H) sensitivity analysis of an SDE, application to finance

Error calculi with respect to parameters $\in \mathbb{R}^n$

. Black-Scholes : sensitivity with respect to σ :

$$dS_t = S_t(\sigma\, dB_t + r\, dt)$$

• For the calls, we may consider that the implicit Black-Scholes-volatility is given by the derivative markets. If we use this volatility σ to price other options, which accuracy have we to take on σ ?

A natural choice is to consider a constant proportional error :

$$\Gamma_\sigma[I] = \varsigma^2\sigma^2$$

where ς is the (historical) agitation of the implicit B&S vol.

(For instance a lognormal homogeneous error structure)

• Now for a European option with payoff $f(S_T)$, the value at time $t \in [0,T]$ of the option is $V_t = F(t, S_t, \sigma, r)$ with

$$F(t,x,\sigma,r) = e^{-r(T-t)} \int_{\mathbb{R}} f\left(xe^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma y\sqrt{T-t}}\right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy.$$

Putting as usual $\mathrm{gamma}_t = \dfrac{\partial^2 F}{\partial x^2}(t, S_t, \sigma, r)$
we get

$$\begin{cases} \Gamma_\sigma[V_0] &= T^2\sigma^2 S_0^4\, \mathrm{gamma}_0^2\, \Gamma_\sigma[I] \\ A_\sigma[V_0] &= T\sigma S_0^2\, \mathrm{gamma}_0\, A_\sigma[I] + \frac{1}{2}\frac{\partial^2 F}{\partial\sigma^2}(0, S_0, \sigma, r)\Gamma_\sigma[I]. \end{cases}$$

I think this bias has significant financial consequences.

(even if we consider that the implicit volatility is "without bias on his spot" i.e. $A_\sigma[I](\sigma_0) = 0$, it remains a bias)

We consider an asset modelled by a diffusion process

$$dX_t = X_t\sigma(t, X_t)\, dB_t + X_t r(t)\, dt.$$

We may study the sensitivity of the financial quantities (option price, hedging) to an error on any parameter of the model.

For finite dimensional parameters we do as above, the most convenient is to proceed with a gradient especially with the sharp.

We shall display the fact that error structures are tools which allow to study as well the sensitivity with respect to the Brownian path, or to other functional quantities.

1. Effect of a lack of accuracy on the Brownian path

2. Error on the function $(t, x) \to \sigma(t, x)$ in the whole model

3. Error on the function $\sigma(t, x)$ not in the whole model but rather only on the hedging formula used by the trader.

Effect of an inaccuracy on the Brownian path : hypotheses

In order to express that the inaccuracy is sometimes larger depending on the period (week-ends, etc.) we choose an O-U-structure with weights :

Let α **be a function on** $\mathbb{R}_+$ **such that** $\alpha(x) \geq 0\ \forall x \in \mathbb{R}_+$ **and** $\alpha \in L^1_{\text{loc}}(\mathbb{R}_+, dx)$**, the O.-U. structure with weights** α **is defined as the generalized Mehler-type structure associated with the semi-group** $p_t u = e^{-\alpha t} u$ **on** $L^2(\mathbb{R}_+)$**.**

It is the mathematical expression of the following perturbation of the Brownian path :

$$\omega(s) = \int_0^s dB_u \quad \longrightarrow \quad \int_0^s e^{-\frac{\alpha(u)}{2}\varepsilon}\, dB_u + \int_0^s \sqrt{1 - e^{-\alpha(u)\varepsilon}}\, d\hat{B}_u,$$

where $\hat{B}$ **is an independent standard Brownian motion.**

This error structure satisfies for $u \in \mathcal{C}_K(\mathbb{R}_+)$**.**

$$\Gamma\left[\int_0^\infty u(s)\, dB_s\right] = \int_0^\infty \alpha(s) u^2(s)\, ds$$

It has a gradient : $D\colon \mathbb{D} \to L^2(\mathbb{P}, \mathcal{H})$ **où** $\mathcal{H} = L^2(\mathbb{R}_+, dt)$

$$D\left[\int u(s)\, dB_s\right](t) = \sqrt{\alpha(t)}\, u(t)\ \forall u \in L^2(\mathbb{R}_+, (1+\alpha)\, dt)$$

