A non classical approach to finance

1. Instantaneous structure of a financial asset

Let us first take the example of a diffusion

$$S_t = S_0 + \int_0^t a(s, S_s)\, dB_s + \int_0^t b(s, S_s)\, ds.$$

let us suppose the instant t is slightly fuzzy, the error $S_{t+h} - S_t$ leads to put

$$\begin{aligned} \Gamma[S_t] &= \lim_{h\to 0} \frac{1}{h}\mathbb{E}[(S_{t+h} - S_t)^2 \mid \mathcal{F}_t] \\ A[S_t] &= \lim_{h\to 0} \frac{1}{h}\mathbb{E}[S_{t+h} - S_t \mid \mathcal{F}_t] \end{aligned}$$

i.e.

$$\begin{aligned} \Gamma[S_t] &= \frac{d\langle S, S\rangle_t}{dt} = a^2(t, S_t) \\ A[S_t] &= b(t, S_t). \end{aligned}$$

Remark 1

$$\left.\begin{array}{l} \textbf{law of } S_t \\ \Gamma[S_t] \\ A[S_t] \end{array}\right\} \textbf{ too much} \qquad\qquad \left\{\begin{array}{l} \textbf{law of } S_t \\ \Gamma[S_t] \end{array}\right.$$

Remark 2

The relation $\Gamma[S_t] = \frac{d\langle S,S\rangle_t}{dt}$ is preserved by some changes of variables.

2. Exemple : homogeneous lognormal error structure

If we start from

$$dS_t = S_t(\sigma\, dB_t + r\, dt) \quad \textbf{c'est à dire } S_t = S_0 \exp\left(\sigma B_t + \left(r - \frac{\sigma^2}{2}\right)t\right)$$

the preceding identification leads to the error structure

$$\Sigma = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \nu, \mathbf{d}, \gamma)$$

where ν is the law of S_t and $\gamma[u](x) = \sigma^2 x^2 u'^2(x)$.

It is the image of the O-U-structure

$$\Sigma_{\text{OU}} = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0,t), H^1(\mathcal{N}(0,t)), \Gamma_1 \colon v \to v'^2)$$

by the map $x \to S_0 \exp\left(\sigma x + \left(r - \frac{\sigma^2}{2}\right)t\right)$.

Hence

$$\mathbf{d} = \{u \in L^2(\nu) \colon x \to x^2 u'^2(x) \in L^1(\nu)\},$$

Σ satisfies the Poincaré inequality:

$$\operatorname{var}_\nu[u] \le \mathbb{E}_\nu[\gamma[u]],$$

the proportional error on S_t is constant

$$\frac{\mathbb{E}[\Gamma[S_t] \mid S_t = x]}{x^2} = \frac{\mathbb{E}[\Gamma[S_t] \mid S_t = 2x]}{(2x)^2} = \sigma^2.$$

and the generator satisfies $I \in \mathcal{D}A$ and

$$A[I](y) = y\frac{1}{2}\left[\frac{\sigma^2}{2} + r - \frac{1}{t}\log\frac{y}{S_0}\right]$$

in particular $A[I](y_0) = 0$ if

$$y_0 = S_0 \exp\left\{\left(r + \frac{\sigma^2}{2}\right)t\right\} = \|S_t\|_{L^2}.$$

3.

Instantaneous error structure associated with a stationary process

Let us mention that there exists generally a symmetric Markov process tangent to a stationary process :

Proposition *Let* $(X_t)_{t\in\mathbb{R}}$ *be a stationary process with values in* $\mathbb{R}^d$ *with continuous sample paths. Let* ν *be the law of* X_0*. Let us suppose that* $\forall f \in \mathcal{C}_K^\infty(\mathbb{R}^d)$*, the limit*

$$\lim_{t\to 0} \frac{1}{2t}\mathbb{E}[f(X_{-t}) - 2f(X_0) + f(X_t) \mid X_0 = x]$$

exists in $L^2(\mathbb{R}^d, \nu)$*. Let* $A[f]$ *be this limit and let* $\Gamma[f]$ *be defined by*

$$\Gamma[f] = A[f^2] - 2fA[f],$$

then $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu, \mathcal{C}_K^\infty(\mathbb{R}^d), \Gamma)$ *is a closable error pre-structure .*

