# THE ALGEBRA OF DIRICHLET STRUCTURES

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## I. Some notations and definitions

### I.1. Dirichlet Structures

$$(\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$$

$(\Omega, \mathcal{F}, m)$ : measured space with $m$ $\sigma$-finite and positive
$\mathcal{E}$ : Dirichlet form with domain $\mathbb{D}$
i.e. Quadratic positive form with dense domain $\mathbb{D}$ in $L^2(m)$ which is

. closed : $\mathbb{D}$ is an Hilbert space under the norm

$$\|f\|_{\mathbb{D}} = [\ \|f\|^2_{L^2(m)} + \mathcal{E}(f,f)\ ]^{1/2}$$

. and s.t. $f \in \mathbb{D} \Longrightarrow (f \wedge 1) \in \mathbb{D}$ and $\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f,f)$.

Notations for different hypotheses:
(**P**) (Probability) $m(\Omega) = 1$
(**M**) (Markovianity) $1 \in \mathbb{D}$ and $\mathcal{E}(1,1) = 0$
(**Γ**) (Existence of a Carré du Champ Operator):

$$\forall f \in \mathbb{D} \cap L^\infty, \exists \tilde{f} \in L^1, \forall h \in \mathbb{D} \cap L^\infty,$$

$$2\mathcal{E}(fh, f) - \mathcal{E}(h, f^2) = \int \tilde{f} h \, dm$$

(**L**) (Locality)

$$\forall f \in \mathbb{D}, \forall F, G \in \mathcal{D}(\mathbb{R})$$

$$\mathrm{supp}F \cap \mathrm{supp}G = \emptyset \implies \mathcal{E}(F(f) - F(0), G(f) - G(0)) = 0$$

(**W**) (Wiener space)

$$\Omega = \{\omega \in C(\mathbb{R}_+, \mathbb{R}^d), \omega(0) = 0\}$$

$\mathcal{F}$ = borelian $\sigma$-field of $\Omega$ with compact convergence
$m$ = Wiener measure
$(\mathcal{E}, \mathbb{D})$ = form associated with the Ornstein-Uhlenbeck semi-group.

## I.2. Basic properties

There is a sub-Markov semigroup associated with a Dirichlet structure.

**Theorem 1** . *Let a Dirichlet structure* $(\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ *be given. There exists a strongly continuous contraction semi-group* $(P_t)_{t\geq 0}$ *symmetric on* $L^2(m)$ *and unique such that*

$$(\star)\begin{cases} \mathbb{D} = \{f \in L^2(m) \ : \ \lim_{t\to 0}(\frac{f-P_t f}{t}, f)_{L^2(m)}\text{exists}\} \\ \forall f \in \mathbb{D} \qquad \mathcal{E}(f,f) = \lim_{t\to 0}(\frac{f-P_t f}{t}, f)_{L^2(m)} \end{cases}$$

*this semi-group is sub-Markov.*
*Conversely, if* $(P_t)$ *is a symmetric strongly continuous contraction semi-group on* $L^2(m)$*, and sub-Markov, the positive quadratic form associated with* $(P_t)$ *by* $(\star)$ *is a Dirichlet form.*

**Definition 2** . *A function* $F$ *from* $\mathbb{R}^n$ *into* $\mathbb{R}$ *is a contraction [resp. a normal contraction] if*

$$\forall x, y \qquad |F(x) - F(y)| \leq \sum_{i=1}^{n} |x_i - y_i|$$

*[resp. and* $F(0) = 0$*]*

**Theorem 3** . $\forall f \in \mathbb{D}$*, if* $F$ *is a normal contraction from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *then*

$$F \circ f \in \mathbb{D} \ \text{ and } \ (\mathcal{E}(F\circ f, F\circ f))^{1/2} \leq \sum_{i=1}^{n}(\mathcal{E}(f_i, f_i))^{1/2}.$$

*Under (***P***)(***M***) the word* normal *can be cancelled.*

The hypothesis $(\mathbf{\Gamma})$ gives rise to a carré du champ operator:

**Theorem 4** . *Under* $(\mathbf{\Gamma})$ *there exists a unique continuous symmetric positive bilinear map from* $\mathbb{D}\times\mathbb{D}$ *into* $L^1(m)$*, denoted by* $\Gamma$ *such that*

$$\forall f, g, h \in \mathbb{D} \cap L^\infty$$

$$\mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(h, fg) = \int h\, \Gamma(f,g)\, dm$$

$\Gamma$ *is the Carré du Champ Operator (CCO) associated with* $\mathcal{E}$*, if* $F$ *is a normal contraction from* $\mathbb{R}$ *to* $\mathbb{R}$

$$\forall f \in \mathbb{D} \qquad \Gamma(F\circ f, F\circ f) \leq \Gamma(f,f) \qquad m-\text{a.e.}$$

## I.3. About hypothesis $(\mathbf{\Gamma})$

Equivalent hypotheses:

