H) sensitivity analysis of an SDE, application to finance
Black-Scholes: sensitivity with respect to $\sigma$:

$$dS_t = S_t(\sigma dB_t + r dt)$$

- For the calls, we may consider that the implicit Black-Scholes-volatility is given by the derivative markets. If we use this volatility $\sigma$ to price other options, which accuracy have we to take on $\sigma$?

A natural choice is to consider a constant proportional error:

$$\Gamma_\sigma[I] = \varsigma^2 \sigma^2$$

where $\varsigma$ is the (historical) agitation of the implicit B&$S$ vol.

(For instance a lognormal homogeneous error structure)

- Now for a European option with payoff $f(S_T)$, the value at time $t \in [0, T]$ of the option is $V_t = F(t, S_t, \sigma, r)$ with

$$F(t, x, \sigma, r) = e^{-r(T-t)} \int_{\mathbb{R}} f \left(x e^{(r-\frac{\sigma^2}{2}) (T-t) + \sigma y \sqrt{T-t}}} \right) \frac{e^{-y^2}}{\sqrt{2\pi}} dy.$$

Putting as usual $\gamma_t = \frac{\partial^2 F}{\partial x^2}(t, S_t, \sigma, r)$ we get

$$\left\{\begin{array}{l}
\Gamma_\sigma[V_0] = T^2 \sigma^2 S_0^4 \gamma_0^2 \Gamma_\sigma[I] \\
A_\sigma[V_0] = T \sigma S_0^2 \gamma_0 A_\sigma[I] + \frac{1}{2} \frac{\partial^2 F}{\partial \sigma^2}(0, S_0, \sigma, r) \Gamma_\sigma[I].
\end{array}\right.$$
Error calculi for a diffusion model

We consider an asset modelled by a diffusion process
\[ dX_t = X_t \sigma(t, X_t) dB_t + X_t r(t) dt. \]

We may study the sensitivity of the financial quantities (option price, hedging) to an error on any parameter of the model.

For finite dimensional parameters we do as above, the most convenient is to proceed with a gradient especially with the sharp.

We shall display the fact that error structures are tools which allow to study as well the sensitivity with respect to the Brownian path, or to other functional quantities.

1. Effect of a lack of accuracy on the Brownian path

2. Error on the function \((t, x) \rightarrow \sigma(t, x)\) in the whole model

3. Error on the function \(\sigma(t, x)\) not in the whole model but rather only on the hedging formula used by the trader.
Effect of an inaccuracy on the Brownian path: hypotheses

In order to express that the inaccuracy is sometimes larger depending on the period (week-ends, etc.) we choose an O-U-structure with weights:

Let \( \alpha \) be a function on \( \mathbb{R}_+ \) such that \( \alpha(x) \geq 0 \ \forall x \in \mathbb{R}_+ \) and \( \alpha \in L^1_{\text{loc}}(\mathbb{R}_+, dx) \), the O.-U. structure with weights \( \alpha \) is defined as the generalized Mehler-type structure associated with the semi-group \( p_t u = e^{-\alpha t} u \) on \( L^2(\mathbb{R}_+) \).

It is the mathematical expression of the following perturbation of the Brownian path:

\[
\omega(s) = \int_0^s dB_u \quad \longrightarrow \quad \int_0^s e^{-\alpha(u) \varepsilon} dB_u + \int_0^s \sqrt{1 - e^{-\alpha(u) \varepsilon}} d\hat{B}_u,
\]

where \( \hat{B} \) is an independent standard Brownian motion.

