Some Results on Lipschitzian Stochastic Differential Equations by Dirichlet Forms Methods.

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Since the impulse given by P. Malliavin, the stochastic calculus of variations has been mainly applied to stochastic differential equations with C^∞ coefficients, see Ocone [?] for a comprehensive exposition.

But it is also important for applications to get regularity results for solutions of SDE with less smooth coefficients and in particular under Lipschitz hypotheses which are, in dimension greater than one, the most natural hypotheses of existence and uniqueness of solutions.

The celebrated integration by parts method cannot apparently be extended beyond the case of functionals in the domain $\mathcal{D}L$ of the Ornstein-Uhlenbeck operator ($\mathbb{D}_{2,2}$ with the notations of Watanabe [?]), so that the regularity of solutions of Lipschitzian SDE must come from specific technics. Especially well adapted are Dirichlet forms methods which allow to exploit intensively the fact that Lipschitz functions operate on $\mathbb{D}_{2,1} = \mathcal{D}\sqrt{-L}$.

We give here an account of results already obtained in this direction by Dirichlet forms methods and we present in details a new example which gives rise to an extension of the stochastic calculus. The first part introduces the framework of the Dirichlet space related to the Ornstein-Uhlenbeck semigroup on the Wiener space and recalls the absolute continuity criterion (cf [?] [?]) for functionals in $\mathbb{D}_{2,1}$ or $\mathbb{D}_{2,1}^{loc}$ and some consequences on Lipschitz SDE.

The second part is devoted to the regularity of solutions of Lipschitz SDE with respect to initial data. It is shown that the solution is differentiable in a slightly weakened sense. That gives for example the following simple result: under these hypotheses, if the initial variable X_0 has a density, then X_t has a density for all t.

After recalling the definition of the capacity associated with the Ornstein-Uhlenbeck Dirichlet form, it is shown in the third part, that the solutions of Lipschitz SDE can be refined, by taking quasi-continuous versions for each t, into processes with continuous paths outside a polar set and unique up to a quasi-evanescent set. The main tool here is an extension of the Kolmogorov theorem on existence of continuous versions to the case where the measure is changed to a capacity.

This allows to study the solutions of Lipschitz SDE under measures which do not charge polar sets. In the last part, using Wiener chaos decompositions of positive distributions, we show that this property allows an extension of the stochastic calculus by constructing a finite energy measure singular with respect to the Wiener measure and for which the coordinates do not build a semimartingale. This answers a conjecture formulated in [?].

I The structure of Dirichlet space on the Wiener space associated with the Ornstein - Uhlenbeck semigroup.

The Wiener space

$$\Omega = \{\omega \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d); \omega(0) = 0\}$$

is equipped with the topology of uniform convergence on compact sets, with its Borelian σ-algebra and with the Wiener measure m which makes the coordinates a standard Brownian motion. $\mathcal{F}$ denotes the m-completed σ-algebra of $\sigma(B_t; t \in \mathbb{R}_+)$, and $\mathcal{F}_t$ the $\mathcal{F}$-m-completed σ-algebra of $\sigma(B_s; s \leq t)$.

We consider on $L^2(m)$ the Ornstein-Uhlenbeck semigroup P_t a strongly continuous symmetric Markovian semigroup characterized by

$$P_t[\exp\{\int h(s).dB_s - \frac{1}{2}\|h\|^2\}] = \exp\{e^{-t/2}\int h(s).dB_s - \frac{1}{2}\|e^{-t/2}h\|^2\}$$

$$\forall h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d).$$

The self-adjoint operator generator of P_t is denoted by L. It corresponds to P_t (cf [?]) a Dirichlet form with domain

$$\mathbb{D} = \mathcal{D}(\sqrt{-L})$$

given by

$$((u,u)) = \|\sqrt{-L}u\|^2_{L^2(m)}.$$

That means that the space $\mathbb{D}$ with the norm $(\|u\|^2_{L^2(m)} + ((u,u))\ )^{1/2}$ is complete and that normal contractions operate: For all $u \in \mathbb{D}$, for all measurable v such that $\forall\omega|v(\omega)| \leq |u(\omega)|$ and $\forall\omega,\omega'|v(\omega) - v(\omega')| \leq |u(\omega) - u(\omega')|$, one has $v \in \mathbb{D}$ and $((v,v)) \leq ((u,u))$.

