Fourth lecture

LINKS WITH STATISTICS AND EMPIRICAL DATA

HOW TO IDENTIFY AN ERROR STRUCTURE ?

In the error calculus by Dirichlet forms, *any quantity is random***. The a priori law has to be understood as the "scope and the use" of the measurement device.**

Once more the idea comes from Gauss : In order to argue about errors, in his famous argument justifying the normal law for the errors, assuming the hypothesis that the arithmetic average gives the best estimate, he supposes that the erroneous quantity X is random and varies in the scope of the measurement device according to an a priori law μ.

The results of the measurement operations are other random variables $X_1, \dots, X_n$ and Gauss supposes that

a) the conditional law of X_1 given X is of the form
$\mathbb{P}\{X_i \in E | X = x\} = \int_E \varphi(x_1 - x)\, dx_1$,

b) the variables $X_1, \dots, X_n$ are conditionally independent given X.

He then computes the conditional law of X given the results of measurement, it has a density with respect to μ and writing that this density is maximal at the arithmetic average,

he obtains the relation

$$\frac{\varphi'(t-x)}{\varphi(t-x)} = a(t-x) + b$$

hence $$\varphi(t-x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(t-x)^2}{2\sigma^2}\right)$$

i.e. the errors follow normal laws.

So, there are two questions in order to identify an error structure

① to determine the a priori law

② to determine Γ.

To determine experimentally the operator Γ of an error structure

The first question is a classical question in statistics.

Let us tackle the second question. Let us suppose the error structure to be identified is on $\mathbb{R}^d$:

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P}, \mathbb{D}, \Gamma)$$

Practically $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$ is the image space of a quantity x which is measured with some accuracy. $\mathbb{P}$ is its a priori law.

We shall consider that performing a measure of the quantity x remains to estimate statistically x as parameter of a family of probability laws $\mathbf{Q}_x$

We then know that if we have an estimator T, say without bias, of x, ($\mathbf{Q}_x[T] = x$) the accuracy of x is limited by the Cramer-Darmois-Fisher-Rao inequality

$$\mathbf{Q}_x[(T-x)(T-x)^t] \geq [J(x)]^{-1}$$

with equality if T is efficient.

This leads us to take

$$\boxed{\Gamma[I](x) = J^{-1}(x)}$$

where $J(x)$ is the Fisher information matrix associated to our prametric model $\mathbf{Q}_x$.

Let us recall the definition of the Fisher information et the inequality of Cramer and al.

Inequality of Cramer and al.

Let be $x \in \mathbb{R}^d$ and let $\mathbf{Q}_x$ be a family of probability measures on some space dominated by the probability measure $\mathbf{Q}$

$$\mathbf{Q}_x = L(x,.)\,\mathbf{Q} \quad \textbf{with } L(x,.) \textbf{ regular in } x$$

Then for every random variable $Y \in L^2(\mathbf{Q})$ we have

$$\mathbb{E}_x[Y - \mathbb{E}_x(Y)]^2 \geq (\mathrm{grad}\mathbb{E}_x(Y))^t [J(x)]^{-1} \mathrm{grad}\mathbb{E}_x(Y)$$

where $J(x)$ is the Fisher information matrix of the model

$$J(x) = \left(\mathbb{E}_x[\frac{\partial \log L(x)}{\partial x_i} \frac{\partial \log L(x)}{\partial x_j}] \right)_{ij}$$

$J(x)$ **behaves as an information**

$\Gamma[I](x)$ **is an accuracy**

We put the fundamental identification :

$$\boxed{\Gamma[I](x) = J^{-1}(x)}$$

In the very simple case where $x \in \mathbb{R}^d$ is estimated as parameter of laws $\mathcal{N}(m(x), \Sigma)$ with $m : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ and Σ invertible $n \times n$-matrix, we obtain

$$\Gamma[I](x) = \left[(m'(x))^t \Sigma^{-1} (m'(x) \right]^{-1}$$

Thus if we choose $\Gamma[I](x) = J^{-1}(x)$**, since** Γ **satisfies the functional calculus, if** $f : \mathbb{R}^d \longrightarrow \mathbb{R}^p$ **is of class** $\mathcal{C}^1 \cap \mathrm{Lip}$**, we have**

$$\Gamma[f](x) = (\mathrm{grad} f)^t.\Gamma[I](x).\mathrm{grad} f$$

In other words the operator Γ **is determined for any image structure of our error structure** $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P}, \mathbb{D}, \Gamma)$**.**

Hence the question occurs to know whether we obtain an accuracy compatible with this calculus when we measure $f(x)$ **instead of** x **for** f **injective.**

The answer is positive.