This error structure satisfies for \( u \in \mathcal{C}_K(\mathbb{R}_+) \).

\[
\Gamma \left[ \int_0^\infty u(s) dB_s \right] = \int_0^\infty \alpha(s) u^2(s) \, ds
\]

It has a gradient: \( D : \mathbb{R} \to L^2(\mathbb{P}, \mathcal{H}) \) où \( \mathcal{H} = L^2(\mathbb{R}_+, dt) \)

\[
D \left[ \int u(s) dB_s \right] (t) = \sqrt{\alpha(t)} u(t) \ \forall u \in L^2(\mathbb{R}_+, (1 + \alpha) dt)
\]

We shall use mainly the sharp, which is a particular gradient with \( \mathcal{H} = L^2(\hat{\Omega}, \hat{A}, \hat{\mathbb{P}}) \) defined by

\[
\left( \int_0^\infty u(s) dB_s \right)^\# = \int_0^\infty \sqrt{\alpha(s)} u(s) d\hat{B}_s, \quad u \in L^2(\mathbb{R}_+, (1 + \alpha) dt),
\]

which satisfies the chain rule and, for a regular adapted process \( H_t \):

\[
\left( \int_0^\infty H_s dB_s \right)^\# = \int_0^\infty \sqrt{\alpha(s)} H_s d\hat{B}_s + \int_0^\infty H_s^\# dB_s.
\]
Propagation of an error on the Brownian motion

From the equation

\[ X_t = X_0 + \int_0^t X_s \sigma(s, X_s) \, dB_s + \int_0^t X_s r(s) \, ds, \]

we draw

\[ X^\#_t = \int_0^t (\sigma(s, X_s) + X_s \sigma_x'(s, X_s)) X^\#_s \, dB_s \]

\[ + \int_0^t \sqrt{\alpha(s)} X_s \sigma(s, X_s) \, d\hat{B}_s + \int_0^t X^\#_s r(s) \, ds. \]

which may be solved in the spirit of the “variation of the constant” method

If we put

\[
\begin{cases}
  K_t = \sigma(t, X_t) + X_t \sigma_x'(t, X_t) \\
  M_t = \exp \left\{ \int_0^t K_s \, dB_s - \frac{1}{2} \int_0^t K_s^2 \, ds + \int_0^t r(s) \, ds \right\},
\end{cases}
\]

we have

\[ X^\#_t = M_t \int_0^t \frac{\sqrt{\alpha(s)} X_s \sigma(s, X_s)}{M_s} \, d\hat{B}_s. \]

The effect of an error from \((B_t)_{t\geq 0}\) on the process \((X_t)_{t\geq 0}\) is given by:

\[
\Gamma[X_t] = M_t^2 \int_0^t \frac{\alpha(s) X_s^2 \sigma^2(s, X_s)}{M_s^2} \, ds
\]

\[
\Gamma[X_s, X_t] = M_s M_t \int_0^{s \wedge t} \frac{\alpha(u) X_u^2 \sigma^2(u, X_u)}{M_u^2} \, du.
\]
Effect on the value of an option

Under the probability which is a martingale-measure, if \( f(X_T) \) is the payoff of a European option, its value at time \( t \) is

\[
V_t = \mathbb{E} \left[ \exp \left( -\int_t^T r(s) \, ds \right) f(X_T) \mid \mathcal{F}_t \right]
\]

where \( (\mathcal{F}_t) \) is the Brownian filtration.

Let us suppose \( f \in \mathcal{C}^1 \cap \text{Lip} \). Let be

\[
Y = \exp \left( -\int_t^T r(s) \, ds \right) f(X_T).
\]

In order to compute \( (\mathbb{E}[Y \mid \mathcal{F}_t])^\# \) we apply the

Lemma: Let \( \Gamma_t \) be defined by

\[
\Gamma_t \left[ \int u(s) \, dB_s \right] = \Gamma \left[ \int 1_{[0,t]}(s) u(s) \, dB_s \right]
\]

and let \( U \rightarrow U_t^# \) be the sharp operator associated with \( \Gamma_t \), then for \( U \in \mathbb{D} \)

\[
(\mathbb{E}[U \mid \mathcal{F}_t])^\# = \mathbb{E}[U_t^# \mid \mathcal{F}_t].
\]