This Dirichlet form is local ([?] p239) and possesses a carré-du-champ operator, i.e. a symmetric bilinear continuous map Γ from $\mathbb{D} \times \mathbb{D}$ into $L^1(m)$ such that $\forall u, v \in \mathbb{D} \cap L^\infty(m)$,

$$2((uv, u)) - ((v, u^2)) = \int v\Gamma(u, u)\, dm.$$

This Dirichlet structure $(\Omega, \mathcal{F}, m; ((.,.)), \mathbb{D})$ is related to the Sobolev spaces which are classically defined on the Wiener space in the following way:

Let $\mathbb{D}_{p,s}, p \in (1, \infty), s \in \mathbb{R}$ the closure of the linear space generated by polynomials in continuous linear forms on Ω for the norm

$$\|F\|_{p,s} = \|(I - L)^{s/2} F\|_p$$

(cf [?]). Then $\mathbb{D} = \mathbb{D}_{2,1}$ and $\|F\|^2_{L^2} + ((F, F)) = \|F\|^2_{2,1}$.

Let

$$\xi_n = \int_0^t \dot{\xi}_n(s)\, ds$$

where $\dot{\xi}_n$is a complete orthonormal system of $H = L^2(\mathbb{R}_+, \mathbb{R}^d)$, then for all $u \in \mathbb{D}$ the following limit exists in probability (cf [?])

$$\nabla_{\xi_n}(u)(\omega) = \lim_{t \downarrow 0} t^{-1}[u(\omega + t\xi_n) - u(\omega)]$$

and one has

$$\Gamma(u, u) = \sum (\nabla_{\xi_n} u)^2.$$

The derivation operator D (cf [?]) which can be defined by

$$Du = \sum_n (\nabla_{\xi_n} u)\dot{\xi}_n$$

and which is continuous from $\mathbb{D} = \mathbb{D}_{2,1}$ into $L^2(\Omega, H)$ is related to the carré-du-champ operator Γ by

$$\Gamma(u, u) =< Du, Du >_H, \; \forall u \in \mathbb{D}_{2,1}.$$

This relation between the carré-du-champ operator and the derivations in the directions of Cameron-Martin vectors ($\xi \in \Omega$ s.t. $\dot{\xi} \in H$) allows, by an extension of the co-area formula of Federer [?], to obtain the following absolute continuity criterion:

Proposition 1 . *Let $u = (u_1, \ldots, u_n) \in (\mathbb{D}_{2,1})^n$, the image by u of the measure*

$$det[\Gamma(u, u^*)].m$$

is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.

When $n = 1$, this result is true for any local Dirichlet space (cf [?]) and also for the local energy part in any Dirichlet space on a locally compact space (cf [?]).

In fact proposition 1 remains valid for u in $(\mathbb{D}_{2,1}^{loc})^n$ defined by

$$\mathbb{D}_{2,1}^{loc} = \{u : \Omega \to \mathbb{R};\ \exists \Omega_n \in \mathcal{F},\ \Omega_n \uparrow \Omega,\ \forall n \exists u_n \in \mathbb{D}_{2,1},\ u = u_n \,\text{on}\, \Omega_n\}$$

and for $u \in \mathbb{D}_{2,1}^{loc}$, $\Gamma(u,u)$ depends only on u.

An important application of the extension of Dirichlet forms methods to the case of the Wiener space is the study of stochastic differential equations. Let us specify the **Lipschitz hypotheses** which will be in force in the sequel:

Two Borelian functions σ, b are given

$$\sigma \,:\, \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$$

$$b \,:\, \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and there exists $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall T \in \mathbb{R}_+,\ \forall t \in [0,T],\ \forall x, y \in \mathbb{R}^n$$

$$|\sigma(t,x)| \vee |b(t,x)| \leq K(T)(1 + |x|)$$
$$|\sigma(t,x) - \sigma(t,y)| \vee |b(t,x) - b(t,y)| \leq K(T)(|x-y|).$$

One is concerned by the equation

$$dX_t = \sigma(t, X_t).dB_t + b(t, X_t)\,dt. \tag{1}$$

From the fact that contractions hence Lipschitz functions operate on the Dirichlet space, it follows (cf [?] [?]) that the solution of (??) is such that the map $t \to X_t$ is continuous from $\mathbb{R}_+$ into $(\mathbb{D}_{2,1})^n$ and by writing down a stochastic differential equation satisfied by the matrix $\Gamma(X_t, X_t^*)$ it is possible to bring out conditions under which X_t has a density by application of proposition 1.

For example if $A_k = \{(t,y)\ :\ \sigma(t,y)\ \text{is of rank}\ k\}$ and if T_k is the essential beginning of A_k for $(X_t)_{t\geq 0}$, one gets that for t such that $m(\{t > T_k\}) > 0$ and for almost all subspace V of $\mathbb{R}^n$ of dimension k, the projection of X_t on V, knowing $\{t > T_k\}$, has a density with respect to the Lebesgue measure on V.