**Theorem 5** .

*a) Let* $P_t^{(1)}$ *be the extension of* $P_t\Big|_{L^1\cap L^\infty}$ *to* $L^1(m)$. $(P_t^{(1)})_{t\geq 0}$ *is a strongly continuous contraction semi-group in* $L^1(m)$ *with generator* $(A^{(1)},\mathcal{D}A^{(1)})$. *It is the smallest closed extension of the restriction of the generator* $A$ *of* $P_t$ *to the set* $\{f\in\mathcal{D}A\cap L^1 :\ Af\in L^1\}$

*b)* $(\mathbf{\Gamma})\iff(\mathbf{\Gamma}')\iff(\mathbf{\Gamma}'')$

$(\mathbf{\Gamma}')$ $\forall f\in\mathcal{D}A\ \ f^2\in\mathcal{D}A^{(1)}$

$(\mathbf{\Gamma}'')$ *There is a sub-space* $H$ *of* $\mathcal{D}A$*, dense in* $\mathbb{D}$ *such that* $\forall f\in H\ f^2\in\mathcal{D}A^{(1)}$

*c) Under* $(\mathbf{\Gamma})$ *it holds*

$$\forall f,g\in\mathcal{D}A \qquad \Gamma(f,g)=A^{(1)}(fg)-fA(g)-gA(f) \qquad m-\text{a.e.}.$$

About the relationship between hypothesis $(\mathbf{\Gamma})$ and the existence of a C.C.O. for Markov processes, we have:

**Theorem 6** . *Suppose* $\Omega$ *be l.c.d.,* $\mathcal{F}$ *its borelian* $\sigma$*-field. Let* $(Q_t)$ *be a Feller semi-group which is symmetric on* $C_c(\Omega)$ *with respect to a Radon positive measure* $m$*, and* $(P_t)$ *the symmetric associated semi-group on* $L^2(m)$.

*1) If* $(Q_t)$ *has a C.C.O. in the sense of Meyer, then the Dirichlet structure associated to* $(P_t)$ *satisfies* $(\mathbf{\Gamma})$.

*2) Conversely, if the Dirichlet structure associated to* $(P_t)$ *satisfies* $(\mathbf{\Gamma})$ *and if the sets of zero potential are* $m$*-negligible, then* $(Q_t)$ *has a C.C.O. in the sense of Meyer.*

## I.4. The locality hypothesis, the functional calculi and the absolute continuity criterion for image measures

**Theorem 7** . $(\mathbf{L}) \Longleftrightarrow (\mathbf{L}') \Longleftrightarrow (\mathbf{L}'')$
$(\mathbf{L}')$ $\mathcal{E}(|f+1|-1, |f+1|-1) = \mathcal{E}(f,f)$
$(\mathbf{L}'')$ $\forall f,g \in \mathbb{D}, \forall a \in \mathbb{R} \qquad (f+a)g = 0 \Longrightarrow \mathcal{E}(f,g) = 0$
*and under* $(\mathbf{P})(\mathbf{M})$ *it is enough to take* $a = 0$

**Theorem 8** . *Suppose* $(\mathbf{\Gamma})(\mathbf{L})$ *:*
*a)* $\forall f \in \mathbb{D} \qquad \mathcal{E}(f,f) = \frac{1}{2}\int \Gamma(f,f)\, dm$
*b)* $\forall f \in \mathbb{D}^m, g \in \mathbb{D}^n, \quad \forall F, G$ *Lipschitz* $C^1$*-maps from* $\mathbb{R}^m [\mathbb{R}^n]$ *into* $\mathbb{R}$ :

$$\Gamma(F(f) - F(0), G(g) - G(0)) = \sum_{i=1}^{m} \sum_{j=1}^{n} F'_i(f) G'_j(g) \Gamma(f_i, g_j) \quad m - \text{a.e.}$$

There is a stronger result in one dimension : the Lipschitz functional calculus:

**Theorem 9** . *Suppose* $(\mathbf{\Gamma})$ $(\mathbf{L})$
*a)* $\forall f \in \mathbb{D} \quad f_\star(\Gamma(f,f).m) << \lambda_1 \quad$ *(*$\lambda_1$ = Lebesgue measure on $\mathbb{R}$*)*
*b) Let be* $f, g \in \mathbb{D}$ *and* $F, G$ *Lipschitz map from* $\mathbb{R}$ *to* $\mathbb{R}$ *and let* $F'$, $G'$ *be versions of their derivatives :*

$$\Gamma(F(f) - F(0), G(g) - G(0)) = F'(f)\, G'\, \Gamma(f,g) \qquad m - \text{a.e.}$$

There is also a criterion of absolute continuity of image laws in the multivariate case:

**Theorem 10** . *Suppose* $(\mathbf{\Gamma})$ $(\mathbf{L})$*, if* $f \in \mathbb{D}^n$ *and if* $\forall 1 \leq i,j \leq n$ $\Gamma(f_i, f_j) \in \mathbb{D}$ *then*
$f_\star[\det \Gamma(f, f^*).m] \ << \ \lambda_n \qquad$ *(*$\lambda_n$ = Lebesgue measure on $\mathbb{R}^n$*)*

**Theorem 11** *Suppose* $(\mathbf{W})$
*if* $f \in \mathbb{D}^n$

$$f_\star[\det \Gamma(f, f^*).m] \ << \ \lambda_n$$

This result can be extended to $\mathbb{D}_{loc}$ with a suitable definition.