We shal use mainly the sharp, which is a particular gradient with $\mathcal{H} = L^2(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$ **defined by**

$$\left(\int_0^\infty u(s)\, dB_s\right)^\# = \int_0^\infty \sqrt{\alpha(s)}\, u(s)\, d\hat{B}_s, \quad u \in L^2(\mathbb{R}_+, (1+\alpha)dt),$$

which satisfies the chain rule and, for a regular adapted process H_t **:**

$$\left(\int_0^\infty H_s\, dB_s\right)^\# = \int_0^\infty \sqrt{\alpha(s)} H_s\, d\hat{B}_s + \int_0^\infty H_s^\#\, dB_s.$$

Propagation of an error on the Brownian motion

From the equation

$$X_t = X_0 + \int_0^t X_s \sigma(s, X_s)\, dB_s + \int_0^t X_s r(s)\, ds,$$

we draw

$$\begin{aligned} X_t^{\#} = & \int_0^t (\sigma(s, X_s) + X_s \sigma'_x(s, X_s)) X_s^{\#}\, dB_s \\ & + \int_0^t \sqrt{\alpha(s)} X_s \sigma(s, X_s)\, d\hat{B}_s + \int_0^t X_s^{\#} r(s)\, ds. \end{aligned}$$

which may be solved in the spirit of the "variation of the constant" method

If we put

$$\begin{cases} K_t & = & \sigma(t, X_t) + X_t \sigma'_x(t, X_t) \\ M_t & = & \exp\left\{ \int_0^t K_s\, dB_s - \frac{1}{2} \int_0^t K_s^2\, ds + \int_0^t r(s)\, ds \right\}, \end{cases}$$

we have

$$X_t^{\#} = M_t \int_0^t \frac{\sqrt{\alpha(s)} X_s \sigma(s, X_s)}{M_s}\, d\hat{B}_s.$$

The effect of an error from $(B_t)_{t \geq 0}$ on the process $(X_t)_{t \geq 0}$ is given by :

$$\begin{aligned} \Gamma[X_t] & = & M_t^2 \int_0^t \frac{\alpha(s) X_s^2 \sigma^2(s, X_s)}{M_s^2}\, ds \\ \Gamma[X_s, X_t] & = & M_s M_t \int_0^{s \wedge t} \frac{\alpha(u) X_u^2 \sigma^2(u, X_u)}{M_u^2}\, du. \end{aligned}$$

Effect on the value of an option

Under the probability which is a martingale-measure, if $f(X_T)$ is the payoff of a European option, its value at time t is

$$V_t = \mathbb{E}\left[\exp\left(-\int_t^T r(s)\,ds\right) f(X_T) \mid \mathcal{F}_t\right]$$

where $(\mathcal{F}_t)$ is the Brownian filtration.

Let us suppose $f \in \mathcal{C}^1 \cap \mathrm{Lip}$. Let be

$$Y = \exp\left(-\int_t^T r(s)\,ds\right) f(X_T).$$

In order to compute $(\mathbb{E}[Y \mid \mathcal{F}_t])^{\#}$ we apply the Lemma : *Let Γ_t be defined by*

$$\Gamma_t\left[\int u(s)\,dB_s\right] = \Gamma\left[\int 1_{[0,t]}(s)u(s)\,dB_s\right]$$

and let $U \rightarrow U^{\#t}$ be the sharp operator associated with Γ_t, then for $U \in \mathbb{D}$

$$(\mathbb{E}[U \mid \mathcal{F}_t])^{\#} = \mathbb{E}[U^{\#t} \mid \mathcal{F}_t].$$

Hence

$$\begin{aligned}(\mathbb{E}[Y \mid \ \mathcal{F}_t])^{\#} &= \\ &\exp\left(-\textstyle\int_t^T r(s)\,ds\right) \mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t] \int_0^t \frac{\sqrt{\alpha(s)}X_s\sigma(s,X_s)}{M_s}\,d\hat{B}_s\end{aligned}$$

and

$$\begin{aligned}\Gamma[V_t] &= \Gamma[\mathbb{E}[Y \mid \mathcal{F}_t]] = \\ &= \exp\left(-2\textstyle\int_t^T r(s)\,ds\right) (\mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t])^2 \int_0^t \frac{\alpha(s)X_s^2\sigma^2(s,X_s)}{M_s^2}\,ds.\end{aligned}$$