Under the hypotheses of a "regular" statistical model, if we consider $y = f(x)$ *where* $f \in \mathcal{C}^1 \cap \mathrm{Lip}$ *is injective, the error structure obtained for* y *is the image of the structure obtained for* x *by the application* f*.*

This means that the obtained error on x **doesn't depend on the way of parametrizing and does possess a physical sense.**

The identification is also stable by products in a natural sense.

Cf. N. B. et Chr. Chorro, ''Structures d'erreur et estimation paramètrique'', *Note C.R.A.S.*, Ser.I338 (2004), 305-310

Remarks on the a priori law

In the fundamental identification, the a priori law $\mathbb{P}$ doesn't occur, except by the technical condition that the obtained structure be closable.

Hence, $\mathbb{P}$ may be taken among a large variety of probability measures, either obtained by statistical methods or coming from physical arguments. Let us note that the bias operator $(A, \mathcal{D}A)$ will depend on the choice of $\mathbb{P}$.

• **If we have no particular reason for an other choice,** *the Jeffrey probability measure* **is interesting because it is stable by changement of parameter and therefore possesses a physical meaning.**

Let $A \subset \mathbb{R}^d$ be such that $\int_A \sqrt{\det J_X(x)}\, dx < +\infty$, we take

$$\mathbb{P} = \frac{\sqrt{\det J_X(x)}}{\int_A \sqrt{\det J_X(t)}\, dt}\, dx$$

• **Now the a priori law may come from physical reasons. It is the case of many dynamical systems because of an argument of Poincaré and Hopf : the so-called "arbitrary functions" argument. In fact, this argument gives not only the a priori law but the whole error structure.**

LIMIT THEOREMS

Natural error structures on dynamical systems, Poincaré-Hopf type theorems

Let us take only the simplest example, that of the harmonic oscillator. Let be a simple pendulum with small oscillations and without damping governed by the equation

$$x(t) = A\cos\omega t + B\sin\omega t.$$

If the pulsation ω is random and follows any probability law μ with density, for large t, the random variable $x(t)$ follows the same law ρ as the random variable:

$$A\cos 2\pi U + B\sin 2\pi U,$$

where U is uniforme on $[0, 1]$. Thus, taking a sample of μ, if we consider a large set of oscillators whose pulsations are drawn according to the law μ, for large t, looking at the instantaneous state of these oscillators we find them distributed according to the law ρ.

The same happens if ω is erroneous and known only with some accuracy.

If the pulsation is defined by the error structure

$$S = (\mathbf{T}_1, \mathcal{B}(\mathbf{T}_1), \mathbb{P}, \mathbb{D}, \Gamma) .$$

where $\mathbb{P}$ has a density, then $x(t)$ possesses a "Dirichlet-law" which, renormalized, converges to the image by $u \longrightarrow A\cos 2\pi u + B\sin 2\pi u$,

of the structure

$$\left(\mathbf{T}_1, \mathcal{B}(\mathbf{T}_1), \lambda_1, H^1(\mathbf{T}_1), u \to u'^2\mathbb{E}\Gamma[I]\right)$$

where $\lambda_1$is the Haar measure on the torus $\mathbf{T}_1$.

See the book:

N. B. *Error Calculus for Finance and Physics, The Language of Dirichlet Forms,* **De Gruyter, 2003**

DONSKER THEOREM AND DIRICHLET FORMS

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Abstract. We use the language of errors to handle local Dirichlet forms with squared field operator (cf [2]). Let us consider, under the hypotheses of Donsker theorem, a random walk converging weakly to a Brownian motion. If, in addition, the random walk is supposed to be erroneous, the convergence occurs in the sense of Dirichlet forms and induces the Ornstein-Uhlenbeck structure on the Wiener space. This quite natural result uses an extension of Donsker theorem to functions with quadratic growth. As an application we prove an invariance principle for the gradient of the maximum of the Brownian path computed by Nualart and Vives.

Keywords : random walk, Brownian motion, Dirichlet form, error

1. INTRODUCTION

In the framework of error calculus and sensitivity analysis of models, the question occurs of the choice of the hypotheses on the size and the correlation or non-correlation of the errors of the parameters.

A general answer is given by the link with statistics through the Fisher information (cf [Bouleau-Chorro])

This answer may be completed by the study of the extension to error structures of the main limit theorems of probability theory, like the central limit theorem or the law of iterated logarithm (cf. [Bouleau-Hirsch] [Chorro])

We study here the very natural question of the extension of the Donsker theorem about the weak limit of a random walk to a Brownian motion. It may be presented in the following way : given a sequence of i.i.d. centered random variables, supposed in addition to be erroneous, the errors being stationary and uncorrelated, does the usual piecewise affine approximation converge to the Brownian motion in the sense of the Dirichlet form describing the errors and if so, which structure does it give on the Wiener space?