II Regularity of solutions of Lipschitz SDE with respect to the initial data

Under these Lipschitz hypotheses, it is known (cf [?]) that there exists a version $(X_t^x(\omega))_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n}$ of the solution of (??) starting at x, such that for almost all ω the map $(t,x) \to X_t^x(\omega)$ is continuous and for all $t \geq 0$ $x \to X_t^x(\omega)$ is an onto homeomorphism of $\mathbb{R}^n$.

If it is supposed further that σ and b are $C^{1,\alpha}$ with respect to x then $x \to X_t^x(\omega)$ is an onto C^1- diffeomorphism.

Under the only Lipschitz hypotheses, X_t^x is of course not C^1 with respect to x in general, but it is possible to show that the Jacobian $\frac{\partial}{\partial x}(X_t^x(\omega))$ exists in a weakened sense and satisfies a SDE which can be written explicitly.

For this, consider the space $\tilde{\Omega} = \mathbb{R}^n \times \Omega$ equipped with the probability $\tilde{m} = h(x)\,dx \times m$ where m is the Wiener measure on Ω and h a strictly positive continuous function such that $\int h(x)\,dx = 1$, $\int |x|^2 h(x)\,dx < +\infty$. The σ-algebras generated by applications B_s, $s \le t$ and completed for $\tilde{m}$ are denoted by $\tilde{\mathcal{F}}_t$.

$(\tilde{\Omega},\, \tilde{m})$ gets a natural Dirichlet form associated with the derivations in directions given by the canonical basis of $\mathbb{R}^n$. In other words the tool is here the form

$$((u,v)) = \int \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i})\,d\tilde{m}$$

with domain and operators $\partial/\partial x_i$ suitably defined.

We denote by $(\tilde{X}_t)_{t\ge 0}$ [resp. $(\tilde{B}_t)_{t\ge 0}$] the class of the process $(X_t^x)_{t\ge 0}$ [resp. of the Brownian motion $(B_t)_{t\ge 0}$] enlarged up to $\tilde{m}$-evanescent sets.

Proposition 2 . *Under the Lipschitz hypotheses,*

a) for m-almost all ω, $\forall t \ge 0$, $X_t^{\bullet}(\omega) \in (H^1_{loc}(\mathbb{R}^n))^n$

b) there exists a process $(M_t(x,\omega))_{t\ge 0}$, $(\tilde{\mathcal{F}}_t)$-adapted, with continuous paths and values in $GL_n(\mathbb{R})$, such that

for m-almost all ω, $\forall t \ge 0$, $[\frac{\partial}{\partial x}(X_t^x(\omega)) = M_t(x,\omega)\ dx - a.e.]$

c) let σ' and b' be fixed Borelian versions of the derivatives $\frac{\partial}{\partial x}\sigma(t,x)$ and $\frac{\partial}{\partial x}b(t,x)$, then M is the unique $(\tilde{\mathcal{F}}_t)$-adapted continuous solution, defined up to an $\tilde{m}$-evanescent set, of the SDE

$$\begin{cases} dM_t &= [\sigma'(t,\tilde{X}_t).M_t]\,dB_t + [b'(t,\tilde{X}_t).M_t]\,dt \\ M_0 &= I \end{cases}$$

It follows from this proposition and from a variant of proposition 1 applied to the Dirichlet structure on $\tilde{\Omega}$ explained above that the equation (1) with initial value a random variable independent of (B_t) possessing a density , has a solution which admits a density for all $t \ge 0$. This was known, apparently, in dimension greater than one, only under $C^{1,\alpha}$ hypotheses.

In dimension 1, there is an explicit solution : if we write as before σ_i', b' for fixed Borelian versions of the derivatives of σ and b with respect to x, the process

$$Y_t^x = \exp\left\{\sum_{i=1}^{d}\left(\int_0^t \sigma_i'(s,X_t^x)\,dB_s^i - \frac{1}{2}\int_0^t [\sigma_i'(s,X_t^x)]^2\,ds\right) + \int_0^t b'(s,X_t^x)\,ds\right\}$$

is such that for m-almost all ω,

$$\forall \alpha,\beta \in \mathbb{R},\, \forall t \ge 0, \quad X_t^{\beta}(\omega) - X_t^{\alpha}(\omega) = \int_{\alpha}^{\beta} Y_t^x(\omega)\,dx.$$

III Regularity, up to a polar set, of the solutions and their flows.