This gives also the cross-error of V_t and V_s, (usefull for instance to compute the error on $\int_0^T h(s)\,dV_s$ or $\int_0^T V_s h(s)\,ds$)

$$\begin{aligned}\Gamma[V_s, V_t] = &\exp\left(-\textstyle\int_s^T r(u)\,du - \int_t^T r(v)\,dv\right) \\ &\mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_s]\mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t] \int_0^{s\wedge t} \frac{\alpha(u)X_u^2\sigma^2(u,X_u)}{M_u^2}\,du.\end{aligned}$$

Effect on the hedging portfolio

The hedging portfolio is the adapted process H_t, which satisfies

$$\tilde{V}_t = V_0 + \int_0^t H_s \, d\tilde{X}_s,$$

where $\tilde{V}_t = \exp(-\int_0^t r(s)\,ds)V_t$ **and** $\tilde{X}_t = \exp(-\int_0^t r(s)\,ds)X_t$**.**

Here it is

$$H_t = \exp\left(-\int_t^T r(s)\,ds\right) \mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t]\frac{1}{M_t}\,.$$

The same method as for V_t **gives**

$$\Gamma[H_t] = \exp\left(-2\int_t^T r(s)\,ds\right) \left(\mathbb{E}\left[\tfrac{M_T}{M_t}(f''(X_T)M_T + f'(X_T)Z_t^T) \mid \mathcal{F}_t\right]\right)^2 \int_0^t \frac{\alpha(u)X_u^2\sigma(u,X_u)}{M_u^2}\,du$$

$$\begin{aligned}
\textbf{avec} \quad Z_t^T &= \int_t^T L_s\,dB_s - \int_t^T K_s L_s M_s\,ds \\
K_s &= \sigma(s,X_s) + X_s\sigma'_x(s,X_s) \\
L_s &= 2\sigma'_x(s,X_s) + X_s\sigma''_{x^2}(s,X_s).
\end{aligned}$$

Effect of an inaccuracy on the Brownian motion : summary

If we introduce the following notation which extends the Black–Scoles case :

$$\begin{aligned}
\mathbf{delta}_t &= H_t = \exp\left(-\int_t^T r(s)\,ds\right)\mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t]\frac{1}{M_t} \\
\mathbf{gamma}_t &= \exp\left(-\int_t^T r(s)\,ds\right)\mathbb{E}\left[\frac{M_T^2}{M_t^2}f''(X_T) + \frac{M_T}{M_t^2}f'(X_T)Z_t^T \mid \mathcal{F}_t\right].
\end{aligned}$$

we may resume the study of the sensitivity with respect to the Brownian path as follows :

$$\begin{aligned}
\Gamma[X_t] &= M_t^2\int_0^t \frac{\alpha(u)X_u^2\sigma^2(u,X_u)}{M_u^2}\,du \\
\Gamma[X_s,X_t] &= M_sM_t\int_0^{s\wedge t} \frac{\alpha(u)X_u^2\sigma^2(u,X_u)}{M_u^2}\,du. \\
V_t^{\#} &= \mathbf{delta}_t\, X_t^{\#} \\
H_t^{\#} &= \mathbf{gamma}_t\, X_t^{\#} \\
\Gamma[V_t] &= \mathbf{delta}_t^2\,\Gamma[X_t] \\
\Gamma[V_s,V_t] &= \mathbf{delta}_s\mathbf{delta}_t\,\Gamma[X_s,X_t] \\
\Gamma[H_t] &= \mathbf{gamma}_t^2\,\Gamma[X_t] \\
\Gamma[H_s,H_t] &= \mathbf{gamma}_s\mathbf{gamma}_t\,\Gamma[X_s,X_t] \\
\Gamma[V_s,H_t] &= \mathbf{delta}_s\mathbf{gamma}_t\,\Gamma[X_s,X_t]
\end{aligned}$$