The answer is positive and the obtained structure is the Ornstein-Uhlenbeck structure. This quite natural result was not yet published. Its proof needs a strict extension of the classical Donsker theorem to functions with quadratic growth. This extension is the main difficulty of the present work.

As a consequence, we give an explicit formula for the limit of the Dirichlet form on the uniform norm which uses the beautiful resukt of Nualart and Vives [11] on the gradient of the maximum of the Brownian path on [0,1].

2. DEFINITIONS AND NOTATION

Convergence in Dirichlet-law.

Let $S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ be an error structure,
and let $\mathcal{E}$ be the associated Dirichlet form.

Let W be a normed vector space equipped with its Borel σ-field $\mathcal{B}(W) = \mathcal{W}$.

Let be given a family of random variables $(U_n)_{n\in\mathbb{N}}$ defined on $(\Omega, \mathcal{A})$ with values in $(W, \mathcal{W})$.

We introduce a notion of convergence adapted for error structures from the convergence in law of random variables.

Definition 1. *We say that $(U_n)_{n\in\mathbb{N}}$ converges in Dirichlet-law if there exists an error structure on $(W, \mathcal{W})$ say $\Sigma = (W, \mathcal{W}, m, \mathbb{D}_0, \Gamma_0)$ such that :*

i) $(U_n)_*\mathbb{P} \to m$ narrowly
i.e. $\forall f : \Omega \mapsto \mathbb{R}$ continuous and bounded $\mathbb{E}[f(U_n)] \longrightarrow \int_{\mathcal{W}} f(w)dm(w)$,

ii) if $F \in \mathcal{C}^1 \cap Lip(W, \mathbb{R})$ then $F \in \mathbb{D}_0$ and $F(U_n) \in \mathbb{D}$ $\forall n$ and
$\mathcal{E}[F(U_n)] \longrightarrow \mathcal{E}_0[F]$ as $n \to \infty$. where $\mathcal{E}_0$ is the form associated to Σ.

Remark 1. Under the hypotheses of definition 1, the U_n's carry the structure S on $(W, \mathcal{W})$:

If we define

$$\begin{aligned} \mathbb{P}_{U_n} &= (U_n)_*\mathbb{P} \quad \text{(loi de } U_n) \\ \mathbb{D}_{U_n} &= \{\varphi \in L^2(\mathbb{P}_{U_n}) \; : \; \varphi(U_n) \in \mathbb{D}\} \\ \Gamma_{U_n}[\varphi](w) &= \mathbb{E}[\Gamma[\varphi(U_n)]|U_n = w] \end{aligned}$$

the term

$$S_{U_n} = (W, \mathcal{W}, \mathbb{P}_{U_n}, \mathbb{D}_{U_n}, \Gamma_{U_n})$$

is an error structure, $C^1 \cap Lip(W, \mathbb{R})$-functions are in $\mathbb{D}_{U_n}$, $\mathbb{P}_{U_n}$ converges narrowly to m on $(W, \mathcal{W})$ and $\mathcal{E}_{U_n}[F] = \frac{1}{2}\int \Gamma_{U_n}[F]\, d\mathbb{P}_{U_n} \to \frac{1}{2}\int \Gamma_0[F]\, dm$ for every $F \in C^1 \cap Lip(W, \mathbb{R})$.

It is natural to call the structure S_{U_n} the Dirichlet-law of U_n.

Remark 2. If in addition there exists a random variable $V \in \mathbb{D}_0$ such that

i) $(U_n)_*\mathbb{P} \to V_*m$ narrowly

ii) $\forall F \in C^1 \cap Lip \quad \mathcal{E}[F(U_n)] \to \mathcal{E}_0[F(V)]$

we shall say that the U_n's converge in Dirichlet-law to V.

3. CONVERGENCE OF AN ERRONEOUS RANDOM WALK.

Let us recall the classical result of Donsker [8] concerning the convergence of a random walk. Let U_n, $n \geq 1$ be a sequence of i.i.d. square integrable random variables, with variance σ^2 and centered. We consider the piecewise linear interpolation of the random walk $\Sigma_{k=1}^{n} U_k$ i.e; the process

$$X_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} U_k + (nt - [nt]) U_{[nt]+1} \right)$$

for $t \in [0,1]$, where $[x]$ denotes the entire part of x.