# II. Image structures

## II.A. Finite dimensional images.

### II.A.1. Definition and basic properties.

**Proposition 12** . *Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a Dirichlet structure satisfying* $(\mathbf{P})$*, and* $1 \in \mathbb{D}$ *.*

*For $U \in \mathbb{D}^d$, let us define*

$$\begin{aligned} \widetilde{\mathbb{D}_U} &= \{f \in L^2(U_* m) : f \circ U \in \mathbb{D}\} \\ \widetilde{\mathcal{E}_U}(f,f) &= \mathcal{E}(f \circ U, f \circ U) \end{aligned}$$

*then $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_* m, \widetilde{\mathcal{E}_U}, \widetilde{\mathbb{D}_U})$ is a Dirichlet structure and the set $\mathcal{L}_d$ of Lipschitz functions from $\mathbb{R}^d$ into $\mathbb{R}$ is in $\widetilde{\mathbb{D}_U}$.*

*Let $\mathbb{D}_U$ be the closure of $\mathcal{L}_d$ in $\widetilde{\mathbb{D}_U}$ and $\mathcal{E}_U = \widetilde{\mathcal{E}_U}\Big|_{\mathbb{D}_U \times \mathbb{D}_U}$*

*then $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_* m, \mathcal{E}_U, \mathbb{D}_U)$ is a* regular *Dirichlet structure (satisfying again* $(\mathbf{P})$*, and* $1 \in \mathbb{D}_U$*).*

**Definition 13** . *The structure $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_* m, \mathcal{E}_U, \mathbb{D}_U)$ will be called the image structure of $S$ and will be denoted by $U_* S$.*

**Notations**. For $\phi \in L^1(m)$ we set

$$\mathbb{E}_m[\phi | U = x] := \frac{dU_*(\phi . m)}{dU_* m}(x) \qquad U_* m\text{-a.e.}$$

then we have

**Proposition 14** .

*1) If $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ possesses a C.C.O. $\Gamma$, the same holds for $\widetilde{U_* S}$ and $U_* S$ and their C.C.O. is given by*

$$\Gamma_U(f,f)(x) = \mathbb{E}_m[\Gamma(f \circ U, f \circ U) | U = x] \qquad \forall f \in \widetilde{\mathbb{D}_U}$$

*2) If $S$ is local, so are $\widetilde{U_* S}$ and $U_* S$ and if $S$ satisfies both* $(\mathbf{L})$ *and* $(\mathbf{\Gamma})$*, $\forall f \in \mathcal{L}_\lceil \cap \mathcal{C}^1(\mathbb{R}^d)$ it holds*

$$\Gamma_U(f,f)(x) = \sum_{i,j}^{d} \mathbb{E}_m[\Gamma(U_i, U_j) | U = x] \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x)$$

**Remark**. There are explicit examples in which

$$\widetilde{U_*S} \neq U_*S.$$

## II.A.2 The Energy Image Density Property.

**Definition 15** . *A Dirichlet structure* $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ *satisfying* $(\mathbf{P})$, $1 \in \mathbb{D}$, $(\mathbf{\Gamma})$, $(\mathbf{L})$ *is said to satisfy the* $(\mathbf{EID})$ *property if*
$\forall n \in \mathbb{N}^*, \forall F \in \mathbb{D}^n$,

$$F_* \left(\det[\Gamma(F, F^*)].m\right) << \lambda_n.$$

A natural question is whether the $(\mathbf{EID})$ property is preserved by image.

**Proposition 16** . *Let* $S$ *satisfying* $(\mathbf{P})$, $1 \in \mathbb{D}$, $(\mathbf{\Gamma})$, $(\mathbf{L})$ *and* $(\mathbf{EID})$. *Let* $U \in \mathbb{D}^d$ *such that one of the following hypotheses holds:*

*a) the matrix* $\Gamma(U, U^*)$ *is* $\sigma(U)$*-measurable*

*b)* $\det[\Gamma(U, U^*)] > 0 \quad m$*-a.e.*

*then the image structure* $U_*S$ *satisfies* $(\mathbf{P})$, $1 \in \mathbb{D}$, $(\mathbf{\Gamma})$, $(\mathbf{L})$ *and* $(\mathbf{EID})$.

With hypothesis a) the proof comes straightforward from the definitions. With hypothesis b) the result is a consequence of the following two lemmas:

**Lemma 17** . *Let* $M$ *be a random matrix which is symmetric and non-negative definite, then*

$$\{\det[\mathbb{E}(M|\mathcal{F})] = 0\} \quad \subset \quad \{\det[M] = 0\}$$

**Lemma 18** . *If* $\det[\Gamma(U, U^*)] > 0$, *for all* $\phi \in (\mathbb{D}_U)^n$ *there exists an* $n \times d$*-matrix* $J$ *which is* $\sigma(U)$*-measurable (up to* $m$*-negligible sets) such that*

$$\Gamma(\phi \circ U, \phi \circ U^*) = J \Gamma(U, U^*) J^* \qquad m - \text{a.e.}$$

II.A.2 The Image Generator.