The Dirichlet form on the Wiener space associated with the Ornstein-Uhlenbeck operator makes it possible to look at properties of the Brownian motion satisfied up to a zero capacity set (cf [?], [?],[?]).

We study here, from this point of view, properties of solutions of Lipschitz SDE. A work in the same spirit was done independently by J. Ren (cf [?]) for equations with C^∞-coefficients and with thin sets associated with $C_{p,s}$-capacities (cf [?]).

We denote by C the capacity associated with the Ornstein-Uhlenbeck Dirichlet form. It is defined by

$$C(G) = inf\{\|u\|_{2,1}^2;\ u \in \mathbb{D},\ u \geq 1\ m - \text{a.e. on } G\}$$

if G is an open set, and by

$$C(G) = inf\{C(G)\ G \text{ open and } G \supset A\}$$

if $A \in \mathcal{F}$.

If $C(A) = 0$, A is said to be a polar set.

$f:\ \Omega \to \mathbb{R}$ is said to be quasi-continuous with respect to the capacity C if $\forall \epsilon > 0, \exists \Omega_\epsilon$ open with $C(\Omega_\epsilon) < \epsilon$ such that f restricted to the complementary Ω_ϵ^c of Ω_ϵ is continuous.

Two processes $(u_\lambda)_{\lambda\in\Lambda}$, $(v_\lambda)_{\lambda\in\Lambda}$ defined on Ω are said to be C-indistinguishable if there exists a polar set A such that $\forall \omega \notin A$, $\forall \lambda \in \Lambda$, $u_\lambda(\omega) = v_\lambda(\omega)$.

Under the Lipschitz hypotheses, we know that the solution $X_t^x(\omega)$ of equation (1) starting at x is such that for fixed t, x, $X_t^x \in \mathbb{D}_{2,1}^n$. It follows that this random variable admits a quasi-continuous version defined up to a polar set . The following extension of the Kolmogorov theorem gives conditions under which it is possible to put these quasi-continuous versions together to get a continuous process outside a polar set.

Proposition 3 . *Let $(u_x)_{x\in\mathbb{R}^r}$ be a family of elements of $\mathbb{D}$ and p, $\alpha_1, \ldots, \alpha_r$ stricly positive real numbers. Suppose the following conditions hold*

- ◇ $\sum_{i=1}^r \frac{1}{\alpha_i} < 1$
- ◇ $\forall x, y \in \mathbb{R}^r\ \ |u_x - u_y|^p \in \mathbb{D}$
- ◇ $\exists L : \mathbb{R}_+ \to \mathbb{R}_+,\ \forall R > 0,\ \forall x, y \in \mathbb{R}^r$
  $|x| \vee |y| \leq \mathbb{R} \Longrightarrow \|\,|u_x - u_y|^p\|_{2,1}^2 \leq L(R) \sum_{i=1}^r |x_i - y_i|^{\alpha_i}$

Then there exists a family $(v_x)_{x\in\mathbb{R}^r}$ such that

i) $x \to v_x(\omega)$ is continuous

ii) for all x v_x is a quasi-continuous version of u_x.

The family (v_x) *is unique up to* C*-indistinguishability and the following uniformity properties hold:*

There exist open sets $(\Omega_\epsilon)_{\epsilon>0}$ *with compact complementary* Ω_ϵ^c *such that*

a) $\forall\epsilon>0,\ C(\Omega_\epsilon)<\epsilon$ *and the map* $(x,\omega)\in\mathbb{R}^r\times\Omega_\epsilon^c\to v_x(\omega)\in\mathbb{R}$ *is continuous*

b) $\forall\beta_i,\ 0<\beta_i<\alpha_i(1-\sum_{j=1}^r 1/\alpha_i)/2p \quad i=1,\ldots,r$

$\exists K>0,\ \forall\epsilon>0,\ \forall R>0,\ \exists\eta>0,$

$(\omega\in\Omega_\epsilon^c,\ |x|\vee|y|\le\mathbb{R},\ |x-y|\le\eta)\ \Rightarrow\ |v_x(\omega)-v_y(\omega)|\le K\sum_{i=1}^r|x_i-y_i|^{\beta_i}.$

This criterion allows to show that under the Lipschitz hypotheses and for a given fixed initial condition $x\in\mathbb{R}^n$, the solution X_t^x of equation (1) can be made more accurate into a process $(\tilde{X}_t)_{t\ge0}$ unique up to C-indistinguishability such that

i) $t\to\tilde{X}_t$ is continuous,

ii) for all t $\tilde{X}_t$ is quasi-continuous and $\tilde{X}_t=X_t^x$ $m-$a.s..