The space $W = C([0,1])$ being equipped with the uniform norm, the random variables X_n with values in W converge in law to a centered Brownian motion with variance $\sigma^2 t$.

It follows that if Φ is a bounded Riemann-integrable function for the Wiener measure,

$$\mathbb{E}[\Phi(X_n)] \to \mathbb{E}[\Phi(B)],$$

where B is a centered Brownian motion with variance $\sigma^2 t$.

Let us suppose now that the U_n's be erroneous, and let us keep the independence and identical distribution hypotheses for the U_n's and their errors.

In other words, let us consider the U_n's are the coordinate maps of a product error structure

$$S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, \mathbf{d}, \gamma)^{\mathbb{N}^*}$$

the structure $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, \mathbf{d}, \gamma)$ being such that the identity map j be in $L^2(\mu)$ centered and in $\mathbf{d}$. Thus the U_n's are i.i.d., with law μ, with variance $\sigma^2 = \mu(j^2)$, satisfy $U_n \in \mathbb{D}$ and

$$\left\{ \begin{array}{rcl} \Gamma[U_n] & = & (\gamma[j])(U_n) \\ \Gamma[U_m, U_n] & = & 0 \text{ si } m \neq n \end{array} \right. \tag{1}$$

The random variables $\Gamma[U_n]$ are in $L^1(\mathbb{P})$, independent and with the same law.

For fixed t the r.v.

$$X_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} U_k + (nt - [nt]) U_{[nt]+1} \right)$$

is in $\mathbb{D}$, and by (1)

$$\Gamma[X_n(s), X_n(t)] = \frac{1}{n} \left[\sum_{k=1}^{[nt]\wedge[ns]} \Gamma[U_k] + \alpha(n, s, t) \right] \tag{2}$$

with

$$\begin{aligned} \alpha(n, s, t) = \; & \Big((ns - [ns]) 1_{\{[ns]<[nt]\}} + (nt - [nt]) 1_{\{[ns]>[nt]\}} \\ & + (ns - [ns])(nt - [nt]) 1_{\{[ns]=[nt]\}} \Big) \Gamma[U_{[ns]\wedge[nt]+1}] \end{aligned}$$

It comes from the strong law of large numbers

$$\frac{1}{n}\sum_{k=1}^{[nt]\wedge[ns]}\Gamma[U_k]\to(s\wedge t)\mathbb{E}[\Gamma[U_1]]\quad \mathbb{P}\text{-p.s. et dans } L^1(\mathbb{P}).$$

On the other hand

$$\frac{|\alpha(n,s,t)|}{n}\to 0\quad \mathbb{P}\text{-p.s. et dans } L^1(\mathbb{P}).$$

Thus $\Gamma[X_n(s),X_n(t)]\ \to\ (s\wedge t)c\quad \mathbb{P}$-p.s. et dans $L^1(\mathbb{P})$ where c is the constant $\mathbb{E}\Gamma[U_1]=\int\gamma[j](x)d\mu(x)$.

From this calculation we deduce the convergence of finite dimensional Dirichlet-laws toward the corresponding marginal laws of the Brownian motion equipped with the Ornstein-Uhlenbeck structure: Let $W=C([0,1])$ be equipped with its Borel σ-field $\mathcal{W}$ and let m be the Wiener measure such that the coordinate map of index t be centered with variance $\sigma^2 t$, let $\mathbb{D}_0$ be the domain of the Ornstein-Uhlenbeck form and Γ_0 be the associated quadratic operator which acts on the first chaos by the formula

$$\forall h\in L^2([0,1])\quad \int_0^1 hdB\in\mathbb{D}_0\ \text{ et }\ \Gamma_0[\int_0^1 hdB]=c\int h^2dt.$$

Proposition 1. *Let be $t_1,\ldots,t_p\in[0,1]$, the random variables $(X_n(t_1),\ldots,X_n(t_p))$ converge in Dirichlet-law to $(B(t_1),\ldots,B(t_p))$ where B is a centered Brownian motion with variance σ^2t equipped with the Ornstein-Uhlenbeck structure* $(W,\mathcal{W},m,\mathbb{D}_0,\Gamma_0)$.

