RESIDUAL RISKS AND HEDGING STRATEGIES IN MARKOVIAN MARKETS *

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Abstract

We prove two explicit formulae for the quadratic residual risk and for the optimal hedging portfolio of a european contingent claim when the underlying stock prices are functions of a Markov process. These expressions allow the practical handling of a great deal of non classical models which are less optimistic than Black and Scholes's one.

PRESENTATION. R.C. Merton [18] begins the history of option pricing with Louis Bachelier (1900) who gave [1] a pricing formula with an underlying stock price modelled by a brownian motion. This idea of representing the chaotic evolution of the stock values by a stochastic process was progressively improved, but it took a really new dimension with the works of Black and Scholes, and Merton, in which appeared the principle of **pricing by simulation** with the underlying stock. Afterwards, by the contributions of Harrison, Kreps, Pliska, Bensoussan, Karatsas, (see [13], [2], [17]) among others, the mathematical framework of the problem was clarified and the essential role of martingale theory and stochastic integration was brought out. The possibility of pricing by simulation is mathematically expressed by the property of representation of the contingent claim as a stochastic integral with respect to the underlying stock price. This is the object of the complete market assumption of Harrison and Pliska.

Nevertheless stochastic calculus allows to go further and to study models in which the claim cannot be completely simulated. Except in some particular situations (see for example [23]), it will be the case in general for models with jumps. For some

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models with jumps, Merton gave pricing formulae in [19] and the existence and characterisation of an optimal strategy minimizing the quadratic risk was brought out by Föllmer and Sondermann [10]. That opens a wide field of models for which a mathematical analysis is possible and which give less optimistic but, in some cases, more realistic results than diffusion models with perfect simulation.

In this study we reinstate a Markovian framework to get explicit formulae. It is not supposed that stock prices constitute a Markov process by themselves but that they are functions of a Markov process. This is a quite general situation which even includes, through Markovian representation, some models based on stationary processes. The introduction of the carré-du-champ operator permits to express the results for almost general Markov processes and these formulae are easy to write down explicitly in each particular case. Thanks to these formulae it is possible to use in a practical real situation a great variety of models which can be completely substituted to the classical Black and Scholes model.

A significant interest of those models is to allow the portfolio manager to evaluate the sensitivity of his hedging to the underlying stock model, and therefore to estimate the instantaneous and residual risks better. To this end, the parameters of the models must be adjusted to the reality and that can be done essentially in two manners:

i) By the statistical way, using historical series. This method gives approximate models which are little sensitive to instantaneous fluctuations and form what can be called the background of the scenery.

ii) By referring to the quoted prices of options in organized markets, as it is usually done with the Black and Scholes model.

The fact that this model is the most widely used by financial traders appears mainly in the structure of relative prices of the options on the same stock or currency. This structure looks **more or less** as if all prices were obtained with the Black and Scholes model with the same volatility σ. But it is also possible to choose a model in a family (determined for example by the method (i)) in such a way that the structure of the option prices obtained from the model coincides at best with the structure of quoted prices.

The formulae of residual risks we establish then permit to use the model chosen by methods (i) or (ii). In the European case, to which we limit the study here, numerical algorithms are classical and essentially involve computations of expectations of functionals of a Markov process. The American case requires the solution of variational inequalities and will be treated elsewhere.

The first part presents the framework and the main results, especially the formulae for residual risks and the estimates for the maximal risk incurred during the management of the portfolio.

The second part deals with the derivation of these results when the underlying stock prices and the conditional claim are functions of a Markov process satisfying the right hypotheses. A family of examples is treated.

The expression of the results is made more accurate in **a third part**, where the Markov process is supposed to be symmetric by using Dirichlet forms. A family of examples is also treated.

In **the last part** we give some consequences of this study about the problem of determining cases of perfect hedging.

The main results were announced in [7]. We thank the members of the workshop "New financial models" of the CERMA for their suggestions and especially O. Chateau for the help he brought to this work.

I Hedging formulae and residual risks.

Before going into the details of hypotheses and proofs, we give here a formal description of the main results which are proved in parts II and III.

The state of the market is represented by a Markov process (X_t) with filtration $(\mathcal{F}_t)$. It is supposed that the discounted underlying stock price S_t is a martingale with respect to $(\mathcal{F}_t)$ which writes

$$S_t = G(t, X_t)$$

where G is a function satisfying suitable hypotheses. We suppose here for simplicity that the contingent claim is European and that its discounted value is of the form $H(S_T)$ where T is the exercise time.

The main tool for the study of residual risks in this setting is the carré-du-champ operator. Such an operator exists under very general assumptions (cf [14], [6], [9] chapter XV §2) and can be computed in terms of the parameters of the model (cf parts II and III). A self-financing hedging strategy for the conditional claim builds a portfolio whose discounted value at time t is

$$V_t = V_0 + \int_{(0,t]} J_s \, dS_s$$

where V_0 is equal to the initial value of the hedging portfolio $F(0, X_0)$, and (J_t) is an $(\mathcal{F}_t)$-predictable process. The **residue** at time T is the lack of hedging

$$R_T = H(S_T) - V_T$$

. Then the following estimate of the variance of R_T holds :

$$\mathbb{E}(R_T^2) \geq \mathbb{E}(V_0 - \mathbb{E}(H(S_T|\mathcal{F}_0))^2 + \mathbb{E}\left[\int_0^T \left[\Gamma(F,F) - \frac{\Gamma(F,G)^2}{\Gamma(G,G)}\right](s, X_s)\, ds\right] \quad (1)$$

where Γ is the carré-du-champ operator which in usual cases can be computed by the formula

$$\Gamma(F, G) = A(FG) - F\,AG - G\,AF,$$

where A is the generator of (X_t) operating on the x variable .

In the estimate (1), equality is obtained for a unique hedging strategy minimizing the quadratic risk $\mathbb{E}[R_T^2]$ and obtained by letting

$$\begin{aligned} V_0 &= \mathbb{E}(H(S_T)|\mathcal{F}_0) \\ J_t^{(optimal)} &= \tfrac{\Gamma(F,G)}{\Gamma(G,G)}(t, X_{t-}). \end{aligned} \tag{2}$$

It is now natural to define the value of the claim at time 0 by :$\mathbb{E}(H(S_T)|\mathcal{F}_0) = F(0, X_0)$, and at time t by : $\mathbb{E}(H(S_T)|\mathcal{F}_t) = F(t, X_t)$. See **??** for ralated arguments.

The residue at time t can then be defined by :

$$R_t = F(t, X_t) - V_t.$$

For the optimal strategy, the martingale (R_t) satisfies

$$< R, R >_t = \int_0^t \left(\Gamma(F, F) - \frac{\Gamma(F, G)^2}{\Gamma(G, G)} \right) (s, X_s)\, ds$$

which gives by Doob's inequality, an estimate for the maximal residue during the optimal management of the portfolio :

$$\mathbb{E}[(R_T^*)^2] \leq 4 \int_0^T P_s \left(\Gamma(F, F) - \frac{\Gamma(F, G)^2}{\Gamma(G, G)} \right) (s, x)\, ds \tag{3}$$

where $x = X_0$ is the starting point of the process (X_t).