There is in general no simple relationship between the initial semi-group and the semi-group of the image structure. Not better for the associated Markov process.

Nevertheless, the generator $(A_U, DA_U)$ of the image structure can be put in relation with the generator $(A, DA)$ of the initial structure:

If $f, g \in \mathbb{D}_U$, and $f \circ U \in DA$, we have

$$\mathcal{E}_U(f,g) = \mathcal{E}(f \circ U, g \circ U) = -(A(f \circ U), g \circ U)_{L^2(m)}$$

$$= -\int \mathbb{E}_m[A(f \circ U)|U = x] g(x)\, dU_* m(x)$$

hence $f \in DA_U$ and $A_U f = \mathbb{E}_m[A(f \circ U)|U = x]$.

But, hypotheses are needed to ensure the space $\mathbb{D}_U \cap \{f : f \circ U \in DA\}$ contains other functions than constants:

**Proposition 19** . *Suppose $S$ satisfies* (**P**), $1 \in \mathbb{D}$, $(\mathbf{\Gamma})$, (**L**). *Let $U \in (DA)^d$ such that $\Gamma(U_i, U_j) \in L^2(m)$ $\forall i, j = 1, \ldots, d$ then*

$$DA_U \supset \{f \in \mathcal{C}^2(\mathbb{R}^d) : \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ bounded}\}$$

*and for such an $f$*

$$\begin{aligned} A_U f(x) &= \mathbb{E}_m[A(f \circ U)|U = x] \\ &= \tfrac{1}{2} \textstyle\sum_{i,j} \alpha_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i \beta_i \frac{\partial f}{\partial x_i} \end{aligned}$$

*with* $\alpha_{ij}(x) = \mathbb{E}[\Gamma(U_i, U_j)|U = x] \quad (\in L^2(U_* m))$
*and* $\beta_i(x) = \mathbb{E}[AU_i|U = x] \quad (\in L^2(U_* m))$

If further, $\det[\Gamma(U, U^*)] > 0$ $m$-a.e. then the function $k = \frac{dU_* m}{d\lambda_n}$ satifies

$$2\beta_i k - \sum_j \frac{\partial}{\partial x_j}(\alpha_{ij} k) = 0 \qquad \forall i = 1, \ldots, n$$

in the sense of distributions.

H. Airault and P. Malliavin have studied the case of Wiener space with

$$U \in W_\infty = \cap_{p,n} D_{p,n}$$

and

$$[\det\Gamma(U, U^*)]^{1/2} \in W_\infty$$

and they show in this case that

$$A_U = \Delta + \nabla_{\vec{u}}$$

where $\Delta$ is the Laplace-Beltrami operator associated to the Riemannian metric with matrix $[(\alpha_{ij})]^{-1}$ and where $\vec{u} = \frac{1}{2}\,\overrightarrow{\text{grad}}\,\log\rho$ with $\rho = \frac{dU_* m}{dv}$ and where $dv = \sqrt{\det(\alpha_{ij})}.\lambda_n$ is the associated area measure.

## II.B. General Images.

II.B.1. U does not need to be supposed in $\mathbb{D}$ or $\mathbb{D}^d$ for defining an image structure, whenever there is enough functions $f \circ U$ in $\mathbb{D}$.

Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ satisfy (**P**) and $1 \in \mathbb{D}$, let $(X, \mathcal{G})$ be a measurable space and let U be a measurable map from $(\Omega, \mathcal{F})$ into $(X, \mathcal{G})$. Let us suppose that there exists a set $\mathcal{A}$ of measurable applications from $X$ into $\mathbb{R}$ such that

. $\mathcal{A}$ is a vector space containing the constants

. $\forall f \in \mathcal{A}, \quad f \circ U \in \mathbb{D}$

. $\mathcal{A}$ is dense in $L^2(U_* m)$

then the form $(\mathcal{E}_{\mathcal{A}}, \mathcal{A})$ defined by $\mathcal{E}_{\mathcal{A}}(f, f) = \mathcal{E}(f \circ U, f \circ U)$ is closable in $L^2(U_* m)$, let $(\mathcal{E}_U, \mathbb{D}_U)$ its closure, we put

$$U_* S = (X, \mathcal{G}, U_* m, \mathcal{E}_U, \mathbb{D}_U).$$

II.B.2. Example.

Let $S = (\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \nu^{\otimes \mathbb{N}}, \mathbb{D}, \mathcal{E})$ be a Dirichlet structure such that

. $\nu = N(0,1)$ is the standard Gaussian measure on $\mathbb{R}$

. $(\mathbb{D}, \mathcal{E})$ is any Dirichlet form on $L^2(\nu^{\otimes \mathbb{N}})$ such that the coordinates $\chi_n$ belong to $\mathbb{D}$.