This result has been extended, by using a Banach valued space $\mathbb{D}_{2,1}$ by D. Feyel and A. de la Pradelle [**?**] to the case of Ito processes of the form

$$X_t=\int_0^t\alpha_s.dB_s+\int_0^t\beta_s.ds$$

with $\alpha,\ \beta\in L^2(\mathbb{R}_+,\mathbb{D})$ and adapted.

The previous criterion of Kolmogorov type, allows also to obtain a quasi-continuous version $\tilde{X}_t^x$ of X_t^x which is for ω outside a polar set, continuous in (t,x) and an onto homeomorphism with respect to x; but for this $C^{1,\alpha}$-hypotheses in x are needed for σ and b (cf [**?**]).

With $C^{2,\alpha}$-hypotheses, the differentiability with respect to x of the flow is obtained with a quasi-continuous regular Jacobian matrix $\frac{\partial}{\partial x}\tilde{X}_t^x(\omega)$ continuously depending on (t,x) for ω outside a polar set (see [**?**] theorems V.1 and V.2 for more precise results).

IV Stochastic calculus under a probability which does not charge polar sets

We keep in the sequel the preceding globally Lipschitz hypotheses and look at the solution of

$$X_t=x+\int_0^t\sigma(s,X_s).dB_s+\int_0^t b(s,X_s)\,ds \tag{2}$$

which is continuous in t, quasi-continuous in ω and unique up to C-indistinguishability.

This process is well defined under any probability measure on the Wiener space which does not charge polar sets.

A. The first case is when the right hand side of (**??**) also makes sense under such a measure ν.

To be precise with the changes of measure we introduce the σ-fields $\mathcal{F}_t^0 = \sigma(B_s,\, s \leq t)$ without any completion.

It can be shown (cf [?]) that there exists an $(\mathcal{F}_t^0)$-adapted solution , $\tilde{X}_t$, of (??) such that, for fixed t, X_t is quasi-continuous in ω, and for quasi every ω, $t \to X_t(\omega)$ is continuous. Then if ν is a probability measure on Ω which does not charge polar sets and such that the process (B_t) is an $(\mathcal{F}_t^0)$-semimartingale under ν, the process $\tilde{X}_t$ is the solution of the same SDE under ν, that is to say $\tilde{X}_t$ satisfies ν-a.e.

$$\forall t \quad \tilde{X}_t = x + \int_0^t \sigma(s, \tilde{X}_s) \overset{\nu}{\cdot} dB_s + \int_0^t b(s, \tilde{X}_s)\, ds$$

where $\int \sigma(s, \tilde{X}_s) \overset{\nu}{\cdot} dB_s$ denotes the stochastic integral under ν.

For a one dimensional Brownian motion ($d = 1$), the law of the Brownian bridge $\mathbb{E}[\ .\ | B_1 = a]$ is an example of such a measure ν which is singular with respect to the Wiener measure (cf [?]). For $d > 1$ the same result is obtained by taking the conditional law of the Brownian motion given that B_1 belongs to an $(n-1)$-dimensional hyperplan with the Gauss measure on it.

B. The case which gives rise to a true extension of the classical stochastic calculus is when under ν (B_t) fails to be a semimartingale so that the right hand side has no direct meaning by itself.

We construct now a family of such measures on the Wiener space in the case $d = 1$ for simplicity.

The idea is to consider a conditional law of the form $\mathbb{E}[\ .\ | \int_0^1 h_0(s) dB_s = 0]$ for $h_0 \in L^2([0,1])$, $\int_0^1 h_0^2(s) ds = 1$.

For using computations by decomposition on the Wiener chaos, we define this object as the positive measure which coincides on $\mathbb{D} \cap C(\Omega)$ with the distribution on the Wiener space

$$\nu = \sqrt{2\pi}\,\delta_0(\tilde{h_0}) \qquad\qquad \tilde{h_0} = \int_0^1 h_0(s) dB_s \tag{3}$$

in the sense of Meyer-Yan [?].

The characteristic functional of ν is

$$U_\nu(\xi) = e^{-\frac{1}{2}<\xi, h_0>^2} \qquad\qquad \xi \in C_c^\infty((0,1]) \tag{4}$$

so that its decomposition on the chaos is written

$$\nu = \sum_n \frac{1}{n!} I_n(f_n)$$

with

$$\begin{cases} f_n &= \frac{(2p)!(-1)^p}{p! 2^p} h_0^{\otimes 2p} & \text{if } n = 2p \\ f_n &= 0 & \text{if } n = 2p+1 \end{cases} \tag{5}$$

where

$$I_n(f) = n! \int_{0<s_1<\ldots<s_n<1} f(s_1,\ldots,s_n) dB_{s_1} \ldots dB_{s_n}$$

for symmetric $f \in L^2([0,1]^n)$.