Proof. We must show that if $f \in C^1 \cap Lip$

$$\int \Gamma[f(X_n(t_1), \ldots, X_n(t_p))]\, d\mathbb{P} \to \int \Gamma_0[f(B(t_1), \ldots, B(t_p))]\, dm.$$

By majoration of the function $\alpha(n, t_i, t_j)$ and by the functional calculus it suffices to study the convergence of the expression

$$(3) \mathbb{E}[f_i'((X_n(t_1), \ldots, X_n(t_p))f_j'((X_n(t_1), \ldots, X_n(t_p))\frac{1}{n} \sum_{k=1}^{[nt_i]\wedge[nt_j]} \Gamma[U_k]]$$

and for this to study the converbence of

$$\mathbb{E}[e^{i(u_1 X_n(t_1)+\cdots+u_p X_n(t_p))}\Gamma[U_k]]$$

for fixed k. But, using $\Gamma[U_k] = (\gamma[j])(U_k)$, by a classical argument, this expression converges to

$$\mathbb{E}[e^{i(u_1 B(t_1)+\cdots+u_p B(t_p))}]c.$$

Hence finally (3) converges to

$$\mathbb{E}[f_i'((B(t_1), \ldots, B(t_p))f_j'((B(t_1), \ldots, B(t_p))]c(t_i \wedge t_j)$$

which proves the proposition.

These results on the finite marginals lead naturally to the following extension of Donsker theorem:

Theorem 1. *The variables X_n converge in Dirichlet-law to the Ornstei Uhlenbeck structure on the Wiener space* $(W, \mathcal{W}, m, \mathbb{D}_0, \Gamma_0)$.

We shall give two proofs of this theorem. The first one is more elementary but needs the additional hypothesis that the function $\gamma[j]$ be in $L^p(\mu)$ for some $p > 1$. It shows the difficulty overcome by the second one which uses a reinforced version of the probabilistic Donsker theorem.

Lemma 1. *If $F \in C^1 \cap Lip(W, \mathbb{R})$ where W is equipped with the uniform norm,*

$$F(x+h) = F(x) + < F'(x), h > + \|h\| \varepsilon_x(h)$$

where $\varepsilon_x(h)$ is bounded (in x and h) and $\varepsilon_x(h) \to 0$ as $h \to 0$ in W, and where $x \mapsto F'(x)$ is bounded continuous from W into the Banach space of Radon measures on $[0, 1]$.

It will be convenient to use the operator $(.)^\#$ which is a particular gradient built with a copy of the initial space (cf. [2] p80).

Let $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$ a copy of $(\Omega, \mathcal{A}, \mathbb{P})$ and $\widehat{U_n}$ the coordinate maps of $\hat{\Omega}$. Choosing a sharp operator for the structure $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, \mathbf{d}, \gamma)$, we deduce a sharp operator for the product structure by puting $U_n^\# = j^\#(U_n, \widehat{U_n})$. Now in order to define the operator $(.)^\#$ from $\mathbb{D}$ into $L^2(\Omega \times \hat{\Omega}, \mathbb{P} \times \hat{\mathbb{P}})$ it suffices to put for $H = h(U_1, \ldots, U_k, \ldots) \in \mathbb{D}$

$$H^\# = \sum_i h_i'(U_1, \ldots, U_n, \ldots) U_i^\#.$$

Then we have

$$\hat{\mathbb{E}}[(H^\#)^2] = \Gamma[H] \qquad \forall H \in \mathbb{D}$$

and therefore $\forall \varphi \in C^1 \cap Lip$, $\forall H_1, \ldots, H_p \in \mathbb{D}$

$$(\varphi(H_1, \ldots, H_p))^\# = \sum_i \varphi_i'(H_1, \ldots, H_q) H_i^\#.$$

Similarly on the Wiener space, we take a copy $(\hat{W}, \hat{\mathcal{W}}, \hat{m})$ and the operator $(.)^\#$ from $\mathbb{D}_0$ into $L^2(W \times \mathcal{W}, m \times \hat{m})$ which satisfies (cf. [2] chap VI §2) $(B_t)^\# = \frac{\sqrt{c}}{\sigma} \hat{B}_t$ and $\forall H \in \mathbb{D}_0 \quad \hat{\mathbb{E}}[(H^\#)^2] = \Gamma_0[H]$.

Lemma 2. *Let be $F \in C^1 \cap Lip(W)$, we have $F(X_n) \in \mathbb{D}$ and*

$$(F(X_n))^\# = \int_{[0,1]} (X_n(s))^\# \, F'(X_n)(ds)$$

and

$$\Gamma[F(X_n)] = \int_{[0,1]} \int_{[0,1]} \Gamma[X_n(s), X_n(t)] F'(X_n)(ds) F'(X_n)(dt).$$

Similarly $F(B) \in \mathbb{D}_0$ and $(F(B))^\# = \int_{[0,1]} B^\#(s) \, F'(B)(ds)$ and

$$\Gamma[F(B)] = \int_{[0,1]} \int_{[0,1]} s \wedge t \, F'(B)(ds) F'(B)(dt) = \int_0^1 < F'(B), 1_{[u,1]} >^2 \, du.$$

The proof is easy in the case where $F(x)$ depends only on a finite number of the values taken by x. Then the general case is obtained by approximation using the fact that the sharp is a closed operator.