Let be $X = \mathcal{C}[0,1]$ and $\mathcal{G}$ be its borelian $\sigma$-field. Let $(\dot{h}_n)$ be a C.O.N.S. of $L^2([0,1])$ and put

$$h_n(t) = \int_0^t \dot{h}_n(s)\, ds$$

Let us consider the map $U$ from $\mathbb{R}^{\mathbb{N}}$ into $X$ defined by

$$(\star) \qquad U(x) = \sum_{n=0}^{\infty} \chi_n(x) h_n \text{ if the serie converges in } \mathcal{C}[0,1],$$

$$U(x) = 0 \qquad \text{elsewhere} \qquad .$$

A vector valued martingale argument shows that

**Lemma 20** . *The serie $(\star)$ converges almost surely and in $L^p_{\mathcal{C}[0,1]}(\nu^{\otimes \mathbb{N}})$ $(1<p<\infty)$, and the law of its sum is the Wiener measure $\mu$ : $\mu = U_*(\nu^{\otimes \mathbb{N}})$.*

Let us denote by $(B_t)$ the brownian motion defined by this Wiener measure on $\mathcal{C}[0,1]$, and let be

$$\widetilde{h_n} = \int_0^1 \dot{h}_n(s)\, dB_s$$

then it can be shown that

$$\widetilde{h_n} \circ U(x) = \chi_n(x) \quad \text{ for } \; \nu^{\otimes \mathbb{N}} - \text{a.e.} x.$$

Hence by hypothesis $\widetilde{h_n} \circ U \in \mathbb{D}$, therefore if $f = F(\widetilde{h_1}, \ldots, \widetilde{h_n})$ with $F$ Lipschitz, we have $f \circ U \in \mathbb{D}$. But $F(\chi_1, \ldots, \chi_n)$ is dense in $L^2(\nu^{\otimes \mathbb{N}})$ hence $F(\widetilde{h_1}, \ldots, \widetilde{h_n})$ is dense in $L^2(\mu)$.

So, the image structure

$$(X, \mathcal{G}, \mu, \mathcal{E}_U, \mathbb{D}_U)$$

is well defined and contains $\{F(\widetilde{h_1}, \ldots, \widetilde{h_n})\}$ for Lipschitz $F$.

# III. Tensor products and projective limits.

## III.A. Finite products.

Let $S_1 = (\Omega_1, \mathcal{F}_1, m_1, \mathcal{E}_1, \mathbb{D}_1)$ and $S_2 = (\Omega_2, \mathcal{F}_2, m_2, \mathcal{E}_2, \mathbb{D}_2)$ be Dirichlet structures.

**Definition 21** .

$$S_1 \otimes S_2 = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, m_1 \times m_2, \mathcal{E}, \mathbb{D})$$

*with*

$$\begin{aligned} \mathbb{D} \;=\; \{f \in L^2(m_1 \times m_2) : \quad & \text{for } m_2\text{−a.e. } y \quad f(.,y) \in \mathbb{D}_1 \\ & \text{for } m_1\text{−a.e. } x \quad f(x,.) \in \mathbb{D}_2 \end{aligned}$$

$$\text{and} \int \mathcal{E}_1(f(.,y), f(.,y))\, dm_1(y) + \int \mathcal{E}_2(f(x,.), f(x,.))\, dm_2(x) < \infty\}$$

*and* $\forall f \in \mathbb{D}$,

$$\mathcal{E}(f,f) = \int \mathcal{E}_1(f(.,y), f(.,y))\, dm_1(y) + \int \mathcal{E}_2(f(x,.), f(x,.))\, dm_2(x).$$

It is indeed easy to see that this form is closed and that contractions operate.

If $S_1$ and $S_2$ satisfy (**P**) and (**M**) the same holds for $S_1 \otimes S_2$.

If $S_1$ and $S_2$ are local, $S_1 \otimes S_2$ is local.

If $S_1$ and $S_2$ satisfy $(\mathbf{\Gamma})$, the same holds for $S_1 \otimes S_2$ and its OCC is given by

$$\Gamma(f,f) = \Gamma_1(f(.,y), f(.,y))(x) + \Gamma_2(f(x,.), f(x,.))(y)$$

Concerning the associated semi-group we have the following :

Let $(P_t^1), (P_t^2)$ be the semi-groups associated with $S_1$ and $S_2$, and let $\widehat{P_t^1}$ and $\widehat{P_t^2}$ be the semi-groups on $L^2(m_1 \times m_2)$ defined by

$$\begin{aligned} \widehat{P_t^1} f(x,y) &= P_t^1(f(.,y))(x) \\ \widehat{P_t^2} f(x,y) &= P_t^2 f(x,.))(y) \end{aligned}$$

which are symmetric, strongly continuous and sub-Markov.

**Proposition 22** . *a) The semi-group associated with* $S_1 \otimes S_2$ *is* $P_t = \widehat{P_t^1}\widehat{P_t^2} = \widehat{P_t^2}\widehat{P_t^1}$

*b) its generator is the smallest closed extension of the operator defined on* $DA_1 \otimes DA_2$ *by* $A(\phi \otimes \psi) = A_1\phi \otimes \psi + \phi \otimes A_2\psi$

*c)* $\mathbb{D}_1 \otimes \mathbb{D}_2$ *is dense in* $\mathbb{D}$.

III.B. Infinite tensor products.