Formulae (1), (2) and (3) extend to the case where there are several stocks (S_t with values in $\mathbb{R}^d$), (see theorem 5. part II).

Remark 1. In this model, the amount of underlying stock which is in the optimal hedging portfolio at time t depends on the value of the process (X_t) which represents the market, that is to say, on the level of economic quantities which govern the evolution of the stock price. If one restricts oneself to hedging strategies for which (J_t) is measurable with respect to the natural filtration of the stock price (S_t), inequality (1) still holds, but the equality (2) may not be reached. The risks are therefore increased which is not surprising regarding the fact that less is used than the available information of the model.

Remark 2. The solution of **the problem of pricing** is not the aim of this study. Nevertheless, as this question is especially difficult in the case of incomplete markets that we are looking at, it is suitable to give some comments to make easier the reading of the sequel.

It is well known that if under some probability $\mathbb{P}$ the discounted stock price S_t is a martingale with respect to a filtration $(\mathcal{F}_t)$, and if there exists a previsible process (J_t) such that the discounted claim C can be written

$$C = K_0 + \int_{(0,T]} J_s . dS_s \qquad (\star)$$

then this property will still be true under a probability $\mathbb{P}'$ equivalent with $\mathbb{P}$, and the value

$$K_0 + \int_{(0,t]} J_s.dS_s \qquad (\star\star)$$

which can be proposed for the (discounted) pricing of C at time t, is the only value of the form $K'_0 + \int_{(0,t]} J'_s.dS_s$ which can be extended to get the equality $(\star)$ at time T, and this is so under any probability $\mathbb{P}' \sim \mathbb{P}$.

On the contrary, if under $\mathbb{P}$ no pair $(K_0, (J_t))$ reaches the equality $(\star)$, the pricing by arbitrage is nolonger possible and if , under $\mathbb{P}$ a strategy $(K_0, (J_t))$ is found which minimizes the quadratic risk

$$\mathbb{E}(C - K_0 - \int_{(0,T]} J_s.dS_s)^2,$$

this strategy and the corresponding residue do depend on $\mathbb{P}$. It should be noted that in an incomplete market there exist in general several probabilities under which the discounted stock price is a martingale. So there is no easy answer to the question of pricing the claim at time t, nor to the question ,the price being chosen, of sharing out the residue of hedging between the seller and the buyer of the claim.

Nevertheless, these questions are in fact somewhat abstract, because in practice it is not really known wether the stock price allows a representation through a model with perfect hedging or not. What is known is the structure of the prices of options in organized markets, and from this point of view the problem of pricing is solved *a priori* and it remains only that of finding the hedging. This can be done following several models and several ways as discussed in the presentation.

II Right hypotheses.

II.1 The carré-du-champ operator.

Let $(\Omega, \mathcal{F}_t, \mathcal{F}, X_t, \mathbb{P}^x)$ be a right Markov process (cf [12]) with state space $(E, \mathcal{E})$. The process (X_t) is said to admit a carré-du-champ operator (cf [20]) if for every initial law μ the square integrable martingales of the filtration $(\mathcal{F}_t^\mu)$ have skew brackets absolutely continuous with respect to Lebesgue measure.

This is equivalent to saying that the domain of the extended generator (defined as in [20], [3], or [9]) is an algebra. According to the problem one is dealing with, it is convenient to change slightly the definition of the extended generator and we shall adopt the following one :

Definition 1 . *Let f be a finite universally measurable function on E.*

i) We shall say that f belongs to $\mathcal{D}_1(A)$ if there exists a universally measurable function g satifying

$$\left(\int_0^t \mid g \mid (X_s)\, ds < +\infty \ \ \forall t \geq 0\right) \qquad \mathbb{P}^x a.s.\ \forall x \in E$$

such that
$$C_t^f = f(X_t) - f(X_0) - \int_0^t g(X_s)\,ds$$
be a local right continuous martingale under $\mathbb{P}^x$ *for all* x *in* E.

ii) We shall say that f *belongs to* $\mathcal{D}_2(A)$ *if* f *belongs to* $\mathcal{D}_1(A)$ *and if* C_t^f *is* $(\mathcal{F}_t^x, \mathbb{P}^x)$*-locally square integrable for all* x *in* E.

The function g which appears in i) is unique up to a zero potential set, it is denoted Af. This definition leads to the following result :

Proposition 2 . *If the Markov process* (X_t) *admits a carré-du-champ operator, and if* f *belongs to* $\mathcal{D}_2(A)$ *then* f^2 *belongs to* $\mathcal{D}_1(A)$ *and*
$$< C^f, C^f >_t = \int_0^t \Gamma(f,f)(X_s)\,ds$$
where
$$\Gamma(f,f) = Af^2 - 2f\,Af$$
is called the carré-du-champ operator.

Proof. Let f be in $\mathcal{D}_2(A)$. Ito's formula applied to the semi-martingale $Y_t = f(X_t)$ gives
$$f^2(X_t) = f^2(X_0) + \int_0^t 2f(X_s)g(X_s)\,ds + [C^f, C^f]_t + 2\int_{(0,t]} Y_{s-}\,dC_s^f.$$
It is known that the bracket $[C^f, C^f]_t$ is an additive functional and by the fact that f belongs to $\mathcal{D}_2(A)$ there exists (cf [21] corollary of theorem 3) an additive functional which is a common version of $< C^f, C^f >_t$ under every measure $\mathbb{P}^x$,. Thanks to the existence of the carré-du-champ operator for the process (X_t), Motoo's theorem gives a universally measurable function h such that
$$< C^f, C^f >_t = \int_0^t h(X_s)\,ds.$$
Then, the process
$$f^2(X_t) - f^2(X_0) - 2\int_0^t f(X_s)g(Xs)\,ds - \int_0^t h(X_s)\,ds$$
is a right continuous local martingale under every $\mathbb{P}^x$.That proves that f^2 belongs to $\mathcal{D}_1(A)$ and $Af^2 = 2f\,Af + h$, which completes the proof. □

For f and g in $\mathcal{D}_2(A)$, $\Gamma(f,g)$ is defined by polarisation.

II.2 Optimal hedging under right hypotheses.

We consider a right Markov process (X_t) with state space $(E, \mathcal{E})$, canonical filtration $(\mathcal{F}_t)$ and transition semi-group (P_t).

(A1) It is supposed that (X_t) admits a carré-du-champ operator.

It is then the same for the process (t, X_t) with values in $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E})$ (cf [20] p162) whose generator is denoted by $\mathcal{A}$.