Now the first proof of the theorem can be given

The first proof

From the preceding lemma and formula (2) we draw

$$\begin{aligned}\Gamma[F(X_n)] = & \int\int\left(\frac{1}{n}\Sigma_{k=1}^{[ns]\wedge[nt]}\Gamma[U_k]\right)F'(X_n)(ds)F'(X_n)(dt)\\ & +\int\int\alpha(n,s,t)F'(X_n)(ds)F'(X_n)(dt)\\ = & (A)+(B).\end{aligned}$$

The second term may be majorized in the following way

$$|(B)| \leq \frac{1}{n}\sup_{k\leq n}\Gamma[U_k]\|F'(X_n)\|^2$$

where $\|F'(X_n)\|$ is the total mass of the measure $F'(X_n)$. It comes from the following lemma that $\mathbb{E}[|(B)|] \to 0$ as n tends to infinity.

Lemma *If the* Y_k *are i.i.d. in* L^1 *and positive,* $\lim_n \mathbb{E}[\frac{1}{n}\sup_{k\leq n} Y_k] = 0$.

Proof of the lemma. We have

$$\mathbb{E}[\frac{1}{n}\sup_{k\leq n} Y_k] = \int_0^\infty \frac{1-((1-\mathbb{P}(Y_1>a))^n}{n}da$$

and

$$\frac{1-((1-\mathbb{P}(Y_1>a))^n}{n} \leq \mathbb{P}(Y_1>a)$$

which is integrable since $Y_1 \in L^1$ hen,ce the lebesgue dominated theorem applies and gives the result.

About the first term (A) let us put $V_k = \Sigma_{i=1}^k (\Gamma[U_i] - \mathbb{E}\Gamma[U_i])$. Supposing $\Gamma[U_1] \in L^p$ for some $p > 1$, we have by Doob inequality ([10] p. 68) applied to the martingale V_k

$$\mathbb{E}[\frac{1}{n} \max_{1\leq k\leq n} |V_k|] \leq \frac{p}{p-1}\frac{1}{n} \max_{1\leq k\leq n} k\|\frac{1}{k}\sum_{i=1}^k (\Gamma[U_i] - \mathbb{E}\Gamma[U_i])\|_p$$

The second member is of the form $\frac{1}{n}\max_{k\leq n} k\varepsilon(k)$ avec $\varepsilon(k) \longrightarrow 0$ hence goes to zero when $n \to \infty$.

Thanks to this majoration $\mathbb{E}[(A)]$ has the same limit as

$$\mathbb{E}\int\int\left(\frac{1}{n}\sum_{k=1}^{[ns]\wedge[nt]} \mathbb{E}[\Gamma[U_k]]\right) F'(X_n)(ds)F'(X_n)(dt)$$

which is equal to

$$c\mathbb{E}\int_0^1 < F'(X_n), 1_{[\frac{[nu]}{n},1]} >^2 \, du$$

Thus, by the Donsker theorem, the mapping $x \mapsto \int < F'(x), 1_{[u,1]} >^2 du$ being bounded and continuous, we have finally

$$\mathbb{E}\Gamma[F(X_n)] \to c\mathbb{E}\Gamma_0[F(B)] \qquad\qquad \text{Q.E.D.}$$

In order to remove the hypothesis $\Gamma[U_1] \in L^p$ pour un $p > 1$, we tackle the question by an other way. From

$$X_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} U_k + (nt - [nt]) U_{[nt]+1} \right)$$

we draw

$$X_n^{\#}(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} U_k^{\#} + (nt - [nt]) U_{[nt]+1}^{\#} \right)$$

thus, by Donsker theorem applied to the pairs $(U_k, U_k^{\#})$ which are i.i.d. we have for G bounded continuous from $W \times W$ into $\mathbb{R}$

$$\mathbb{E}\hat{\mathbb{E}}[G(X_n, X_n^{\#})] \to \mathbb{E}\hat{\mathbb{E}}[G(B, B^{\#})].$$

To prove the theorem by applying this idea to $F(X_n)^{\#} = \int X_n^{\#}(s) F'(X_n)(ds$ we should have the Donsker theorem not only for bounded continuous functions but for G such that $|G(x)| \leq K_1 \|x\|^2 + K_2$. It is what we prove here :

Theorem 2. *Let $X_n(t)$ be as in the Donsker theorem, and let $B(t)$ be a Brownian motion with variance $\sigma^2 t$, then*

$$\mathbb{E}[\Phi(X_n)] \to \mathbb{E}[\Phi(B)]$$

for any Φ continuous from W into $\mathbb{R}$ such that $|\Phi(x)| \leq K_1 \|x\|^2 + K_2$.