4.
From an instantaneous error structure to a pricing model

We consider an asset S_t with values in $\mathbb{R}^d$, a process with the following hypotheses :

• For each fixed t, the random variable S_t is erroneous and has an image error structure of the form

$$(\mathbb{R}^d, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$$

with

$$\Gamma_S[X^i, X^j](x) = \alpha_{ij}(x)$$

where $I = (X^1, \ldots, X^d)$ is the identity map of $\mathbb{R}^d$. and where $(\alpha_{ij}(x))$ is a symmetric matrix.

(a priori $\mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma$ depends on t)

• At time zero, for the observed spot value s_0, the random variable S_0 is without bias :

$$A_{S_0}[X^i](s_0) = 0 \quad \forall i = 1, \ldots, d$$

• similarly at any time

$$A_{S_t}[X^i](S_t(\omega)) = 0 \quad \forall i = 1, \ldots, d$$

(hence $\mathbb{D}, \Gamma$ and A do depend on t and on ω)

We have replaced the global hypothesis of martingale measure, by the local hypothesis of "process without bias on the spot" which expresses the fundamental idea of Bachelier that at present time we dont know whether the rate is increasing or decreasing and the present value is the best forecasting for the small interval of time $(t, t+h)$.

But now if S is without bias, $F(S)$ has a bias :

$$A_S[F] = \sum_i F'_i A_S[X^i] + \frac{1}{2}\sum_{ij} F''_{ij}\Gamma_S[X^i, X^j]$$

hence on the spot :

$$A_S[F] = \frac{1}{2}\sum_{ij} \alpha_{ij} F''_{ij}$$

in other words the value at present time of the quantity $F(S)$ at time $t+h$ isn't $F(S)$ but

$$F(S) + A_S[F](S)h = F(S) + \frac{h}{2}\sum_{ij} \alpha_{ij}(S)F''_{ij}(S).$$

In order to know the value at time 0 of the option $F(S_T)$ at time T, we share the interval $[0,T]$ and we argue going downstairs back by computing the value at

$$T - \frac{1}{n},\ T - \frac{2}{n}, \ldots$$

Putting F_t the function to take on the spot in order to get the value of the option at time t, we find

$$F_0 = \lim_{n\to\infty}\left(I - \frac{T}{n}B\right)^{-n} F = e^{TB} = Q_T F$$

where $(Q_t)_{t\geq 0}$ is the semi-group with generator

$$B[u](x) = \frac{1}{2}\sum_{ij} \alpha_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

6.

• **If the interest rate varies, we have to replace the notion of "process without bias on the spot" by the one of "process whose bias on the spot is equal to the interest rate".**

This leads to the following pricing formula :

$$\begin{cases} \dfrac{\partial F_t}{\partial t} + \dfrac{1}{2}\sum_{ij} \alpha_{ij}(x)\dfrac{\partial^2 F_t}{\partial x_i \partial x_j} + r(t)\sum_i x_i \dfrac{\partial F_t}{\partial x_i} - r(t)F_t = 0 \\ \\ F_T = F. \end{cases}$$

• **About the hedging, we find (coming back to the case** $r = 0$**) that an exact hedging occurs if, in addition to the preceding hypotheses we suppose** S_t **be a continuous semi-martingale satisfying**

$$\frac{d\langle S^i, S^j\rangle_t}{dt} = \alpha_{ij}(S_t),$$

because then, thanks Ito formula, the equation satisfied by F **implies**

$$F_t(S_t) = F_0(S_0) + \sum_i \int_0^t \frac{\partial F}{\partial x_i}(S_s)\, dS_s.$$

• **The semi-martingale case with**

$$\frac{d\langle S, S\rangle_t}{dt} \neq \Gamma_t[I](S_t)$$

where a lack of hedging occurs is also interesting.