Hence

\[
(\mathbb{E}[Y \mid \mathcal{F}_t])^\# = \exp \left( -\int_t^T r(s) \, ds \right) \mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t] \int_0^t \sqrt{\alpha(s)X_s\sigma(s,X_s)} \, d\hat{B}_s
\]

and

\[
\Gamma[V_t] = \Gamma[\mathbb{E}[Y \mid \mathcal{F}_t]] = \exp \left( -2 \int_t^T r(s) \, ds \right) (\mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t])^2 \int_0^t \frac{\alpha(s)X_s^2\sigma^2(s,X_s)}{M_s^2} \, ds.
\]

This gives also the cross-error of \( V_t \) and \( V_s \), (usefull for instance to compute the error on \( \int_0^T h(s) \, dV_s \) or \( \int_0^T V_s h(s) \, ds \))

\[
\Gamma[V_s, V_t] = \exp \left( -\int_s^t r(u) \, du - \int_t^T r(v) \, dv \right) \mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_s] \mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t] \int_{s \wedge t}^t \frac{\alpha(u)X_u^2\sigma^2(u,X_u)}{M_u^2} \, du.
\]
Effect on the hedging portfolio

The hedging portfolio is the adapted process $H_t$, which satisfies

$$\tilde{V}_t = V_0 + \int_0^t H_s d\tilde{X}_s,$$

where $\tilde{V}_t = \exp(-\int_0^t r(s) \, ds) V_t$ and $\tilde{X}_t = \exp(-\int_0^t r(s) \, ds) X_t$.

Here it is

$$H_t = \exp \left( - \int_t^T r(s) \, ds \right) \mathbb{E} \left[ f'(X_T) M_T \mid \mathcal{F}_t \right] \frac{1}{M_t}.$$

The same method as for $V_t$ gives

$$\Gamma[H_t] = \exp \left( -2 \int_t^T r(s) \, ds \right) \left( \mathbb{E} \left[ \frac{M_T}{M_t} (f''(X_T) M_T + f'(X_T) Z_T) \mid \mathcal{F}_t \right] \right)^2 \int_0^t \frac{\alpha(u) X_u^2 \sigma(u, X_u)}{M_u^2} \, du$$

avec

$$Z_T = \int_t^T L_s \, dB_s - \int_t^T K_s L_s M_s \, ds$$

$$K_s = \sigma(s, X_s) + X_s \sigma'_x(s, X_s)$$

$$L_s = 2\sigma'_x(s, X_s) + X_s \sigma''_x(s, X_s).$$
Effect of an inaccuracy on the Brownian motion: summary

If we introduce the following notation which extends the Black–Scoles case:

\[
\delta_t = H_t = \exp \left( - \int_t^T r(s) \, ds \right) \mathbb{E}[f'(X_T) M_T \mid \mathcal{F}_t] \frac{1}{M_t}
\]

\[
\gamma_t = \exp \left( - \int_t^T r(s) \, ds \right) \mathbb{E} \left[ \frac{M_T^2}{M_t^2} f''(X_T) + \frac{M_T}{M_t^2} f'(X_T) Z_t \mid \mathcal{F}_t \right].
\]

we may resume the study of the sensitivity with respect to the Brownian path as follows:

\[
\Gamma[X_t] = M_t^2 \int_0^t \frac{\alpha(u) X_u^2 \sigma(u, X_u)}{M_u^2} \, du
\]

\[
\Gamma[X_s, X_t] = M_s M_t \int_0^{s \wedge t} \frac{\alpha(u) X_u^2 \sigma^2(u, X_u)}{M_u^2} \, du.
\]

\[
V_t^\# = \delta_t X_t^\#
\]

\[
H_t^\# = \gamma_t X_t^\#
\]

\[
\Gamma[V_t] = \delta_t^2 \Gamma[X_t]
\]

\[
\Gamma[V_s, V_t] = \delta_s^2 \delta_t \Gamma[X_s, X_t]
\]

\[
\Gamma[H_t] = \gamma_t^2 \Gamma[X_t]
\]

\[
\Gamma[H_s, H_t] = \gamma_s \gamma_t \Gamma[X_s, X_t]
\]

\[
\Gamma[V_s, H_t] = \delta_s \gamma_t \Gamma[X_s, X_t]
\]