So ν is a distribution of Watanabe and putting $\nu_{2p} = \frac{1}{(2p)!} I_{2p}(f_{2p})$ one has

$$\|\nu_{2p}\|_{L^2}^2 = \frac{(2p)!}{(p!)^2 2^{2p}}$$

It follows that ν has a finite energy that is to say

$$\|\nu\|_{2,-1}^2 = \sum_{p=0}^{\infty} \frac{1}{1+2p} \|\nu_{2p}\|^2 \ \leq \ 1 + \sum_{p=1}^{\infty} \frac{1}{1+2p} . \frac{1}{\sqrt{ep}} \ < +\infty.$$

Formula (4) extends to $\xi \in L^2([0,1])$

$$<\nu, \mathcal{E}(\xi)> = e^{-\frac{1}{2}<\xi,h_0>^2} \tag{6}$$

where

$$\mathcal{E}(\xi) = \exp[\int_0^1 \xi_s \, dB_s - \frac{1}{2} \int_0^1 \xi_s^2 \, ds].$$

Let $h_0, h_1, \ldots, h_n, \ldots$ be a complete orthonormal system of $L^2([0,1])$, it follows from (??) that if g is a polynomial one has

$$<\nu, g(\tilde{h_0}, \tilde{h_1}, \ldots, \tilde{h_n})> = < m, g(0, \tilde{h_1}, \ldots, \tilde{h_n})> \tag{7}$$

and ν is then a positive distribution of Watanabe hence ν is a measure (cf [?]).

For $t \in [0,1]$ we consider also the distrbutions ν_t defined by

$$U_{\nu_t}(\xi) = <\nu, \mathcal{E}(\xi 1_{[0,t]})> = e^{-\frac{1}{2}<\xi, h_0 1_{[0,t]}>^2}.$$

A similar computation as the previous one gives

$$\|\nu_t\|_{2,-1} \leq \|\nu\|_{2,-1}$$

The family (ν_t) is a distribution martingale in the sense of Yan [?] and the ν_t's are probabilities on Ω.

Following the notations of [?] we compute now the adapted projection of the distribution $D\nu$ where D is the gradient operator. We have

$$(D\nu)^{ad}(t) = \sum_{q=1}^{\infty} \frac{1}{(2q-1)!} I_{2q-1} \left((h_0 1_{[0,t]})^{\otimes 2q-1} \frac{(2q)!(-1)^q}{q! 2^q} h_0(t) \right)$$

and by using the formulae of Shigekawa [?]

$$\tilde{h} I_n(h^{\otimes n}) = I_{n+1}(h^{\otimes(n+1)}) + n\|h\|^2 I_{n-1}(h^{\otimes(n-1)})$$

we obtain

$$(D\nu)^{ad}(t) = 1_{\{t<a\}} \frac{-h_0(t)}{1 - \|h_0 1_{[0,t]}\|^2} (\widetilde{h_0 1_{[0,t]}}).\nu_t \tag{8}$$

where $\| \, . \, \|$ is the norm of $L^2([0,1])$ and

$$(\widetilde{h_0 1_{[0,t]}}) = \int_0^t h_0(s)\, dB_s,$$

and with

$$a = \inf\{t : \int_0^t h_0^2(s)\, ds = 1\}.$$

If we write (**??**) in the following form

$$(D\nu)^{ad}(t) = \zeta(t).\nu_t$$

the formula of Ito-Ustunel [**?**] gives

$$\nu = m + \delta(\zeta(.)\nu_.)$$

From now on, we suppose $a < 1$ and h_0 with bounded variation. $(\widetilde{h_0 1_{[0,t]}})$ possesses then a version which is an $(\mathcal{F}_t^0)$-adapted process continuous in (t,ω) on $[0,1] \times \Omega$. In the sequel ζ is supposed to be defined from this version. (The following construction is also possible without supposing the variation of h_0 to be bounded. A version of $(\widetilde{h_0 1_{[0,t]}})$ should be chosen continuous in t and quasi-continuous in ω, what is always possible).

That leads to the following lemma:

Lemma 4 *.The process*

$$M_t = B_t - \int_0^t \zeta_s\, ds$$

(where the integral is, for $t \geq a$, a semi-convergent integral) is an $(\mathcal{F}_t^0)$-Brownian motion under ν.