(A2) For all x, the discounted stock price S_t is supposed to be a $(\mathcal{F}_t, \mathbb{P}^x)$-martingale of the form $S_t = G(t, X_t)$ with $G \in \mathcal{D}_2\mathcal{A}$.

(A3) It is supposed that the contingent claim is of the form $H(S_t)$ and that the function $f = H(G(T, .))$ satisfies $P_T f^2(x) < +\infty$ for all x in$E$.

For all x, the value of the contingent claim at time $t \leq T$ is given by the martingale

$$M_t = \mathbb{E}^x(H(S_T) \mid \mathcal{F}_t) = F(t, X_t)$$

where $F(t, x) = P_{T \wedge t} f(x)$.

By assumption **(A3)** the martingale M_t is square integrable under every $\mathbb{P}^x$and it follows that F belongs to $\mathcal{D}_2(\mathcal{A})$.

With any predictable process (J_t) such that

$$\mathbb{E}^x \left(\int_0^T J_t^2 \, d < S, S >_t \right) < +\infty$$

a self financing portfolio is associated, whose initial value is $V_0 = F(0, X_0)$ and whose value at time t is

$$V_t = V_0 + \int_{(0,t]} J_s \, dS_s.$$

The residue which corresponds to this strategy is given by

$$R_t = F(t, X_t) - F(0, X_0) - \int_{(0,t]} J_s \, dS_s.$$

Minimizing $\mathbb{E}^x(R_T^2)$ amounts to projecting the martingale $F(t, X_t) - F(0, X_0)$ on the stable subspace generated by (S_t), which leads for J_t to take a previsible version of the density of $< M, S >$ with respect to $< S, S >$ and that gives, thanks to paragraph II.1, the following result:

Theorem 3 . *The process $(J_t)_{0 \leq t \leq T}$ of optimal hedging is given, under every $\mathbb{P}^x$, by*

$$J_t = \frac{\Gamma(F, G)}{\Gamma(G, G)}(t, X_{t-}) \quad 0 \leq t \leq T$$

where Γ is the carré-du-champ operator of the process (t, X_t) and the associated residue satisfies

$$< R, R>_t = \int_0^t \left[\Gamma(F,F) - \frac{\Gamma(F,G)^2}{\Gamma(G,G)}\right](s, X_s)\, ds.$$

It is to be noted that the previsible set $\{(\omega, t) : \Gamma(G,G)(t, X_{t-}) = 0\}$ is not charged by the measure $d{<}S, S{>}_t$. As usual the left limit X_{t-} is to be taken in a Ray compactification of E (cf [9] chapter XV). The expression of $< R, R >_t$ permits to compute the variance of R_t and therefore by Doob's inequality to estimate the maximal residue during the time interval $[0, T]$:

$$\mathbb{E}^x[(R_T^*)^2] \leq 4\mathbb{E}^x[R_T^2] = 4\int_0^T P_s\left[\Gamma(F,F)(s,.) - \frac{\Gamma(F,G)^2}{\Gamma(G,G)}(s,.)\right](x)\, ds$$

where $R_T^* = \sup_{0 \leq t \leq T} \mid R_t \mid$.

II.3 Multivariate case.

For handling models with several stocks or currencies, we replace the assumptions **(A2)** by the following one :

(A2 bis) It is supposed that, for all x the vector S_t of stock prices is a $(\mathcal{F}_t)$-martingale with value in $\mathbb{R}^d$ under $\mathbb{P}^x$ whose components are of the form $S_t^i = G^i(t, X_t)$ with $G_t^i \in \mathcal{D}_2(\mathcal{A})$ for $i = 1, \ldots, d$.

In the same way, the function H of assumption **(A3)** will be a borelian function from $\mathbb{R}^d$ to $\mathbb{R}$, and we keep the same assumptions on f as before.

We denote G the column array with components $G^1, \ldots, G^d$; $\Gamma(G, G^*)$ the matrix with coefficients $\Gamma(G^i, G^j)$ $(0 \leq i, j \leq d)$ and $\Gamma(G, F)$ [resp. $\Gamma(F, G^*)$] the column [resp. row] array with components $\Gamma(G^i, F)$ $(0 \leq i \leq d)$ [resp. $\Gamma(F, G^j)$ $(0 \leq j \leq d)$].

Lemma 4 . *i) The matrix $\Gamma(G, G^*)(t, x)$ is positive definite outside a zero potential set for the process (t, X_t).*

ii) For (t, x) outside a zero potential set, the vector $\Gamma(G, F)(t, x)$ is in the range of $\Gamma(G, G^*)(t, x)$.

Proof. For i) it is sufficient to remark that for every vector Λ with rational coordinates $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}$, we have

$$\Lambda^*\Gamma(G, G^*)\Lambda = \sum_{i,j} \lambda_i\lambda_j\Gamma(G_i, G_j) = \Gamma\left(\sum_{i=1}^d \lambda_i G_i, \sum_{i=1}^d \lambda_i G_i\right)$$

and to use the positivity property of Γ.

For ii), it is shown that $\Gamma(G,F)$ is orthogonal to the kernel of $\Gamma(G,G^*)$ by using the inequality

$$\left|\sum_{i=1}^d \lambda_i \Gamma(G_i,F)\right| \le \sqrt{\Gamma\left(\sum_{i=1}^d \lambda_i G_i, \sum_{i=1}^d \lambda_i G_i\right)}\sqrt{\Gamma(F,F)}$$

which follows from the fact that

$$\Lambda^*\Gamma(G,G^*)\Lambda + 2\mu\Lambda^*\Gamma(G,F) + \mu^2\Gamma(F,F)$$

is positive for all rationals $\mu,\lambda_1,\ldots,\lambda_d$ outside a zero potential set hence also for real $\mu,\lambda_1,\ldots,\lambda_d$ outside the same set.

Theorem 5 . *The optimal hedging process $J_t = (J_t^1,\ldots,J_t^d)$ is given, under every measure $\mathbb{P}^x$, by*

$$J_t = \lim_{\epsilon\to 0} \Gamma(F,G^*)\left(\Gamma(G,G^*)+\epsilon I\right)^{-1}(t,X_{t-}) \tag{4}$$

and the corresponding residue satisfies

$$<R,R>_t = \int_0^t \left[\Gamma(F,F) - \lim_{\epsilon\to 0}\Gamma(F,G^*)\left(\Gamma(G,G^*)+\epsilon I\right)^{-1}\Gamma(G,F)\right](s,X_s)\,ds. \tag{5}$$

Proof. Thanks to the preceding lemma, the algebraic lemma below (and the similar result obtained by transposition) gives that the projection of the martingale $F(t,X_t) - F(0,X_0)$ on the stable subspace generated by (S_t) (cf [15] chapter IV) is given by

$$\int_0^t \lim_{\epsilon\to 0}\Gamma(F,G^*)\left(\Gamma(G,G^*)+\epsilon I\right)^{-1}(s,X_{s-})\,dS_s$$

and formula 5 follows likewise.