It is also possible to compute the biases: in the Black-Scholes case (with \( \Gamma[S_t] = S_t^2 \sigma^2 t \)) we have:

\[
A[S_t] = -S_t \sigma B_t + \frac{1}{2} \sigma^2 S_t t
\]

\[
A[V_t] = \delta_t A[S_t] + \frac{1}{2} \gamma_t \Gamma[S_t]
\]

\[
A[H_t] = \gamma_t A[S_t] + \frac{1}{2} \frac{\partial^3 F}{\partial x^3}(t, S_t, \sigma, r) \Gamma[S_t].
\]
Erroneous volatility: model with an inaccuracy on $\sigma$

We suppose that the model of the asset is

$$X_t = x + \int_0^t X_s \sigma(s, X_s) \, dB_s + \int_0^t X_s r(s) \, ds$$

and that in this equation, the function $\sigma$ is endowed with an error such that the following formula holds

$$(\sigma(t, Y))^\# = \sigma^\#(t, Y) + \sigma_x'(t, Y) Y^\#$$

where $Y$ is a random variable, eventually correlated with $\sigma$, such that $\sigma(t, Y) \in \mathbb{D}$.

From the equation

$$X_t = x + \int_0^t X_s \sigma(s, X_s) \, dB_s + \int_0^t X_s r(s) \, ds$$

we draw

$$X_t^\# = \int_0^t (X_s^\# \sigma(s, X_s) + X_s \sigma^\#(s, X_s) + X_s \sigma_x'(s, X_s) X_s^\#) \, dB_s + \int_0^t X_s^\# r(s) \, ds$$

which has the solution

$$X_t^\# = M_t \int_0^t \frac{X_s \sigma^\#(s, X_s)}{M_s} (dB_s - K_s \, ds).$$

with

$$K_s = \sigma(s, X_s) + X_s \sigma_x'(s, X_s)$$

and

$$M_t = \exp \left\{ \int_0^t K_s \, dB_s - \frac{1}{2} \int_0^t K_s^2 \, ds + \int_0^t r(s) \, ds \right\}.$$
First case. \( \sigma(t, x) \) is represented by a series of functions \( \psi_n(t, x) \) regular in \( x \):

\[
\sigma(t, x) = \sum_n a_n \psi_n(t, x)
\]

the coefficients \( a_n \) are erroneous random variables with laws such that a.s. only a finite number of \( a_n \) dont vanish.

\[
\Gamma[a_n] = a_n^2 \\
\Gamma[a_m, a_n] = 0 \quad \text{for } m \neq n \\
a_n^\# = a_n - \hat{\mathbb{E}}a_n \\
\beta_n = \sqrt{\hat{\mathbb{E}}(\hat{a}_n - \hat{\mathbb{E}}a_n)^2},
\]

\[
\sigma^\#(s, X_s) = \sum_n a_n^\# \psi_n(s, X_s)
\]

\[
X_t^\# = \sum_n M_t \int_0^t \frac{X_s \psi_n(s, X_s)}{M_s} (dB - K_s ds) a_n^\#
\]

\[
\Gamma[X_t] = \sum_n M_t^2 \left( \int_0^t \frac{X_s \psi_n(s, X_s)}{M_s} (dB - K_s ds) \right)^2 a_n^2.
\]