The preceding construction extends without any problem to the infinite tensor products (countable or not) :

$$\bigotimes_{i\in I}(E_i,\mathcal{F}_i,\mu_i,\mathcal{E}_i,\mathbb{D}_i)$$

where the factors are supposed to satisfy (**P**).

That comes mainly from the fact that the limit of an increasing sequence of Dirichlet forms is a Dirichlet form:

**Lemma 23** . *Let $(\Omega,\mathcal{F},m)$ be a measured space equipped with Dirichlet forms $\mathcal{E}^{(n)},\mathbb{D}^{(n)}$ such that*

. $\mathbb{D}^{(n)}\downarrow$ *as* $n\uparrow$

. $\mathcal{E}^{(n)}\uparrow$ *as* $n\uparrow$: $\forall f\in\mathbb{D}^{(n)}\quad \mathcal{E}^{(n+1)}(f,f)\geq\mathcal{E}^{(n)}(f,f)$

*then* $\mathbb{D}=\bigcap\mathbb{D}^{(n)},\quad \mathcal{E}(f,f)=\lim\mathcal{E}^{(n)}(f,f)$ *is a Dirichlet form.*

If the $S_i$'s are local, so is $\otimes_i S_i$.

if each $S_i$ possesses a CCO, the same holds for $\otimes_i S_i$.

Suppose now that the family $S_i$ is countable and that each finite product

$$\bigotimes_{i=0}^{n} S_i$$

satisfies the (**EID**) property, then $S=\otimes_{i=0}^{\infty}S_i$ satifies (**EID**).

That comes directly from the definitions.

As an example let us take

$$S_i=(\mathbb{R},\mathcal{B}(\mathbb{R}),h_i(x)dx,\int\nabla^2\ \ h_i(x)dx,\mathbb{D}_i)$$

where $h_i$ satisfies the Hamza condition and $\int h_i(x)dx=1$.

Then by the coarea formula of Federer, the finite products satisfy (**EID**) and therefore the infinite product structure (which is in general non Gaussian) satisfies (**EID**).

**Remark**. In this example, putting

$$\mu^i=\bigotimes_{\substack{j\in\mathbb{N}\\ j\neq i}}(h_i dx)$$

and

$$\mathcal{E}_i(f,f) = \int_{\mathbb{R}} (\nabla f)^2 h_i dx$$

we have

$$\mathbb{D} = \{f \in L^2(m) : \forall i \in \mathbb{N} \quad f(.,y) \in \mathbb{D}_i \text{ for } \mu^i - \text{a.e. } y$$

$$\text{and} \sum_{i=0}^{\infty} \int \mathcal{E}_i(f(.,y), f(.,y)) d\mu^i(y) < \infty\}$$

and for $f \in \mathbb{D}$

$$\Gamma(f,f) = \sum_{i=0}^{\infty} \Gamma_i(f,f) = \sum_{i=0}^{\infty} f_i'^2$$

where $\Gamma_i$ acts only on the $i$-th variable.

Therefore if for $f \in \mathbb{D}$ we put

$$Df = (f_i')_{i \in \mathbb{N}}$$

this defines a continuous operator from $\mathbb{D}$ into $L^2(m, \ell^2)$ and we have

$$\Gamma(f,f) = < Df, Df >_{\ell^2}$$

We shall see later that this allows to develop a conditional Dirichlet calculus. These product structures are examples of Classical Dirichlet forms in the sense of Albeverio-Röckner.

## III.C. Projective limits.

**1**. General projective system of Dirichlet structures can be defined in an obvious way.

But there is a difficulty for passing to the limit unless some uniform closability property is known (which is the case for products). Here is an example of projective system without limit :

Example

Let $\mu$ be the Gauss measure on $\mathbb{R}$. Let us consider the structures

$$S^{(n)} = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^{\otimes n}, \mathcal{E}^{(n)}, \mathbb{D}^n)$$

with

$$\mathbb{D}^{(n)} = \{f \in L^2(\mu^{\otimes n}) : \frac{\partial f}{\partial x_i} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int (\sum_{i=0}^{n} \frac{1}{a_i}\frac{\partial f}{\partial x_i})^2 \, d\mu^{\otimes n} < \infty\}$$

and

$$\mathcal{E}^{(n)}(f,f) = \int (\sum_{i=0}^{n} \frac{1}{a_i}\frac{\partial f}{\partial x_i})^2 \, d\mu^{\otimes n}$$

where the numbers $a_i$ are chosen such that

$$a_i > 0, \qquad \lim_{i\to\infty} a_i = 0$$

The $S^{(n)}$'s define a compatible system of Dirichlet structures, but if $h_n(x) = a_n x_n$ we have $\|h_n\|_{L^2(\mu^{\otimes \mathbb{N}})} \to 0$ and the candidate $\tilde{\mathcal{E}}$ satisfies

$$\tilde{\mathcal{E}}(h_n - h_m, h_n - h_m) = \int (\frac{1}{a_n}a_n - \frac{1}{a_m}a_m)^2 \, d\mu^{\otimes \mathbb{N}} = 0$$

and $\tilde{\mathcal{E}}(h_n, h_n) = 1$ therefore $\tilde{\mathcal{E}}$ is not closable.