Lemma 6 . *Let B be a symmetric positive $d\times d$-matrix, let U be a vector in the range of B and V be a vector such that $U = BV$. Then*

$$W = \lim_{\epsilon\to 0}(B+\epsilon I)^{-1}U$$

exists, $U = BW$ and $V^*BV = W^*BW$. Further more W is given by $W = P^B V$ where P^B is the orthogonal projection on the range of B.

Proof. That is easily seen by taking a basis of $\mathbb{R}^d$ consisting of a basis of Ker(B) and a basis of Range(B). □

II.4 Example.

Before specifying further, let us give a result which opens a wide field of applications.

Proposition 7 . *Let (Y_t) be a process with stationary independent increments (PSII) with values in $\mathbb{R}^d$ and let a be a Lipschitz function mapping from $\mathbb{R}^n$ to $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$. The Markov process with values in $\mathbb{R}^n$, associated with the stochastic differential equation :*

$$dX_t = a(X_{t-})\, dY_t \tag{6}$$

possesses a carré-du-champ operator.

Proof. Let $\mathcal{G}_t^\mu$ be the canonical σ-fields of the PSII Y considered as a Markov process. Denote $\mathcal{H}_t = \mathcal{G}_t^{\epsilon_0}, \mathbb{P} = \mathbb{P}^{\epsilon_0}$ the law of Y when starting from zero. The fact that bounded C^2-functions with bounded derivatives belong to the domain of the extended generator of Y and constitute an algebra stable by the resolvent family, which is easy to verify, implies that on $(\Omega, \mathcal{H}_t, \mathbb{P})$ every square integrable martingale M_t has an absolutely continuous bracket $<M, M>$.

On $(\Omega, \mathcal{H}_t, \mathbb{P})$ equation (6) has a unique strong solution $X_t(x, \omega)$ starting from x. If we set

$$P_t f(x) = \mathbb{E}[f(X_t(x,\omega))]$$

for $f \in C_b(\mathbb{R}^n)$ the flow $X_t(x,\omega)$ has the Markov property with respect to the filtration $(\mathcal{H}_t)$:

$$\mathbb{E}[f(X_{t+u}(x,\omega)) \mid \mathcal{H}_t] = P_t f(X_u(x,\omega)).$$

On the other hand the function a being globally Lipschitz, there exists (cf [22]) an x-continuous version of the flow $X_t(x,\omega)$. It follows that if we set $W = \mathbb{R}^n \times \Omega$, $\mathcal{K}_t = \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{H}_t$, $\mathbb{P}^\mu = \mu \otimes \mathbb{P}$ where μ is a probability on $\mathbb{R}^n$, the process $X_t(w) = X_t(x,\omega)$ is a Feller Markov process with respect to the σ-fields $\mathcal{K}_t$ with semi-group (P_t). Let $(U_p)_{p>0}$ be the resolvent family of (P_t) and $(\mathcal{F}_t)$ be the canonical σ-fields of (X_t). To show that X possesses a carré-du-champ operator, let us consider universally measurable bounded functions f and g such that $f = U_p(pf - g)$ and let us remark that

$$C_t^{p,f} = e^{-pt} f(X_t) - f(X_0) - \int_0^t e^{-ps}(g - pf)(X_s)\, ds = \mathbb{E}^\mu \left[\int_0^\infty e^{-ps}(pf - g)(X_s)\, ds \mid \mathcal{K}_t\right]$$

is a martingale with respect not only to $(\mathcal{F}_t)$ but also to $(\mathcal{K}_t)$.

By the preceding the process $<C^{p,f}, C^{p,f}>_t^{(\mathcal{K}_t)}$ is absolutely continuous, but by a result of Çinlar [8] (see also [16]) this continuous increasing additive functional is in fact $(\mathcal{F}_t)$-adapted, hence is the skew bracket of $(C_t^{p,f})$ with respect to $(\mathcal{F}_t)$. That is enough to imply that X possesses a carré-du-champ operator. □

In order to completely specify an example, let us write the characteristic function of the PSII Y :

$$\mathbb{E} \exp i(u, Y_t) = e^{-t\psi(u)} \quad \forall u \in \mathbb{R}^d$$

with
$$\psi(u) = i(\mu, u) + \frac{1}{2}(\Sigma u, u) + \int_{\mathbb{R}^d} (1 - e^{i(y,u)} + i(y,u)1_{\{|y|\leq 1\}})\, d\nu(y)$$
where $\mu \in \mathbb{R}^d$, Σ is a symmetric positive definite matrix, and ν is a positive measure on $\mathbb{R}^d$ such that $\int 1 \wedge |y|^2\, d\nu(y) < +\infty$.

We suppose that ν has a compact support in $\mathbb{R}^d$ and satisfies $\mu = \int y 1_{\{|y|>1\}}\, d\nu(y)$. Then Y is a martingale with exponential moments.

We introduce d functions $\alpha_1, \ldots, \alpha_d$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ vanishing at zero, Lipschitzian with coefficients k_i respectively, and we suppose that the compact support of ν is contained in
$$\prod_{i=1}^{d} (-\frac{1}{k_i}, +\infty).$$
The market is represented by the following system of stochastic differential equations :
$$\left.\begin{array}{l} dX_t^i = \alpha_i(X_{t-}^i)\, dY_t^i \quad i = 1, \ldots, d \\ X_0^i = x^i > 0 \end{array}\right\} \tag{7}$$

Lemma 8 . *The unique solution of system (7) takes on values in* $(0, \infty)^d$.

Proof. Extending the α_i's by 0 on $\mathbb{R}_-$, and using the usual iterative method, it is easy to prove the existence and uniqueness of a solution with values in $\mathbb{R}^d$ which is a square integrable martingale. To show that X_t^i never vanishes, we consider the sequence of stopping times $T_n = \inf\{t > 0 : X_t^i < \frac{X_0^i}{n}\}$. Then $X_{T_n-}^i \geq \frac{X_0^i}{n}$, and by the condition on the support of ν and the k_i's $X_{T_n}^i \geq \frac{1-\rho}{n} X_0^i$ holds for some $\rho \in [0, 1)$. Then Ito's formula applied to the martingale $(X_{T_n\wedge t}^i)$ and the log function yields:
$$\begin{aligned} \mathbb{E}\left(\log \frac{X_{T_n\wedge t}^i}{X_0^i}\right) &= -\frac{1}{2}\mathbb{E}\int_0^{T_n\wedge t} \sigma_{ii} \frac{\alpha_i^2(X_u^i)}{(X_u^i)^2}\, du \\ &\quad + \mathbb{E}\int_0^{T_n\wedge t} du \int \nu(dy) \left(\log \frac{X_u^i + \alpha_i(X_u^i) y^i}{X_u^i} - \frac{\alpha_i(X_u^i)}{X_u^i} y^i\right). \end{aligned}$$
Therefore $\mathbb{E}\left(\log \frac{X_{T_n\wedge t}^i}{X_0^i}\right) \geq -ct$ for some positive constant c which does not depend on n, and that gives the announced result. $\square$

By proposition (7), the process X_t is a Feller process admitting a carré-du-champ operator. The generator can be computed on C^2-functions with bounded derivatives by Ito's formula :
$$\begin{aligned} Af(x) &= \lim_{t\to 0} \frac{1}{t} \mathbb{E}^x (f(X_t) - f(X_0)) \\ &= \frac{1}{2} \sum_{i,j} \alpha_i(x) \sigma_{ij} \alpha_j(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \end{aligned}$$

$$+\int_{\mathbb{R}^d}[f(x+\alpha(x)\otimes y)-f(x)-(\mathrm{grad}f(x),\alpha(x)\otimes y)]\;d\nu(y)$$

where $\alpha(x)\otimes y$ is the vector with coordinates $\alpha_i(x)y_i \quad i=1,\ldots,d$.