Then the error on the value of a European option is

\[
\Gamma[V_t] = \sum_n (V_t^n)^2 a_n^2.
\]

with

\[
V_t^n = \exp \left( - \int_t^T r(s) ds \right) \mathbb{E} \left[ f'(X_T) M_T \int_0^T \frac{X_s \psi_n(s, X_s)}{M_s} (dB - K_s ds) \mid \mathcal{F} \right]
\]

which may be computed by Monte Carlo, the same for the hedging portfolio.
Second case. We suppose the volatility is locale and stochastic
\[ \sigma(t, y, w) \]
given by a diffusion processes independent of \((B_t)_{t \geq 0}\).
\[ \sigma(t, y, w) = \sigma^y_t(w) \]
where \(\sigma_t\) is solution of
\[
\begin{align*}
    d\sigma_t &= a(\sigma_t)\,dW_t + b(\sigma_t)\,dt \\
    \sigma_0 &= c(y)
\end{align*}
\]
with \((W_t)_{t \geq 0}\) a Brownian motion independent of \((B_t)_{t \geq 0}\).

If functions \(a, b\) et \(c\) are regular, the map \(y \to \sigma(t, y, w)\) is regular and we suppose that the formula
\[
(\sigma(t, Y))^\# = \sigma^\#(t, Y) + \sigma'_y(t, Y)Y^\#
\]
holds at each step of the computation.

If \(W\) is endowed with an error of O-U-type, putting
\[
m^y_t = \exp \left\{ \int_0^t a'(\sigma^y_s)\,dW_s - \frac{1}{2} \int_0^t a'^2(\sigma^y_s)\,ds + \int_0^t b'(\sigma^y_s)\,ds \right\}
\]
we obtain
\[
\Gamma[X_t] = M^2_t \int_0^t \left( \int_u^t \frac{X_s c'(X_s)m^X_s a(\sigma^X_s)}{M_s m^X_u} (dB_s - K_s\,ds) \right)^2 \,du.
\]
The computation of \(\Gamma[V_t]\) et \(\Gamma[H_t]\) is done in a similar way and leads to computable formulae by Monte Carlo method.
Third case. We suppose the volatility is local and stochastic, \( \sigma(t, y) \) being a stationary process independent of \((B_t)_{t \geq 0}\).

For instance, let \( k \) be regular functions \( \eta_1(y), \ldots, \eta_k(y) \) and let us put

\[
\sigma(t, y) = \sigma_0 e^{Y(t,y)}
\]

with

\[
Y(t, y) = \sum_{i=1}^{k} Z_i(t) \eta_i(y)
\]

where \( Z(t) = (Z_1(t), \ldots, Z_k(t)) \) is a stationary process with values in \( \mathbb{R}^k \).

In order to have a real process, we may take for example

\[
Z_i(t) = \sum_j \int_0^\infty \xi_{ij}(\lambda)(\cos \lambda t \, dU_\lambda + \sin \lambda t \, dV_\lambda)
\]

where \( \xi_{ij} \in L^2(\mathbb{R}_+) \) et \( U_\lambda^1, \ldots, U_\lambda^k, V_\lambda^1, \ldots, V_\lambda^k \) are independent Brownian motions.

We put an error of O-U-type on these Brownian motions.

\[
X_t^\# = M_t \int_0^t \frac{X_s \sigma(s, X_s) \dot{Y}(s, X_s)}{M_s} (dB_s - K_s \, ds).
\]

this leads for \( \Gamma[X_t] \) to a sum of squares:

\[
\Gamma[X_t] = \int_0^t M_t^2 \sum_{i,j} \left[ \left( \int_0^t \frac{X_s \sigma(s, X_s)}{M_s} y_i(X_s) \xi_{ij}(\lambda) \cos \lambda s (dB_s - K_s \, ds) \right)^2 + \left( \int_0^t \frac{X_s \sigma(s, X_s)}{M_s} y_i(X_s) \xi_{ij}(\lambda) \sin \lambda s (dB_s - K_s \, ds) \right)^2 \right] ds.
\]

etc.
Error on $\sigma$ due to the trader: preliminary remark

• Let us consider a probabilistic model with a parameter $\lambda$ (here the volatility). If a quantity, because of mathematical relations of the model, may be written in two ways:

\[
X = \varphi(\omega, \lambda) \\
X = \psi(\omega, \lambda)
\]

and if we consider that $\lambda$ is erroneous (in the whole model), the error will be the same when $X$ is computed by $\varphi$ or by $\psi$.