**2**. <u>An important special case where the limit exists</u>

Consider a Dirichlet structure $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ and a family $(U_n)$ of applications such that

$$U_n : \Omega \longrightarrow \mathbb{R}^d \quad \text{and } U_n \in \mathbb{D}^{d_n} \quad \forall n$$

then the image structure of $S$ by $(U_0, \ldots, U_n)$ defines a Dirichlet structure $S^{(n)}$ with state space

$$\prod_{i=0}^{n} \mathbb{R}^{d_i}$$

These structures $S^{(n)}$ define a projective system which always possesses a limit. This comes easily from the fact that the initial form $(\mathcal{E}, \mathbb{D})$ is closed.

The limit can be called <u>the image structure by the process</u> $(U_n)_{n\in\mathbb{N}}$.

The same would be true, mutatis mutandis, for uncountable families.

# IV. D-independence.

## IV.A. Definition and examples.

Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a Dirichlet structure satisfying $(\mathbf{P})$ and $1 \in \mathbb{D}$.

If $U \in \mathbb{D}^p$ the image structure $U_*S$ will be called the D-law of $U$.

**Definition 24** . *If $U \in \mathbb{D}^p$, $V \in \mathbb{D}^q$, $U$ and $V$ will be said to be D-independent if the D-law of $(U,V)$ is the product of the D-laws of $U$ and $V$.*

**Proposition 25** . *A necessary and sufficient condition for independent $U$ and $V$ to be D-independent is*

$$\begin{aligned}&\forall f_1,\ f_2 \in \mathcal{C}^1_c(\mathbb{R}^p),\ \forall g_1,\ g_2 \in \mathcal{C}^1_c(\mathbb{R}^q)\\ &\mathcal{E}(f_1 \circ U g_1 \circ V, f_2 \circ U g_2 \circ V)\\ &= \mathcal{E}(f_1 \circ U, f_2 \circ U)(g_1 \circ V, g_2 \circ V)_{L^2(m)} + \mathcal{E}(g_1 \circ V, g_2 \circ V)(f_1 \circ U, f_2 \circ U)_{L^2(m)}\end{aligned}$$

If E is local and possesses a CCO we have the more explicit result

**Proposition 26** . *If $S$ satisfies $(\mathbf{P})$, $1 \in \mathbb{D}$,$(\mathbf{L})$, $(\mathbf{\Gamma})$, for $U \in \mathbb{D}^p$ and $V \in \mathbb{D}^q$ to be D-independent it is necessary and sufficient that*

*1) $U$ and $V$ are independent,*

*2) $\forall i, k \quad \mathbb{E}[\Gamma(U_i, V_k)|U, V] = 0 \quad m$-a.e.*

*3) $\forall i, j \quad \mathbb{E}[\Gamma(U_i, U_j)|U, V] = \mathbb{E}[\Gamma(U_i, U_j)|U] \quad m$-a.e.*

*4) $\forall l, k \quad \mathbb{E}[\Gamma(V_k, V_l)|U, V] = \mathbb{E}[\Gamma(V_k, V_l)|V] \quad m$-a.e.*

Remark. These conditions are fulfilled as soon as

. $\Gamma(U_i, V_k) = 0$ for all $i, k$

. $(U, \Gamma(U_i, U_j))$ is independent of $V$ for all $i, j$

. $(V, \Gamma(V_k, V_l))$ is independent of $V$ for all $k, l$.

Examples.

1) If $U$ and $V$ are random variables in the first chaos on the Wiener space, they are D-independent as soon as they are independent i.e. orthogonal.

2) Let $f \in L^2_{sym}(\mathbb{R}^p_+)$ and $g \in L^2_{sym}(\mathbb{R}^q_+)$. By a result of Ustunel and Zakaï if the multiple Wiener integrals $I_p(f)$ and $I_q(g)$ are independent so are the $\sigma$-fields

$$\sigma(I_p(f), < DI_p(f), h_{1,1} >, \dots, < D^{p-1}I_p(f), h_{p-1,1} \otimes \dots \otimes h_{p-1,p-1} >)$$

and

$$\sigma(I_q(g), < DI_q(g), k_{1,1} >, \dots, < D^{q-1}I_q(g), k_{q-1,1} \otimes \dots \otimes k_{q-1,q-1} >)$$

Therefore $I_p(f)$ and $I_q(g)$ are D-independent as soon as they are independent and $\Gamma(I_p(f), I_q(g)) = 0$.

3) This extends to the case of multivariate random variables whose components are multiple Wiener integrals.

IV.B. Convergence in D-law.

Let as before $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$be a Dirichlet structure satisfying (**P**) and $1 \in \mathbb{D}$.

**Definition 27** . *Let $(U_n)$ be a sequence in $\mathbb{D}^d$ and $V \in \mathbb{D}^d$. The sequence $(U_n)$ is said to converge in D-law to $V$*

$$U_n \xrightarrow{D-L} V$$

*if*

*. $U_{n*}m$ converges to the law of $V$ in the narrow sense*
*. $\forall f \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d) \qquad \mathcal{E}(f \circ U_n, f \circ U_n) \longrightarrow \mathcal{E}(f \circ V, f \circ V)$*

in other words that means convergence of the D-laws on bounded continuous functions for the measures, on $\mathcal{C}^1$-Lipschitz functions for the forms.