Let us suppose that the stock price be modelled by a vector in $\mathbb{R}^n$ which is a linear function of the components of X :

$$S_t = BX_t$$

where B is a $n\times d$-matrix with positive coefficients. Assumptions **(A1)** and **(A2bis)** are fulfilled.

The expression of the carré-du-champ operator on the function

$$F(t,x)=P_{T-t}H\circ B(x),$$

where H satisfies assumption **(A3)**, is easy to obtain by Ito's formula if F is supposed to be C^1 in t and C^2 in x with bounded derivatives (which often, in practice, will come from the regularizing property of P_t) and then one has :

$$\begin{aligned}
\Gamma(F,F)(t,x) &= \sum_{i,j}\alpha_i(x)\sigma_{ij}\alpha_j(x)\frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_j}(x)\\
&\quad+\int_{\mathbb{R}^d}(F(t,x+\alpha(x)\otimes y)-F(t,x))^2\;d\nu(y)\\
\Gamma(G,G^*)(t,x) &= B\underline{\underline{\alpha}}(x)\left[\Sigma+\int_{\mathbb{R}^d}yy^*\,d\nu(y)\right]\underline{\underline{\alpha}}(x)B^*
\end{aligned}$$

where $\underline{\underline{\alpha}}$ is the diagonal matrix $(\alpha_i)\;\; i=1,\ldots,d$ and

$$\begin{aligned}
\Gamma(G,F)(t,x) &= B\underline{\underline{\alpha}}(x)\Sigma\underline{\underline{\alpha}}(x)\mathrm{grad}_x F(t,x)\\
&\quad+B\int_{\mathbb{R}^d}\alpha(x)\otimes y\,[F(t,x+\alpha(x)\otimes y)-F(t,x)]\;d\nu(y).
\end{aligned}$$

Remark. It is seen on this example that computing the carré-du-champ operator of (t,X_t) on regular functions amounts to letting the carré-du-champ operator of (X_t) act on the partial function $F_t : x\longrightarrow F(t,x)$. But this is not always the case. It will become valid under symmetric hypotheses that we now introduce.

III Symmetric hypotheses.

III.1 Symmetric processes and the carré-du-champ operator

We shall reinforce the assumptions of paragraph (II.2) in the following way. We suppose the Markov process (X_t) is a Hunt process with values in a locally compact space with a denumerable basis, equipped with its borel σ-field $\mathcal{E}$ and that it is symmetric with respect to a positive σ-finite measure m on $(E,\mathcal{E})$ (cf [9], [6], [11]).

The semi-group (P_t) of (X_t) induces a symmetric strongly continuous semi-group on $L^2(m)$ whose generator will be denoted by A. The operator $(-A)$ is positive self adjoint and admits the spectral representation

$$-A = \int_{[0,\infty)} \lambda \, dE_\lambda .$$

The scalar product on $L^2(m)$ is denoted $(.,.)$ and we set

$$\mathbb{D} = \{u \in L^2(m) \mid \lim_{t \to 0} \frac{1}{t}(u - P_t u, u) < +\infty\}.$$

$\mathbb{D}$ is the domain of the operator $\sqrt{-A}$ and is a Hilbert space when equipped with the graph norm : $\| u \|_1 = [(u,u) + (\sqrt{-A}u, \sqrt{-A}u)]^{\frac{1}{2}}$.

Classically $\mathbb{D}$ is called the Dirichlet space and the bilinear form on $\mathbb{D}$

$$((u,v)) = (\sqrt{-A}u, \sqrt{-A}v)$$

is the Dirichlet form associated with (X_t) (cf [11]). Let $C_0(E)$ be the space of continuous functions with compact support from E to $\mathbb{R}$; $\mathbb{D}$ is said to be regular if $\mathbb{D} \cap C_0(E)$ is dense both in $C_0(E)$ equipped with the uniform norm and in $\mathbb{D}$ equipped with the norm $\| \;.\; \|_1$. Under these conditions it can be proved (cf [11] chapter 3) that every function u in $\mathbb{D}$ admits a quasicontinuous version which will be denoted $\tilde{u}$.

The additive functional $\tilde{u}(X_t) - \tilde{u}(X_0)$ admits, under the measure $\mathbb{P}^m$, a unique decomposition of the form $\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + A_t^u$ where M_t^u is a martingale additive functional with finite energy and A_t^u is a continuous additive functional with null energy :It is the Fukushima decomposition (cf [11] chapter 5).

We now introduce the following assumption which reinforces **(A1)** :

(SA1). $\mathbb{D}$ is supposed to be regular and (X_t) is supposed to admit a carré-du-champ operator.

In this symmetric context the existence of a carré-du-champ operator will always refer to the following definition ([6] proposition 2.2) :

$$\forall f \in \mathbb{D} \cap \mathrm{L}^\infty, \exists \tilde{f} \in \mathrm{L}^1, \forall h \in \mathbb{D} \cap \mathrm{L}^\infty,$$
$$2((fh, f)) - ((h, f^2)) = \int h \tilde{f} \, dm. \qquad (8)$$

And then the carré-du-champ of f is $\Gamma(f,f) = \tilde{f}$.

The following two propositions illustrate the utility of the carré-du-champ operator for computations of skew brackets.

Proposition 9 . *Under assumption* **(SA1)**, *the carré-du-champ operator can be defined as a continuous bilinear form from $\mathbb{D} \times \mathbb{D}$ into $\mathrm{L}^1(m)$ and for u in $\mathbb{D}$ the increasing process $< M^u, M^u >_t$ is given, under $\mathbb{P}^m$, by*

$$< M^u, M^u >_t = \int_0^t \Gamma(u,u)(X_s) \, ds.$$

Proof. The definition and continuity of Γ on $\mathbb{D} \times \mathbb{D}$ is established in [6] proposition 2.2. The expression of $< M^u, M^u >_t$ can be obtained by approximating u by potentials.

