mathbb{R}_+)$, $\int_0^\infty f(s)\,d(N_s - s) \in \mathbb{D}$ and*

$$\Gamma\left[\int_0^\infty f(s)\,d(N_s - s)\right] = \int_0^\infty f'^2(s)\,dN_s.$$

c) *For $f \in H^1(\mathbb{R}_+)$ with $f'(0) = 0$ and $f'' \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ we have*

$$\int_0^\infty f(s)\,d(N_s - s) \in \mathcal{D}A$$

and

$$A\left[\int_0^\infty f(s)\,d(N_s - s)\right] = \frac{1}{2}\int_0^\infty f''(s)\,dN_s.$$

3.2.5. Application to internalization. The construction discused above is indispensable for studying random variables that depend on an infinite number of T_n. Nevertheless, it also gives results in finite dimension, which could be elementarily proved using the fact that random variables $T_{n+1} - T_n$ are i.i.d. with exponential law. We have, for instance, the following results.

Lemma. *Let $g \in \mathcal{C}^1(\mathbb{R}_+)$ with polynomial growth and vanishing at zero. Let $F \in \mathcal{C}^1 \cap \mathrm{Lip}(\mathbb{R}^n)$. Then*

$$\mathbb{E}\left[\sum_{i=1}^{n} g(T_i) F_i'(T_1, \ldots, T_n)\right] = \mathbb{E}\left[\left(g(T_n) - \sum_{i=1}^{n} g'(T_i)\right) F(T_1, \ldots, T_n)\right].$$

Proof. Let us first consider an f as in Corollary VI.30. The proof of this corollary yields:

$$\mathbb{E}\left[\sum_{i=1}^{n} f'(T_i) F_i'(T_1, \ldots, T_n)\right] = \mathbb{E}\left[\left(f'(T_n) - \sum_{i=1}^{n} f''(T_i)\right) F(T_1, \ldots, T_n)\right].$$

This relation now extends to the hypotheses of the statement by virtue of dominated convergence.

◇

With the same hypotheses on F, the lemma directly yields the following formula

$$\frac{d}{d\alpha}\mathbb{E}\big[F(\alpha T_1, \ldots, \alpha T_n)\big] = \frac{1}{\alpha}\mathbb{E}\big[(T_n - n) F(\alpha T_1, \ldots, \alpha T_n)\big]. \tag{5}$$

Exercise. Provide a formula without derivation for

$$\frac{d}{d\alpha}\mathbb{E}\big[F(\alpha h(T_1), \ldots, \alpha h(T_n))\big].$$

Exercise. Consider the random variable with values in $\mathbb{R}^2$ $X = \big(N(f_1), N(f_2)\big)$ for $f_1, f_2 \in L^1 \cap L^2(\mathbb{R}_+)$; show that if

$$\det \Gamma\big[N(f_i), N(f_j)\big] = 0 \quad \mathbb{P}\text{-a.s.},$$

then the law of X is carried by a straight line.

3.3. The Carlen–Pardoux error structure. For the classical Poisson process on $\mathbb{R}_+$, E. Carlen and E. Pardoux have proposed and studied an error structure which possesses a gradient and a δ with attractive properties.

As previously mentioned, if T_n are the jump times of the Poisson process, random variables $E_n = T_n - T_{n-1}, n > 1, E_1 = T_1$, are i.i.d. with exponential law. Since the knowledge of all E_n is equivalent to the knowledge of the process path, we can start with the E_n's and place an error structure on them.

Consider the error structure

$$S = \left(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), e^{-x}\,dx, \mathbf{d}, u \xrightarrow{\gamma} xu'^2(x)\right),$$

closure of the pre-structure defined on $C_k^\infty(\mathbb{R}_+)$, and define

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = S^{\otimes \mathbf{N}^*}$$

with the random variables E_n being the coordinate mappings. We have:

$$\begin{aligned} \Gamma\big[E_n\big] &= E_n \quad n \geq 1 \\ \Gamma\big[E_m, E_n\big] &= 0 \quad m \neq n. \end{aligned}$$

Lemma. *Setting*

$$D\big[E_n\big] = -1_{]T_{n-1},T_n]}(t)$$

defines a gradient with value in $\mathcal{H} = L^2(\mathbb{R}_+)$.

Indeed

$$\int_0^\infty 1_{]T_{n-1},T_n]}(t)\,dt = E_n = \Gamma\big[E_n\big].$$

Among the attractive properties of this gradient is the following.

Proposition VI.32. *Let $U = \varphi\big(E_1, \ldots, E_n\big)$ for $\varphi \in \mathcal{C}^1 \cap \mathrm{Lip}(\mathbb{R}^n)$, then*

$$U = \mathbb{E}U + \int_0^\infty K_s d\big(N_s - s\big),$$

where K_s is the predictable projection of the process $D[U](s)$.

For the proof we refer to Bouleau–Hirsch [1991], Chapter V, Section 5.

Corollary VI.33. *The adjoint operator δ coincides with the integral with respect to $N_t - t$ on predictable stochastic processes of $L^2(\mathbb{P}, \mathcal{H})$.*

Proof. If H_s is a predictable process in $L^2(\mathbb{P}, \mathcal{H})$, the proposition implies the equality

$$\mathbb{E}\left[U \int_0^\infty H_s\, d\big(N_s - s\big)\right] = \mathbb{E}\left[\int_0^\infty D[U](s) H_s\, ds\right]. \tag{6}$$

It then follows that $H_s \in \operatorname{dom}\delta$ and $\delta[H] = \int_0^\infty H_s d\big(N_s - s\big)$. $\diamond$

Although this error structure yields new integration by parts formulae different from the preceding ones, on very simple random variables it yields the same internalization formula.

Let $X = F\big(\alpha T_1, \dots, \alpha T_n\big)$, $F \in \mathcal{C}^1 \cap \text{Lip}$ as before. Then

$$\frac{d}{d\alpha}\mathbb{E}\big[F\big(\alpha T_1, \dots, \alpha T_n\big)\big] = \mathbb{E}\left[\sum_{i=1}^n T_i F_i'\big(\alpha T_1, \dots, \alpha T_n\big)\right],$$

whereas

$$D[X] = -\sum_{i=1}^n \alpha F_i'\big(\alpha T_1, \dots, \alpha T_n\big) 1_{]0,T_i]}(s)$$

such that

$$\begin{aligned}\frac{d}{d\alpha}\mathbb{E}[X] &= -\frac{1}{\alpha}\mathbb{E}\int_0^{T_n} D[X](s)\,ds \\ &= -\frac{1}{\alpha}\mathbb{E}\int_0^{\infty} D[X]1_{]0,T_n]}(s)\,ds\end{aligned}$$

which gives according to (6)

$$\begin{aligned}&= -\frac{1}{\alpha}\mathbb{E}\left[X\int_{]0,T_n]} d\big(N_s - s\big)\right] \\ &= \frac{1}{\alpha}\mathbb{E}\big[X\big(T_n - n\big)\big].\end{aligned}$$

which is exactly (5).