It is also possible to compute the biases : in the Black-Scholes case (with $\Gamma[S_t] = S_t^2\sigma^2 t$) we have :

$$\begin{aligned}
A[S_t] &= -S_t\sigma B_t + \frac{1}{2}\sigma^2 S_t t \\
A[V_t] &= \mathbf{delta}_t A[S_t] + \frac{1}{2}\mathbf{gamma}_t\,\Gamma[S_t]
\end{aligned}$$

$$A[H_t] = \mathbf{gamma}_t A[S_t] + \frac{1}{2}\frac{\partial^3 F}{\partial x^3}(t,S_t,\sigma,r)\Gamma[S_t].$$

Erroneous volatility : model with an inaccuracy on σ

We suppose that the model of the asset is

$$X_t = x + \int_0^t X_s \sigma(s, X_s)\, dB_s + \int_0^t X_s r(s)\, ds$$

and that in this equation, the function σ is endowed with an error such that the following formula holds

$$(\sigma(t, Y))^{\#} = \sigma^{\#}(t, Y) + \sigma'_x(t, Y) Y^{\#}$$

where Y is a random variable, eventually correled with σ, such that $\sigma(t, Y) \in \mathbb{D}$.

From the equation

$$X_t = x + \int_0^t X_s \sigma(s, X_s)\, dB_s + \int_0^t X_s r(s)\, ds$$

we draw

$$X_t^{\#} = \int_0^t (X_s^{\#} \sigma(s, X_s) + X_s \sigma^{\#}(s, X_s) + X_s \sigma'_x(s, X_s) X_s^{\#})\, dB_s + \int_0^t X_s^{\#} r(s)\, d$$

which has the solution

$$X_t^{\#} = M_t \int_0^t \frac{X_s \sigma^{\#}(s, X_s)}{M_s} (dB_s - K_s\, ds).$$

with

$$K_s = \sigma(s, X_s) + X_s \sigma'_x(s, X_s)$$

and

$$M_t = \exp\left\{ \int_0^t K_s\, dB_s - \frac{1}{2} \int_0^t K_s^2\, ds + \int_0^t r(s)\, ds \right\}.$$

First case. $\sigma(t,x)$ **is represented by a series of functions** $\psi_n(t,x)$ **regular in** x**:**

$$\sigma(t,x) = \sum_n a_n \psi_n(t,x)$$

the coefficients a_n **are erroneous random variables with laws such that a.s. only a finite number of** a_n **dont vanish.**

$$\begin{aligned}
\Gamma[a_n] &= a_n^2 \\
\Gamma[a_m, a_n] &= 0 \quad \textbf{for } m \neq n \\
a_n^\# &= a_n \frac{\hat{a}_n - \hat{\mathbb{E}}\hat{a}_n}{\beta_n} \qquad \beta_n = \sqrt{\hat{\mathbb{E}}(\hat{a}_n - \hat{\mathbb{E}}\hat{a}_n)^2},
\end{aligned}$$

$$\sigma^\#(s, X_s) = \sum_n a_n^\# \psi_n(s, X_s)$$

$$X_t^\# = \sum_n M_t \int_0^t \frac{X_s \psi_n(s, X_s)}{M_s} (dB_1 - K_s\, ds) a_n^\#$$

$$\Gamma[X_t] = \sum_n M_t^2 \left(\int_0^t \frac{X_s \psi_n(s, X_s)}{M_s} (dB_s - K_s\, ds) \right)^2 a_n^2.$$

Then the error on the value of a European option is

$$\Gamma[V_t] = \sum_n (V_t^n)^2 a_n^2.$$

with

$$V_t^n = \exp\left(-\int_t^T r(s)\, ds\right) \mathbb{E}\left[f'(X_T) M_T \int_0^T \frac{X_s \psi_n(s, X_s)}{M_s} (dB_s - K_s\, ds) \mid \mathcal{F}\right.$$

which may be computed by Monte Carlo, the same for the hedging portfolio.