Proof. Using the fact that the measure ν restricted to the σ-field $\mathcal{F}_t^0$ is the measure ν_t, we break up the interval $[0,1]$ in $[0,a)$, $\{a\}$, $(a,1]$.

a) First if $t < a$, the distribution ν_t is a random variable in L^2, in other words the measure ν_t has a density in L^2 with respect to m.

Indeed setting $\theta = \int_0^t h_0^2(s)\, ds$,

$$\begin{aligned} \|\nu_t\|_{L^2}^2 &= \sum_{p=0}^{\infty} \frac{(2p)!(-1)^p}{p!2^p} \theta^{2p} \\ &= \frac{1}{\sqrt{1-\theta^2}} < +\infty. \end{aligned}$$

It follows then from (8) that if we set $n_t = \frac{d\nu_t}{dm}$ we have

$$n_t = 1 + \int_0^t \zeta_s n_s\, dB_s \qquad \text{for } t < a \tag{9}$$

and because $\mathbb{P}[\int_0^t \zeta_s^2\, ds < +\infty] = 1$ for $t < a$ we get

$$n_t = \exp[\int_0^t \zeta_s\, dB_s - \frac{1}{2}\int_0^t \zeta_s^2\, ds]$$

hence $n_t > 0$ and in fact ν_t and m are equivalent, ν_t being a probability it holds $\mathbb{E}n_t = 1$ and the classical Girsanov theorem (cf [?]) applies and gives the result.
b) The study of the limit of M_t for $t \uparrow a$ is obvious under ν, and M_t is an $\mathcal{F}_t^0$-Brownian motion under ν on $[0, a]$.
c) At last it follows easily from formula (7) that under ν σ-fields $\mathcal{F}_a^0$ and $\sigma(B_s - B_a, a \leq s \leq 1)$ are independent and that

$$< \nu, GF >=< \nu, G >< m, F >$$

if G is $\mathcal{F}_a^0$-measurable and if F is $\sigma(B_s - B_a, a \leq s \leq 1)$-measurable, what gives the result by

$$M_t = M_a + B_t - B_a \qquad \text{for } t > a.$$

□

Writing $B_t = M_t + \int_0^t \zeta_s\, ds$, we see that in order that the coordinates (B_t) fail to be a semimartingale, it is sufficient to choose h_0 in such a way that the continuous process $\int_0^t \zeta_s\, ds$ fails to have a finite variation in the neighbourhood of a under ν.

As m and ν are mutually singular, we must express ζ_s in terms of the Brownian motion M_t under ν.

Lemma 5 . *For $t < a$, it holds*

$$\zeta_t = -h_0(t)\int_0^t \frac{h_0(s)}{\int_s^a h_0^2(u)\, du} dM_s$$

Proof. This comes from the fact that the relation $M_t = B_t - \int_0^t \zeta_s\, ds$ yields

$$M_t = B_t + \int_0^t \left[\frac{h_0(s)}{\int_s^a h_0^2(u)\, du}\int_0^s h_0(u)\, dB_u\right] ds$$

and it is not difficult to see that this relation can be turned into the following

$$M_t = B_t + \int_0^t \left[h_0(s)\int_0^s \frac{h_0(u)}{\int_u^a h_0^2(v)\, dv} dM_u\right] ds.$$

□

To show that it is possible to choose h_0 in such a way that

$$I := \int_0^a |\zeta_s|\, ds = +\infty \qquad \nu - \text{a.s.} \tag{10}$$

we perform some transformations:

Let u be a function from $[0,\infty)$ into $(0,\infty)$ such that

$$\int_0^\infty \frac{u^2(t)}{t+1}\,dt = a. \tag{11}$$

The map $y \to a - \int_y^\infty \frac{u^2(t)}{t+1}\,dt$ being stricly increasing, we can define a function $\xi\ :\ [0,a) \to [0,\infty)$ by

$$\forall s \in [0,a) \qquad a - \int_{\xi(s)}^\infty \frac{u^2(t)}{t+1}dt = s. \tag{12}$$

Then if we set

$$h_0(s) = \frac{\sqrt{\xi'(s)}}{\xi(s)+1} \tag{13}$$

it holds $\int_0^a h_0^2(t)\,dt = 1$ and $\int_s^a h_0^2(t)\,dt = \frac{1}{\xi(s)+1}$ hence

$$\xi(s) = \frac{\int_0^s h_0^2(u)\,du}{\int_s^a h_0^2(u)\,du}.$$

But the process

$$Y_t = \int_0^t \frac{h_0(s)}{\int_s^a h_0^2(u)\,du} dM_s$$

is a continuous martingale with bracket

$$< Y, Y >_t = \frac{\int_0^t h_0^2(u)\,du}{\int_t^a h_0^2(u)\,du}.$$

Therefore there exists a Brownian motion (W_t) such that $Y_t = W_{\xi(t)}$ and the integral to be studied can be written

$$\begin{aligned} I &= \int_0^a |h_0(t)|\,|W_{\xi(t)}|\,dt \\ &= \int_0^\infty \frac{|h_0(\xi^{-1}(s))|}{\xi'(\xi^{-1}(s))}\,|W_s|\,ds. \end{aligned}$$