Proposition 10 . *Let f belong to $\mathrm{L}^2(m)$ and let T be a positive number. For every real $t \in [0, T[$, the function $F_t = P_{T-t} f$ is in $\mathbb{D}$ and the martingale*

$$M_t = \mathbb{E}^m(f(X_t) \mid \mathcal{F}_t) = P_{T-t} f(X_t)$$

defined for $t \leq T$, satisfies

$$< M, M >_t = \int_0^t \Gamma(F_s, F_s)(X_s)\, ds \quad \forall t < T, \ \mathbb{P}^m a.s.$$

This proposition is easily obtained by showing that $M_t^2 - \int_0^t \Gamma(F_s, F_s)(X_s)\, ds$ is a martingale under $\mathbb{P}^m$, which follows easily, by using the stationarity of the process, from the following lemma :

Lemma 11 . *For every $f \in \mathrm{L}^2(m)$, the function $\Phi : [0, T[\longrightarrow \mathrm{L}^1(m)$ defined by $\Phi(t) = P_t(P_{T-t} f)^2$ is continuously differentiable and*

$$\frac{d\Phi}{dt} = P_t \Gamma(P_{T-t} f, P_{T-t} f).$$

Proof. a) Let us note first that the symmetry of the semi-group implies by spectral representation that the application $t \longrightarrow P_t f$ is C^1 (and even analytic) from $(0, \infty)$ into $\mathrm{L}^2(m)$and also into $\mathcal{D}A$ or $\mathbb{D}$.

b) Suppose $f \in \mathrm{L}^2(m) \cap \mathrm{L}^\infty(m)$, then by writing

$$\frac{\Phi(t+h) - \Phi(t)}{h} = \frac{(P_{t+h} - P_t)(P_{T-t-h} f)^2}{h} + P_t \frac{(P_{T-t-h} f)^2 - (P_{T-t} f)^2}{h}$$

it is clear that, for all $t < T$,

$$\lim_{h \to 0} \frac{\Phi(t+h) - \Phi(t)}{h} = A P_t (P_{T-t} f)^2 - 2 P_t [(P_{T-t} f) A P_{T-t} f] \tag{9}$$

in $\mathrm{L}^2(m)$.

Now, it follows from the definition of the carré-du-champ operator (8) that

$$A P_t g^2 - 2 P_t (g\, Ag) = P_t(\Gamma(g, g)) \quad \forall g \in \mathcal{D}A \cap \mathrm{L}^\infty(m). \tag{10}$$

From (9) and (10) we obtain that the following limit holds in $\mathrm{L}^2(m)$:

$$\lim_{h \to 0} \frac{\Phi(t+h) - \Phi(t)}{h} = P_t \Gamma(P_{T-t} f, P_{T-t} f) \quad \forall f \in \mathrm{L}^2(m) \cap \mathrm{L}^\infty(m). \tag{11}$$

But the right hand side being a continuous map from $[0, T)$ into $\mathrm{L}^1(m)$, we have :

$$\frac{d}{dt}(P_t(P_{T-t} f)^2) = P_t \Gamma(P_{T-t} f, P_{T-t} f)$$

in $\mathrm{L}_1(m)$ and this relation extend to all functions f in $\mathrm{L}^2(m)$. $\square$

III.2 The space $\mathbb{D}_{\mathrm{loc}}$.

Concerning the underlying stock price, it is important to be able to consider functions on E not bounded at infinity. Hence we shall introduce a local boundedness assumption for the jumps which allows to define the space $\mathbb{D}_{\mathrm{loc}}$. The Levy kernel (cf [9] chapter XV) will be denoted by N.

Proposition 12 . *Let U and V be two open sets in E such that*

$$\overline{U} \subset V \text{ and } \left(x \in \overline{U} \Longrightarrow N(x, V^c) = 0\right).$$

If two functions f and g in $\mathbb{D}$ coincide in V, then

$$\Gamma(f, f) = \Gamma(g, g)$$

holds m-a.e. in U.

Proof. Let us set $u = f - g$, and let us write the Fukushima decomposition for u :

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + A_t^u.$$

If

$$T = \inf\{t > 0 : X_t \in U^c\}$$

then $X_t \in U$ for all $t \in (0, T)$ hence $X_{t-} \in \overline{U}$ for $t \in (0, T]$ and $X_t \in V$ a.s. for t in $(0, T]$ by the assumption on the Levy kernel. It follows that $M_{t\wedge T}^u + A_{t\wedge T}^u = 0 \quad \forall t$ a.s.. But A_t^u is a continuous additive functional with zero energy hence with vanishing quadratic variation, and that implies $M_{t\wedge T}^u = 0$, hence $< M^u, M^u >_{t\wedge T} = 0$, which yields, by proposition (9), $\int \Gamma(u, u)(X_s)\, ds = 0$ a.s..

From this we obtain for almost all $s < T$,

$$\mathbb{E}^m[\Gamma(u, u)(X_s) 1_{\{s<T\}}] = 0.$$

But by the symmetry of the process killed at time T ([11] lemma 4.2.3 p97)

$$\mathbb{E}^m(\Gamma(u, u)(X_s) 1_{\{s<T\}}) = \mathbb{E}^m(\Gamma(u, u)(X_0) 1_{\{s<T\}}).$$

Hence $\Gamma(u, u)(X_0) 1_{T>0} = 0 \quad \mathbb{P}^m$ a.e., and therefore $\Gamma(u, u) = 0$ m-a.e. in U which implies the equality $\Gamma(f, f) = \Gamma(g, g)$ m-a.e. in U. □

This proposition leads to the following assumption in order to be able to define $\mathbb{D}_{\mathrm{loc}}$.

(SA2) (local boundedness of the jumps) It is supposed there exists a sequence (U_n) of relatively compact open sets such that

$$\overline{U_n} \subset U_{n+1},\ \cup_{n\in\mathbb{N}} U_n = E$$

and

$$\forall n \in \mathbb{N},\ \exists m > n,\ \forall x \in U_n \ \ N(x, U_m^c) = 0.$$

Definition 13 . *A function f will be said to belong to $\mathbb{D}_{\text{loc}}$ if there exists a sequence of functions f_n in $\mathbb{D}$ and a sequence of open sets $\mathcal{O}_n$ increasing to E such that $f = f_n$ m-a.e. on $\mathcal{O}_n$.*

Under assumption **(SA2)** one can extend Γ as an operator from $\mathbb{D}_{\text{loc}} \times \mathbb{D}_{\text{loc}}$ into $\mathrm{L}^1_{\text{loc}}(m)$.

III.3 Applications

a) Let us associate now with the process (X_t) satisfying **(SA1)** and **(SA2)** the following financial model :

(SA3) It is supposed that the discounted stock price S_t is a martingale (under $\mathbb{P}_x$ for all x) of the form $S_t = h(t)k(X_t)$ with $k \in \mathbb{D}_{\text{loc}}$ and h locally bounded with values in $\mathbb{R}$.