Second case. **We suppose the volatility is locale and** *stochastic*

$$\sigma(t, y, w)$$

given by a diffusion processes independent of $(B_t)_{t\geq 0}$**.**

$$\sigma(t, y, w) = \sigma_t^y(w)$$

where σ_t **is solution of**

$$\begin{cases} d\sigma_t &= a(\sigma_t)\, dW_t + b(\sigma_t)\, dt \\ \sigma_0 &= c(y) \end{cases}$$

with $(W_t)_{t\geq 0}$ **a Brownian motion independent of** $(B_t)_{t\geq 0}$**.**

If functions a**,** b **et** c **are regular, the map** $y \to \sigma(t, y, w)$ **is regular and we suppose that the formula**

$$(\sigma(t, Y))^{\#} = \sigma^{\#}(t, Y) + \sigma'_y(t, Y) Y^{\#}$$

holds at each step of the computation.

If W **is endowed with an error of O-U-type, putting**

$$m_t^y = \exp\left\{ \int_0^t a'(\sigma_s^y)\, dW_s - \frac{1}{2} \int_0^t a'^2(\sigma_s^y)\, ds + \int_0^t b'(\sigma_s^y)\, ds \right\}$$

we obtain

$$\Gamma[X_t] = M_t^2 \int_0^t \left(\int_u^t \frac{X_s c'(X_s) m_s^{X_s} a(\sigma_u^{X_s})}{M_s m_u^{X_s}} (dB_s - K_s\, ds) \right)^2 du.$$

The computation of $\Gamma[V_t]$ **et** $\Gamma[H_t]$ **is done in a similar way and leads to computable formulae by Monte Carlo method.**

Third case. **We suppose the volatility is local and stochastic, $\sigma(t,y)$ being a stationary process independent of $(B_t)_{t\geq 0}$.**

For instance, let k be regular functions $\eta_1(y),\ldots,\eta_k(y)$ and let us put

$$\sigma(t,y)=\sigma_0 e^{Y(t,y)}$$

with

$$Y(t,y)=\sum_{i=1}^{k} Z_i(t)\eta_i(y)$$

where $Z(t)=(Z_1(t),\ldots,Z_k(t))$ is a stationary process with values in $\mathbb{R}^k$.

In order to have a real process, we may take for example

$$Z_i(t)=\sum_j \int_0^\infty \xi_{ij}(\lambda)(\cos\lambda t\, dU_\lambda^j+\sin\lambda t\, dV_\lambda^j)$$

where $\xi_{ij}\in L^2(\mathbb{R}_+)$ et $U_\lambda^1,\ldots,U_\lambda^k$, $V_j^1,\ldots,V_\lambda^k$ are independent Brownian motions.

We put an error of O-U-type on these Brownian motions.

$$X_t^{\#}=M_t\int_0^t \frac{X_s\sigma(s,X_s)\hat{Y}(s,X_s)}{M_s}(dB_s-K_s\,ds).$$

this leads for $\Gamma[X_t]$ to a sum of squares:

$$\Gamma[X_t]=\int_0^t M_t^2\sum_{ij}\Big[\left(\int_0^t \frac{X_s\sigma(s,X_s)}{M_s}y_i(X_s)\xi_{ij}(\lambda)\cos\lambda s(dB_s-K_s\,ds)\right)^2$$

$$+\left(\int_0^t \frac{X_s\sigma(s,X_s)}{M_s}y_i(X_s)\xi_{ij}(\lambda)\sin\lambda s(dB_s-K_s\,ds)\right)^2\Big]ds.$$

etc.