Since by (12) it holds $\xi'(\xi^{-1}(s)) = \frac{s+1}{u^2(s)}$ one gets with (13)

$$I = \int_0^\infty \frac{u(s)}{(s+1)^{3/2}}\,|W_s|\,ds. \tag{14}$$

Hence it is enough to find a function $u > 0$ satisfying (11) and such that in (14) one gets $+\infty$. For this we use the following version of a lemma of Jeulin [?]:

Lemma 6 . *Let R_t be a positive measurable real process on a probability space $(\Omega, \mathbb{P})$ such that*

1) the law ν of R_t does not depend on t
2) $\nu(\{0\}) = 0$
3) $\int x\, d\nu(x) < +\infty$

then for any positive Radon measure μ on $\mathbb{R}_+$

$$\begin{array}{lll} i) & \int_0^\infty d\mu(t) < +\infty \;\Rightarrow & \int_0^\infty R_t\, d\mu(t) \in L^1(\mathbb{P}) \\ ii) & \int_0^\infty d\mu(t) = +\infty \;\Rightarrow & \int_0^\infty R_t\, d\mu(t) = +\infty \quad \mathbb{P}\text{a.s.} \end{array}$$

Proof. The point *i)* is clear because

$$\mathbb{E}R_t = \int x\, d\nu(x) < +\infty.$$

For the second point let $n \in \mathbb{N}$ and $J_n = \{\int_0^\infty R_t\, d\mu(t) \le n\}$. Suppose $\mathbb{P}(J_n) > 0$, then

$$\begin{aligned} \mathbb{E}[1_{J_n} R_t] &= \int_0^\infty du\, \mathbb{E}[1_{J_n} 1_{\{R_t > u\}}] = \int_0^\infty du \mathbb{E}[(1_{J_n} - 1_{\{R_t \le u\}})^+] \\ &\ge \int_0^\infty du (\mathbb{P}(J_n) - \nu([0,u]))^+ \end{aligned}$$

and by the hypothesis 2) $\lim_{u\to 0} \nu([0,u]) = 0$, hence the last integral is equal to $a_n > 0$. By integration

$$n\mathbb{P}(J_n) \ge a_n \int_0^\infty d\mu(t)$$

what gives *ii)* by contraposition. □

It follows by taking $R_t = \frac{|W_t|}{\sqrt{t}}$ that $I = +\infty$ as soon as

$$\int_0^\infty \frac{u(s)\sqrt{s}}{(s+1)^{3/2}}\, ds = +\infty \tag{15}$$

There are several functions satifying (11) and (15), for example $u(s) = 1/(\frac{1}{a} + \log(s + 1))$, which gives

$$h_0(t) = \frac{e^{\frac{1}{2a} - \frac{1}{2(a-t)}}}{a - t} 1_{[0,a)}(t).$$

Let us **summarize** the preceding discussion. Let h_0 associated with u by (12) and (13) and let ν be the distribution on the Wiener space associated with h_0 by (3) and (4). ν is a distribution of Watanabe in $\mathbb{D}_{2,-1}$ and is also a positive measure which does not charge polar sets.

For $t < a$, on the σ-field $\mathcal{F}_t^0$ the measures m and ν are equivalent, $(B_s)_{s \le t}$ is an $(\mathcal{F}_t^0)$-Brownian motion under m and an $(\mathcal{F}_t^0)$-semimartingale under ν.

For $t \ge a$, the measures m and ν are mutually singular on the σ-field $\mathcal{F}_t^0$, $(B_s)_{s \le t}$ is not an $(\mathcal{F}_t^0)$-semimartingale under ν, nevertheless the process

$$B_s - \int_0^a \left[\frac{h_0(v)}{\int_v^a h_0^2(u) du} \int_0^v h_0(u)\, dB_u \right] dv$$

is an $(\mathcal{F}_s^0)$-Brownian motion under ν.

It is possible to build examples of measures which do not charge polar sets and for which the singularity which is here at the point a, appears along a whole interval. Such measures are solutions in sense of distributions of Watanabe of the stochastic differential equation which defines the exponential of Doléans. This will be published elsewhere.

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