• If, on the contrary, in order to take a practical decision, we use a particular explicit formula of the model and if we make an error on $\lambda$ when using this formula, then the error depends on the formula we use.

Let us take a simple example.

Let $L$ be the length of the projection of a triangle with edges of lengths $a_1$, $a_2$, $a_3$ and with polar angles $\theta_1 + \alpha$, $\theta_2 + \alpha$, $\theta_3 + \alpha$.

The length $L$ satisfies

\[
L = \max_{i=1,2,3} |a_i \cos(\theta_i + \alpha)| \\
L = \frac{1}{2} \sum_{i=1,2,3} |a_i \cos(\theta_i + \alpha)|.
\]

If the user is wrong on $a_1$, and only on $a_1$ (without to try to respect the triangle) then the first formula gives an error on $L$ different from zero only when the term $|a_1 \cos(\theta_1 + \alpha)|$ dominates the other ones, but not the second formula.
Error on $\sigma$ due to the trader

Hence we have to specify which formula we deal with.

In order to manage an option with payoff $f(X_T)$ we suppose that the trader performs a correct pricing but that his hedging is imperfect, because an error on $\sigma$.

The hedging equation is

$$\tilde{V}_t = V_0 + \int_0^T H_s d\tilde{X}_s$$

where $\tilde{V}_t = \exp \left( -\int_0^t r(s) \, ds \right) V_t$ and $\tilde{X}_t = \exp \left( -\int_0^t r(s) \, ds \right) X_t$

with $V_t = \exp \left( -\int_t^T r(s) \, ds \right) \mathbb{E}[f(X_T) \mid \mathcal{F}_t]$ which gives:

$$H_t = \exp \left( -\int_t^T r(s) \, ds \right) \mathbb{E}[f'(X_T) M_T \mid \mathcal{F}_t] \frac{1}{M_t}. \quad (*)$$

By doing an error on $H_t$, the trader doesn’t realizes at time $T$ the discounted payoff $\tilde{V} = \exp \left( -\int_0^T r(s) \, ds \right) f(X_T)$ but

$$\tilde{P}_T = V_0 + \int_0^T [H_s] \, d\tilde{X}_s$$

where $H_s$ is computed by $(*)$ with a wrong function $\sigma$ what we denote by brackets $[H_s]$. We must make formula $(*)$ completely explicit:

The trader uses the Markovian character of $X_t$ in order to write the conditional expectation under the form

$$\mathbb{E} \left[ f'(X_T) \frac{M_T}{M_t} \mid \mathcal{F}_t \right] = \Psi(t, X_t).$$

When computing $\Psi$, he does an error on $\sigma$ and he is correct on $X_t$, which is given by the market (since we suppose that the model is correct).
Error on $\sigma$ due to the trader

The function $\Psi$ is given by

$$\Psi(t, x) = \frac{\partial \Phi}{\partial x}(t, x)$$

where $\Phi$ is the function giving the value of the option from the stock price $X_t$:

$$V_t = \exp \left( - \int_t^T r(s) \, ds \right) \mathbb{E}[f(X_T) \mid \mathcal{F}_t] = \Phi(t, X_t).$$

It satisfies

$$\begin{cases}
\Phi(T, x) = f(x) \\
\frac{\partial \Phi}{\partial t} + A_t \Phi - r(t) \Phi = 0
\end{cases}$$

where $A_t$ is the operator

$$A_t u(x) = \frac{1}{2} x^2 \sigma^2(t, x) \frac{\partial^2 u}{\partial x^2}(x) + xr(s) \frac{\partial u}{\partial x}(x).$$