(SA4) The discounted contingent claim is supposed to be of the form $H(S_T)$ with H such that the function $f = H(h(T)k(.))$ be in $\mathrm{L}^2(m)$.

Proposition 14 . *Under assumptions* **(SA1)** *to* **(SA4)** *the formula giving the optimal hedging process takes the form*

$$J_t = \frac{\Gamma(F_t, G_t)}{\Gamma(G_t, G_t)}(X_{t-}) \qquad\qquad 0 < t < T$$

and the residue R_t satisfies

$$< R, R >_t = \int_0^t \left[\Gamma(F_s, F_s) - \frac{\Gamma(F_s, G_s)^2}{\Gamma(G_s, G_s)} \right] (X_s)\, ds \tag{12}$$

where $F_t = P_{T-t} f$ and $G_t = h(t)k$.

This proposition follows from the computation of skew brackets made in paragraph III.1. It can be extended to the multivariate case in the same spirit as in part II.

b) Example.

Let us consider a real PSII (X_t) whose Levy measure is symmetric with respect to the origin. Its characteristic function can then be written

$$\mathbb{E}[e^{iuX_t} \mid X_0 = x] = e^{iux} e^{-t\psi(u)}$$

with

$$\psi(u) = \frac{\sigma^2}{2} u^2 + \int_{\mathbb{R}} (1 - \cos uy)\, d\nu(y).$$

(X_t) is a Feller process symmetric with respect to Lebesgue measure.

Denote the Fourier transform of f :

$$\hat{f}(u) = \int e^{iux} f(x)\, dx.$$

Then we have :

Proposition 15 . *a) The Dirichlet form on* $\mathrm{L}^2(\mathbb{R}, dx)$ *associated with* (X_t) *is given by*

$$\mathbb{D} = \{f \in \mathrm{L}^2(\mathbb{R}, dx) \ :\ \int_{\mathbb{R}} |\hat{f}(u)|^2 \psi(u)\, du \leq +\infty\}$$

$$((f,g)) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(u)\overline{\hat{g}(u)}\psi(u)\, du \tag{13}$$

b) There exists a carré-du-champ operator given for f *and* g *in* $\mathbb{D}$ *by*

$$\Gamma(f,g)\hat{}(u) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{\hat{g}(t)}\hat{f}(t+u)\, [\psi(t) - \psi(u) + \psi(u+t)]\, dt \tag{14}$$

Proof. Point a) is classical ([11] §1.4) and the definition of the carré-du-champ operator with the explicit form obtained in a) gives (14) for f and g in $\mathbb{D} \cap \mathrm{L}^\infty$ and extends to $\mathbb{D} \times \mathbb{D}$ by continuity. □

In particular the bound $|\psi(u)| \leq c(1 + |u|^2)$ shows that functions f belonging to $\mathrm{H}^1(\mathbb{R})$ are in $\mathbb{D}$.

Let us suppose, for simplicity, that the measure ν has a compact support in $\mathbb{R}$, then the local boundedness assumption for jumps is fulfilled, and the space $\mathbb{D}_{\mathrm{loc}}$ contains the functions whose first derivatives in the sense of distributions are in ${\mathrm{L}_{\mathrm{loc}}}^2(\mathbb{R})$ hence contains C^1-functions. And for such a function f we have :

$$\Gamma(f,f) = \sigma^2 f'^2(x) + \int_{\mathbb{R}} [f(x+y) - f(x)]^2\, d\nu(y). \tag{15}$$

Observing, then, that if

$$\begin{aligned} k(x) &= ae^{\lambda x} + be^{-\lambda x} \\ h(t) &= e^{\psi(i\lambda)t} \qquad\qquad a, b, \lambda \in \mathbb{R} \end{aligned}$$

the function $h(t)k(x)$ is harmonic for the process (t, X_t), which implies, thanks to the local boundedness of jumps, that $h(t)k(X_t)$ is a martingale, we see that we can define the stock price of our model by

$$S_t = h(t)k(X_t)$$

with

$$\begin{aligned} h(t) &= \exp\left[-t\left(\frac{\sigma^2\lambda^2}{2} + \int_{\mathbb{R}} (1 - e^{\lambda y})\, d\nu(y)\right)\right] \\ k(x) &= ae^{\lambda x} + be^{-\lambda x} \\ a, b, \lambda &\in \mathbb{R}_+. \end{aligned}$$

S_t is then a positive martingale with $k \in \mathbb{D}_{\text{loc}}$ because k is C^1, assumptions **(SA1)**, **(SA2)** and **(SA3)** are therefore fulfilled and so is **(SA4)** if we take for the function H, which defines the claim, a function in $\mathrm{L}^2(\mathbb{R}, dx)$ with compact support, and proposition (14) applies.

In the case where $k(x) = ae^{\lambda x}, \quad \lambda \in \mathbb{R}, \;\; a \in \mathbb{R}_+$, by setting

$$f(x) = H(h(t)k(x)),$$

formula (12) gives after integration

$$\mathbb{E}^x < R, R >_T = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \overline{\hat{f}(u)} \hat{f}(v) e^{i(u-v)x} \left[e^{-T\psi(u-v)} - e^{-T[\psi(u)+\psi(v)]} \right] N_\lambda(u,v)\, du\, dv \tag{16}$$

with

$$N_\lambda(u,v) = 1 - \frac{[\psi(u) + \psi(i\lambda) - \psi(u+i\lambda)][\psi(v) + \psi(i\lambda) - \psi(v+i\lambda)]}{2\psi(i\lambda)[\psi(u) - \psi(u-v) + \psi(v)]}$$

and the amount of stock to have in the hedging portfolio can be computed by the fact that if one sets

$$\chi(x) = ah(T)\, e^{\lambda x} \frac{\Gamma(F_t, G_t)}{\Gamma(G_t, G_t)}(x) \tag{17}$$

then

$$J_t = \chi(X_t -) \frac{e^{-\lambda X_{t-}}}{ah(T)}$$

and $\chi(x)$ is computable by

$$\hat{\chi}(u) = \hat{f}(u) \frac{[\psi(i\lambda) + \psi(u+i\lambda) + \psi(u)]}{2\psi(i\lambda)} e^{-(T-t)[\psi(u)-\psi(i\lambda)]}.$$

If one wishes, as it is usual in practice when the asset S_t contains all the information of the market, to express the predictable process

$$J_t = \frac{\Gamma(F_t, G_t)}{\Gamma(G_t, G_t)}(X_{t-})$$

by means of the asset itself S_{t-}, one has to replace x by $\frac{1}{\lambda} \log \frac{y}{ah(t)}$ in formula (17).