Error on σ due to the trader : preliminary remark

• **Let us consider a probabilistic model with a parameter λ (here the volatility). If a quantity, because of mathematical relations of the model, may be written in two ways :**

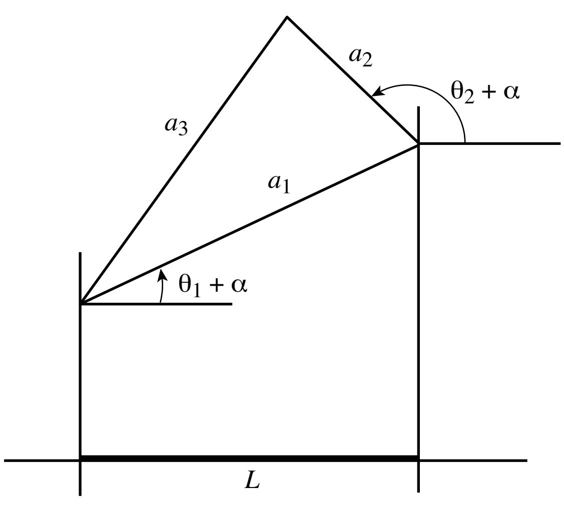
$$\begin{aligned} X &= \varphi(\omega, \lambda) \\ X &= \psi(\omega, \lambda) \end{aligned}$$

and if we consider that λ is erroneous (in the whole model), the error will be the same when X is computed by φ or by ψ.

• **If, on the contrary, in order to take a practical decision, we use a particular explicit formula of the model and if we make an error on λ when using this formula, then the error depends on the formula we use.**

Let us take a simple example.

Let L be the length of the projection of a triangle with edges of lengths a_1, a_2, a_3 and with polar angles $\theta_1+\alpha, \theta_2+\alpha, \theta_3+\alpha$.



The length L satisfies

$$L = \max_{i=1,2,3} |a_i \cos(\theta_i + \alpha)|$$

$$L = \frac{1}{2} \sum_{i=1,2,3} |a_i \cos(\theta_i + \alpha)|.$$

If the user is wrong on a_1, and only on a_1 (without to try to respect the triangle) then the first formula gives an error on L different from zero only when the term $|a_1 \cos(\theta_1+\alpha)|$ dominates the other ones, but not the second formula.

Hence we have to specify which formula we deal with.

In order to manage an option with payoff $f(X_T)$ we suppose that the trader performs a correct pricing but that his hedging is imperfect, becaus an error on σ.

The hedging equation is

$$\tilde{V}_t = V_0 + \int_0^T H_s \, d\tilde{X}_s$$

where $\tilde{V}_t = \exp\left(-\int_0^t r(s)\,ds\right) V_t$ and $\tilde{X}_t = \exp\left(-\int_0^t r(s)\,ds\right) X_t$ with $V_t = \exp\left(-\int_t^T r(s)\,ds\right) \mathbb{E}[f(X_T) \mid \mathcal{F}_t]$ which gives :

$$H_t = \exp\left(-\int_t^T r(s)\,ds\right) \mathbb{E}[f'(X_T)M_T \mid \mathcal{F}_t]\frac{1}{M_t}. \qquad (*)$$

By doing an error on H_t, the trader doesn't realizes at time T the discounted payoff $\tilde{V} = \exp\left(-\int_0^T r(s)\,ds\right) f(X_T)$ but

$$\tilde{P}_T = V_0 + \int_0^T [H_s] \, d\tilde{X}_s$$

where H_s is computed by (*) with a wrong function σ what we denote by brackets $[H_s]$. We must make formula (*) completely explicit :

The trader uses the Markovian character of X_t in order to write the conditional expectation under the form

$$\mathbb{E}\left[f'(X_T)\frac{M_T}{M_t} \mid \mathcal{F}_t\right] = \Psi(t, X_t).$$

When computing Ψ, he does an error on σ and he is correct on X_t, which is given by the market (since we suppos that the model is correct).

Error on σ due to the trader

The function Ψ is given by

$$\Psi(t,x) = \frac{\partial \Phi}{\partial x}(t,x)$$

where Φ is the function giving the value of the option from the stock price X_t :

$$V_t = \exp\left(-\int_t^T r(s)\,ds\right) \mathbb{E}[f(X_T) \mid \mathcal{F}_t] = \Phi(t, X_t).$$

It satisfies

$$\begin{cases} \Phi(T,x) = f(x) \\ \\ \dfrac{\partial \Phi}{\partial t} + A_t\Phi - r(t)\Phi = 0 \end{cases}$$

where A_t is the operator

$$A_t u(x) = \frac{1}{2}x^2\sigma^2(t,x)\frac{\partial^2 u}{\partial x^2}(x) + xr(s)\frac{\partial u}{\partial x}(x).$$