We are concerned by the calculation of

$$(\tilde{P}_T)^\# = \int_0^T \Psi^\#(t, X_t) \, d\tilde{X}_t = \int_0^T \frac{\partial \Phi^\#}{\partial x}(t, X_t) \, d\tilde{X}_t.$$

The function $\Phi^\#(t, x)$ satisfies:

$$\begin{cases}
\Phi^\#(T, x) = 0 \\
\frac{\partial \Phi^\#}{\partial t} + A_t \Phi^\# + A_t^\# \Phi - r(t) \Phi^\# = 0
\end{cases}$$

where $A_t^\#$ is the operator

$$A_t^\# u(x) = \frac{1}{2} x \sigma(t, x) \sigma^\#(t, x) \frac{\partial^2 u}{\partial x^2}(x).$$
Error on $\sigma$ due to the trader

Using these formula and the Ito formula applied to

$$\exp \left( - \int_0^t r(s) \, ds \right) \Phi^#(t, X_t).$$

we obtain

$$\int_0^t \frac{\partial \Phi^#}{\partial x}(s, X_s) \, d\tilde{X}_s = -\Phi^#(0, X_0) + \int_0^t \exp \left( - \int_0^s r(u) \, du \right) \left( A^#_s \Phi \right)(s, X_s)$$

hence eventually

$$(\tilde{P}_T)^# = -\Phi^#(0, X_0) + \frac{1}{2} \int_0^T \tilde{X}_s \sigma(s, X_s) \sigma^#(s, X_s) \frac{\partial^2 \Phi}{\partial x^2}(s, X_s) \, ds.$$

• $\sigma^#$ is not yet specified. The preceding calculus is valid, for instance, when we modelize the error done by the trader by one of the three case discussed above:

First case

$$\sigma(t, y) = \sum_n a_n \psi_n(t, y)$$

$$\sigma^#(t, y) = \sum_n a^#_n \psi_n(t, y).$$

Second case. $\sigma$ is an independent diffusion

$$\sigma^#(t, y) = c'(y) m^y_\cdot \int_0^t \frac{a(s^y)}{m^s_\cdot} \, d\hat{W}_s.$$

Third case. $\sigma$ is an independent stationary process

$$\sigma(t, x) = \sigma_0 \exp (Y(t, y))$$

$$\sigma^#(t, x) = \sigma(t, x) \hat{Y}(t, y).$$
Error on $\sigma$ due to the trader

Let us make some comments on the formula we obtained:

$$(\tilde{P}_T)^\# = -\Phi^\#(0, X_0) + \frac{1}{2} \int_0^T \tilde{X}_s \sigma(s, X_s) \sigma^\#(s, X_s) \frac{\partial^2 \Phi}{\partial x^2}(s, X_s) ds.$$  

- The first term: $-\Phi^\#(0, X_0)$ comes from the fact that the trader is supposed to perform a correct pricing. Hence his pricing isn’t coherent with the stochastic integral he uses to hedge. $\Phi^\#(0, X_0)$ is the difference between the pricing that the trader would have proposed and the true one (that one of the model).

- In the second term

$$\frac{1}{2} \int_0^T \tilde{X}_s \sigma(s, X_s) \sigma^\#(s, X_s) \frac{\partial^2 \Phi}{\partial x^2}(s, X_s) ds$$

the quantity $\sigma^\#(t, x)$ is a random derivative in the sense of Dirichlet forms. Often it may be interpreted as a directional derivative.

So, we see that if the payoff is a (regular and ) convex function of the price of the asset and if $\sigma(t, x)$ has a positive directional derivative in the direction of $y(t, x)$, then this second term is positive.

In other words, if the trader hedges with a function $\sigma$ disturbed in the direction of such a function $y$, his final lost is weaker than the difference between the pricing he would have proposed and that one of the market (= that one of the model here) since the second term is positive.