Proof. Let us put

$$Z_n = \max_t |X_n(t)] = \frac{1}{\sqrt{n}} \max_{1\le k\le n} |\sum_{j=1}^{k} U_j|.$$

a) It suffices to show that the r.v. Z_n^2 are uniformly integrable.

Indeed, given $\varepsilon > 0$, this uniform integrability implies we can find an $a > 0$ such that

$$|\mathbb{E}[(\Phi(B) \wedge a) \vee (-a)] - \mathbb{E}[\Phi(B)]| \le \varepsilon/3$$

et que $\forall n$

$$|\mathbb{E}[(\Phi(X_n) \wedge a) \vee (-a)] - \mathbb{E}\Phi(X_n)| \le \mathbb{E}[|\Phi(X_n)| 1_{|\Phi(X_n)|>a}] \le \varepsilon/3.$$

Then choosing, by Donsker theorem, n large enough

$$|\mathbb{E}[(\Phi(X_n) \wedge a) \vee (-a)] - \mathbb{E}[(\Phi(B) \wedge a) \vee (-a)]| \le \varepsilon/3$$

we have $|\mathbb{E}\Phi(X_n) - \mathbb{E}\Phi(B)| \le \varepsilon$.

b) In order to show that the r.v. Z_n^2 are uniformly integrable, we put $S_n = \Sigma_{i=1}^n U_i$ and we use the following majoration ([1] p.69)

$$\mathbb{P}\{\max_{i \leq n} |S_i| \geq \lambda\sigma\sqrt{n}\} \leq 2\mathbb{P}\{\frac{|S_n|}{\sigma\sqrt{n}} \geq \frac{\lambda}{2}\} \quad \text{si } \lambda \geq 2\sqrt{2}$$

hence

$$\mathbb{P}\{Z_n^2 \geq \alpha\} \leq 2\mathbb{P}\{\frac{|S_n|}{\sigma\sqrt{n}} \geq \frac{\sqrt{\alpha}}{2\sigma}\} \quad \text{si } \alpha \geq 8\sigma^2.$$

From

$$\mathbb{E}[Z_n^2 1_{Z_n^2 \geq \alpha}] = \alpha\mathbb{P}\{Z_n^2 \geq \alpha\} + \int_\alpha^\infty \mathbb{P}\{Z_n^2 \geq t\}dt$$

we get

$$\mathbb{E}[Z_n^2 1_{Z_n^2 \geq \alpha}] \leq 2\alpha\mathbb{P}\{\frac{|S_n|}{\sigma\sqrt{n}} \geq \frac{\sqrt{\alpha}}{2\sigma}\} + 2\mathbb{E}[(4\frac{S_n^2}{n} - \alpha)^+].$$

It comes now from the central limit theorem, and from the fact that $\frac{S_n^2}{n}$ are uniformly integrable, that if $\alpha \geq 8\sigma^2$,

$$(4) \quad \limsup_n \mathbb{E}[Z_n^2 1_{Z_n^2 \geq \alpha}] \leq 2\alpha\mathbb{P}\{|N| \geq \frac{\sqrt{\alpha}}{2\sigma}\} + 2\mathbb{E}(4N^2 - \alpha)^+$$

where N is a reduced normal variable. Hence

$$\lim_{\alpha\uparrow\infty} \limsup_n \mathbb{E}[Z_n^2 1_{Z_n^2 \geq \alpha}] = 0$$

and this implies the uniform integrability of the Z_n^2. Q.E.D.

Let us come back to the second proof of theorem 1.

The function $G(x,y) = \int_{[0,1]} y(s)\, F'(x)(ds)$ is continuous from $W \times W$ into $\mathbb{R}$ and satisfies $|G(x,y)| \leq \|y\|^2 \sup_x \|F'(x)\|^2$. Theorem 2, extended to 2-dimensional variables which is easy, applies and gives thanks to the lemma 2

$$\mathbb{E}\Gamma[F(X_n)] = \mathbb{E}\hat{\mathbb{E}}[((F(X_n))^{\#})^2] \rightarrow \mathbb{E}[\Gamma_0[F(B)]]. \qquad \text{Q.E.D.}$$

By the properties of the narrow convergence for Riemann-integrable functions this proof gives also :

Corollary 1. *Let $\Phi(x,y)$ be a function from $W \times \hat{W}$ into $\mathbb{R}$ continuous outside a negligible set for $m \times \mu$ where m is the law of B and μ the law of $B^{\#}$, such that*

$$|\Phi(x,y)| \leq K_1\|x\|^2 + K_2\|y\|^2 + K_3$$

then $\mathbb{E}\hat{\mathbb{E}}\Phi(X_n, X_n^{\#}) \rightarrow \mathbb{E}\hat{\mathbb{E}}\Phi(B, B^{\#})$.