We are concerned by the calculation of

$$(\tilde{P}_T)^{\#} = \int_0^T \Psi^{\#}(t, X_t)\,d\tilde{X}_t = \int_0^T \frac{\partial \Phi^{\#}}{\partial x}(t, X_t)\,d\tilde{X}_t.$$

The function $\Phi^{\#}(t,x)$ satisfies :

$$\begin{cases} \Phi^{\#}(T,x) = 0 \\ \\ \dfrac{\partial \Phi^{\#}}{\partial t} + A_t\Phi^{\#} + A_t^{\#}\Phi - r(t)\Phi^{\#} = 0 \end{cases}$$

where $A_t^{\#}$ is the operator

$$A_t^{\#} u(x) = \frac{1}{2}x\sigma(t,x)\sigma^{\#}(t,x)\frac{\partial^2 u}{\partial x^2}(x).$$

Error on σ due to the trader

Using these formula and the Ito formula applied to

$$\exp\left(-\int_0^t r(s)\,ds\right)\Phi^{\#}(t,X_t).$$

we obtain

$$\int_0^t \frac{\partial\Phi^{\#}}{\partial x}(s,X_s)\,d\tilde{X}_s = -\Phi^{\#}(0,X_0)+\int_0^t \exp\left(-\int_0^s r(u)\,du\right)(A_s^{\#}\Phi)(s,X_s)$$

hence eventually

$$(\tilde{P}_T)^{\#} = -\Phi^{\#}(0,X_0) + \frac{1}{2}\int_0^T \tilde{X}_s\sigma(s,X_s)\sigma^{\#}(s,X_s)\frac{\partial^2\Phi}{\partial x^2}(s,X_s)\,ds.$$

• $\sigma^{\#}$ **is not yet specified. The preceding calculus is valid, for instance, when we modellize the error done by the trader by one of the three case discussed above :**

First case

$$\begin{aligned}\sigma(t,y) &= \sum_n a_n\psi_n(t,y)\\ \sigma^{\#}(t,y) &= \sum_n a_n^{\#}\psi_n(t,y).\end{aligned}$$

Second case. σ **is an independent diffusion**

$$\sigma^{\#}(t,y) = c'(y)m_t^y\int_0^t \frac{a(\sigma_s^y)}{m_s^y}\,d\hat{W}_s.$$

Third case. σ **is an independent stationary process**

$$\begin{aligned}\sigma(t,x) &= \sigma_0\exp(Y(t,y))\\ \sigma^{\#}(t,x) &= \sigma(t,x)\hat{Y}(t,y).\end{aligned}$$

Let us make some comments on the formula we obtained :

$$(\tilde{P}_T)^\# = -\Phi^\#(0, X_0) + \frac{1}{2}\int_0^T \tilde{X}_s \sigma(s, X_s)\sigma^\#(s, X_s)\frac{\partial^2\Phi}{\partial x^2}(s, X_s)\, ds.$$

• **The first term: $-\Phi^\#(0, X_0)$ comes from the fact that the trader is supposed to perform a correct pricing. Hence his pricing isn't coherent with the stochastic integral he uses to hedge.**

$\Phi^\#(0, X_0)$ **is the difference between the pricing that the trader would have proposed and the true one (that one of the model).**

• **In the second term**

$$\frac{1}{2}\int_0^T \tilde{X}_s \sigma(s, X_s)\sigma^\#(s, X_s)\frac{\partial^2\Phi}{\partial x^2}(s, X_s)\, ds$$

the quantity $\sigma^\#(t, x)$ is a random derivative in the sense of Dirichlet forms. Often it may be interpretaed as a directional derivative.

So, we see that if the payoff is a (regular and) convex function of the price of the asset and if $\sigma(t, x)$ has a positive directioal dervative in the direction of $y(t, x)$, then this second term is positive.

In other words, if the trader hedges with a function σ disturbed in the direction of such a function y, his final lost is weaker than the difference between the pricing he would have proposed and that one of the market (= that one of the model here) since the second term is positive.