IV Cases of existence of strategies without risk

The following proposition results from stochastic calculus without any Markovian hypothesis :

Proposition 16 . *Suppose the discounted underlying stock price is a* continuous *local martingale with values in* $\mathbb{R}^d$ *with respect to a filtration* $(\mathcal{G}_t)$ *and that the contingent claim* H *be such that the conditional expectation* $\mathbb{E}[H \mid \mathcal{G}_t]$ *can be expressed in the form*

$$\mathbb{E}[H \mid \mathcal{G}_t] = V(t, S_t)$$

where V *is a difference of convex functions on* $\mathbb{R}_+ \times \mathbb{R}^d$*. Then there exists a* $\mathcal{G}_t$*-predictable process* K_s *such that*

$$H = H_0 + \int_{(0,T]} K_s \, dS_s$$

with H_0 $\mathcal{G}_0$*-measurable.*

Proof. Applying Ito's formula for convex functions (cf [4]) to V and the semimartingale $Y_t = (t, S_t)$, one has

$$V(t, S_t) - V(0, S_0) = \int_0^t V^*(Y_s) \, dY_s + C_t(Y, Y^*, V)$$

where $C_t(Y, Y^*, V)$ is a continuous process with finite variation and where V^* is some (any, cf [5]) borel section of the sub-differential of V.

Then we see that in the relation

$$V(t, S_t) - V(0, S_0) - \sum_{i=1}^{d} \int_0^t V_i^*(s, S_s) \, dS_s^i = \int_0^t V_t^*(s, S_s) \, ds + C_t(Y, Y^*, V)$$

both sides are equal to zero because the left hand side is a local martingale and the right hand side a continuous process with finite variation. That gives the result. □

The preceding study allows to improve this result under Markovian hypotheses.

Assume that symmetric hypotheses of part III hold and that the process (X_t) has continuous sample paths. Let us recall the notations:

$$\begin{array}{rclcrcl} S_t & = & h(t)k(X_t), & \qquad & G_t(x) & = & h(t)k(x) \\ f & = & H(h(T)k(.)), & & F_t(x) & = & P_{T-t}f(x). \end{array}$$

Proposition 17 . *Let us suppose* F_t *can be written*

$$F_t(x) = \chi(t)\xi(G_t(x))$$

with χ *measurable and finite and* ξ *Lipschitzian from* $\mathbb{R}$ *into* $\mathbb{R}$*, then there exists a hedging strategy without risk.*

Proof. That comes from the Lipschitzian functional calculus (cf [6]). In formula (12) we have

$$\Gamma(F_t, G_t) = \chi(t)\xi'(G_t)\Gamma(G_t, G_t) \qquad m - \text{a.e.}$$

and

$$\Gamma(F_t, F_t) = \chi^2(t)\xi'^2(G_t)\Gamma(G_t, G_t) \qquad m - \text{a.e.}$$

where ξ' is a version of the derivative of ξ. □

There is a similar result in the multivariate case if ξ is supposed to be C^1 (cf [6]).

References

[1] L. Bachelier. Théorie de la spéculation. Ann. Sci. Ec. Normale,17, (1900), *21-86.*

[2] A. Bensoussan. On the theory of option pricing. Acta Applic. Math. 2, (1984), *139-158.*

[3] N. Bouleau. Propriétés d'invariance du domaine du générateur étendu d'un processus de Markov. In Sem. Prob.XV, *167-188*, (Lect. Notes in M. 850, Springer, 1981).

[4] N. Bouleau. Semimartingales à valeurs $\mathbb{R}^d$ et fonctions convexes. Note C.R.Acad.Sci. Paris, t292, (1981).

[5] N. Bouleau. Formules de changement de variables. Annales I.H.P. vol20, n^o2, (1984), *133-145.*

[6] N. Bouleau, F. Hirsch. Formes de Dirichlet générales et densité des variables aléatoires réelles sur l'espace de Wiener. J. of Func. Analysis 69, n^o2, (1986),*229-259.*

[7] CERMA. Sur les risques résiduels des stratégies de couverture des actifs conditionnels. C.R.Acad.Sci. Paris,307, sI, (1988), *625-630* .

[8] E. Cinlar. Markov additive processes. Zeit. f. Wahrscheinlichkeit., 24, (1972),*85-93.*

[9] C. Dellacherie, P.A. Meyer. *Probabilités et Potentiels, Processus de Markov.* Chap. XII à XVI, Hermann 1987.

[10] H. Föllmer, D. Sondermann, Hedging of non redundant contingent claims. In *Contributions to Mathematical Economics in Honor of Gerard Debreu.* W. Hildebrand, A. Mas-Colell ed., North Holland, 1986.

[11] M. Fukushima, *Dirichlet Forms and Markov processes.* North Holland, Kodansha, 1980.

[12] R.K. Getoor. *Markov processes, Ray processes and right processes.* Lect. notes in M. 440, Springer, 1975.

[13] J. Harrison, S. Pliska. Martingales and stochastic integrals in the theory of continuous trading. Stoch. Processes and Appl. II, (1981),*215-260.*

[14] F. Hirsch. L'opérateur carré du champ d'après Roth. Sém. Bourbaki, $29^{ème}$ année, In Lect. notes in M. 677, *167-182*, Springer, 1978.

[15] J. Jacod. *Calcul stochastique et Problème de Martingales.* Lect. Notes in M. 714, Springer, 1979.

[16] J. Jacod, B. Maisonneuve. Remarques sur les fonctionnelles additives non adaptées des processus de Markov. In Sém. Prob. XIV, Lect. notes in M. 784, Springer, 1980.

[17] I. Karatsas. Applications of stochastic calculus in financial economics, to appear in Lect. Notes in Control and inf. Sci., Springer.

[18] R.C. Merton. Theory of rational option pricing. Bell J. of Econ. and Manag. Sci., 4, (1973), *141-183.*

[19] R.C. Merton. Option pricing when underlying stock returns are discontinuous. J. Financial Economics, 3, (1976), *125-144.*

[20] P.A. Meyer. Démonstration probabiliste de certaines inégalités de Littlewood-Paley. In Sém. Prob. X, *125-183*, Lect. Notes in M. 551, Springer, 1976.

[21] P.A. Meyer. Martingales locales fonctionnelles additives I, In Sém. Prob. XII, Lect. Notes in M. 649, *775-785*, Springer, 1978.

[22] P.A. Meyer. Flot d'une équation différentielle stochastique. Sém. Prob. XV, In Lect. notes in M. 850, *103-117*, Springer, 1981.

[23] C. Stricker. Representation prévisible et changement de temps. Annals of Prob. 14, $n^o 3$, (1986), *1070-1074.*

Short title:

HEDGING STRATEGIES

Abstract:

We prove two explicit formulae for the quadratic residual risk and for the optimal hedging portfolio of a european contingent claim when the underlying stock prices are functions of a Markov process. These expressions allow the practical handling of a great deal of non classical models which are less optimistic than Black and Scholes's one.

Keywords:

Right Markov processes
Symmetric Markov processes
Carré-du-champ operator
Processes with independent increments
Options
Hedging strategies
Portfolios