Application. Let us suppose for the sake of simplicity that $\sigma^2 = c = 1$ so that $B^\# = \hat{B}$. the uniform norm $N(w) = \|w\|$ which is continuous and Lipschitz belongs to $\mathbb{D}_0$ (cf. [7] with the Feyel-La Pradelle method [9], or [12] p.90), similarly $M(w) = \sup_t w(t)$.

By the results of Nualart and Vives [11] the operators $(.)^\#$ and Γ_0 may be computed on these functionals :

i) $M^\#(w, \hat{w}) = \hat{B}_\Sigma = \hat{w}(\Sigma(w))$ où $\Sigma = \inf\{t : B(t) = \sup_s B(s)\}$ and thus $\Gamma_0[M] = \Sigma$.

ii) $N^\#(w, \hat{w}) = \operatorname{sign}(B_{\mathcal{T}})\hat{B}_{\mathcal{T}}$ où $\mathcal{T} = \inf\{t : |B(t)| = \sup_s |B(s)|\}$ and thus $\Gamma_0[N] = \mathcal{T}$.

The set of the Brownian paths which reach several times their maximum is negligible and outside this set it is not difficult to see that the map $w \mapsto \Sigma(w)$ is continuous, then by corollary 1, when $n \uparrow \infty$:

$$\mathbb{E}\Gamma[\sup_t X_n(t)] = \mathbb{E}\Gamma[\frac{1}{\sqrt{n}} \max_{1\leq k\leq n} S_k] \to \mathbb{E}\hat{\mathbb{E}}[M^{\#2}] = \mathbb{E}[\Sigma]$$

and similarly

$$\mathbb{E}\Gamma[\|X_n(t)\|_\infty] = \mathbb{E}\Gamma[\frac{1}{\sqrt{n}} \max_{1\leq k\leq n} |S_k|] \to \mathbb{E}\hat{\mathbb{E}}[N^{\#2}] = \mathbb{E}[\mathcal{T}].$$

Thus we can state with the above notation :

Proposition 2. *Let* $F : \mathbb{R}^2 \mapsto \mathbb{R}$ *be of class* $C^1 \cap Lip$*, then on the one hand*

$$\mathbb{E}[F^2(\frac{1}{\sqrt{n}} \max_{1\leq k\leq n} S_k, \frac{1}{\sqrt{n}} \max_{1\leq k\leq n} |S_k|)] \to \mathbb{E}[F^2(M,N)]$$

and on the other hand

$$\mathbb{E}\Gamma[F(\tfrac{1}{\sqrt{n}} \max_{1\leq k\leq n} S_k, \tfrac{1}{\sqrt{n}} \max_{1\leq k\leq n} |S_k|)]$$
$$\to \mathbb{E}[F_1'^2(M,N)\mathcal{T}] + 2\mathbb{E}[F_1'(M,N)F_2'(M,N)\,\mathcal{T}\wedge\Sigma] + \mathbb{E}[F_2'^2(M,N$$

Remarque finale. Terminons par quelques mots sur le résultat principal lui-même. Supposons que les U_n soient simulées par une méthode de Monte Carlo avec une certaine précision, de telle façon que l'hypothèse d'indépendance et de stationarité des variables et de leurs erreurs puisse être considérée comme acceptable. Contrairement certains théorèmes limites comme la loi des grands nombres qui effacent les erreurs (cf [4]), la normalisation faite pour la convergence en loi vers le brownien ne conduit sur celui-ci ni à une erreur nulle ni à une erreur infinie mais à l'erreur d'Ornstein-Uhlenbeck. Que ce soit cette structure d'erreur qu'on obtienne se conçoit bien car, d'après la formule de Mehler (cf [12] p49 et [2] p116 §2.5.9), l'erreur qu'elle décrit est transversale et stationnaire. Nous voyons donc que pour obtenir d'autres structures d'erreur sur l'espace de Wiener, telles que des structures de Mehler généralisées (cf [2] p113 §2.5), il faut supposer que les erreurs sur les U_n sont correlées.

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