The central limit theorem becomes the following :

**Theorem 28** . *Let us suppose $S$ satisfies* (**P**)*,* $1 \in \mathbb{D}$*,* (**L**)*,* (**Γ**)*. Let $(U_n)$ be a sequence of functions in $\mathbb{D}^d$ which are centered, with the same D-law, and D-independent, then*

$$V_n = \frac{1}{\sqrt{n}}(U_1 + \ldots + U_n)$$

*converges in D-law and the limit Dirichlet structure is*

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu, \hat{\mathcal{E}}, \hat{\mathbb{D}})$$

*with*

$$\nu = N_d(0, \Sigma)$$

$$\forall f \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d) \quad \hat{\mathcal{E}}(f,f) = \sum_{i,j} \int \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} a_{ij} \, d\nu$$

*where $\Sigma_{ij} = \int x_i x_j \, d\mu$ ($\mu$ being the common law of the $U_n$'s) and $a_{ij} = \mathcal{E}(U_{n,i}, U_{n,j}) = \mathcal{E}_{U_n}(x_i, x_j)$ (which does not depend on $n$).*

The main step of the proof is the following lemma

**Lemma 29** . *Let $U_1, \ldots, U_n$ be in $\mathbb{D}^d$ and D-independent then* $\forall f \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d)$

$$\mathcal{E}(f(U_1 + \ldots + U_n), f(U_1 + \ldots + U_n))$$

$$= \frac{1}{2} \sum_{ij} \int \frac{\partial f}{\partial x_i}(y_1 + \ldots + y_n) \frac{\partial f}{\partial x_j}(y_1 + \ldots + y_n) \left( \sum_{\ell=1}^{n} a_{ij}^{\ell}(y_\ell) \right) \; d\mu_1(y_1) \ldots d\mu_n(y_n)$$

*where* $\mu_n = U_{n*}m$ *is the law of* $U_n$ *and*
$a_{ij}^{\ell}(y_\ell) = \mathbb{E}[\Gamma(U_{\ell,i}, U_{\ell,j}) | U_\ell = y_\ell] \;\; (= \Gamma_{U_\ell}(x_i, x_j)(y_\ell)$.

Let $(Z^{(n)})_{n\in\mathbb{N}}$ be a sequence of discrete time processes

$$Z^{(n)} = (Z_1^{(n)}, \ldots, Z_k^{(n)}, \ldots)$$

defined on a Dirichlet structure $S$, we shall say that $(Z^{(n)})$ converges in D-law to the process $Z$, if the marginal D-laws of $Z^{(n)}$ converge to those of $Z$.
Example. Let us take $S = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, \int \nabla^2\, d\mu, H^1(\mathbb{R}, \mu))$ with $\mu = N(0,1)$ and let us consider the "standard Gaussian product space"

$$S^{\otimes\mathbb{N}} = (\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^n), \mu^{\otimes\mathbb{N}}, \mathcal{E}, \mathbb{D}).$$

Let $X_i$ be the coordinates and let us put

$$Y_k^{(n)} = \frac{X_k\sqrt{n}}{\sqrt{X_1^2 + \ldots + X_n^2}}$$

then the process

$$Z^{(n)} = (Y_1^{(n)}, \ldots, Y_n^{(n)}, 0, 0 \ldots)$$

converges in D-law toward

$$X = (X_1, \ldots, X_n, \ldots)$$

.

That is an extension of the Gateaux-Lévy theorem which states the same result with only probability structures.

# V.Dirichlet sub-spaces and conditioning.

## V.A. Dirichlet sub-spaces.

**Definition 30** . *Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a Dirichlet structure satisfying* $(\mathbf{P})$ *and* $1 \in \mathbb{D}$, *a sub-vector space* $\mathrm{d\!I}$ *of* $\mathbb{D}$ *will be called a Dirichlet sub-space if it is closed in* $\mathbb{D}$ *and stable under composition by Lipschitz functions on* $\mathbb{R}$.

**Proposition 31** . *If* $\mathrm{d\!I}$ *is a Dirichlet sub-space, it holds*

$$\overline{\mathrm{d\!I}}^{L^2(m)} = L^2(m, \sigma(\mathrm{d\!I}))$$

*and so* $S_{\mathrm{d\!I}} = (\Omega, \sigma(\mathrm{d\!I}), m, \mathcal{E}|_{\mathrm{d\!I}\times\mathrm{d\!I}}, \mathrm{d\!I})$ *is a Dirichlet structure.*

In particular, $\mathrm{d\!I}$ is stable by composition by Lipschitz functions of several variables.

For example if $X_i \in \mathbb{D}, \ \ \forall i \in I$, the space

$$\mathbb{D}(X_i, i \in I) = \overline{\{G(X_{i_1}, \dots, X_{i_n}) \ i_k \in I, \ G \in \mathcal{C}^1(\mathbb{R}^n)\}}^{\mathbb{D}}$$

is a Dirichlet sub-space which will be called the Dirichlet sub-space generated by the family $(X_i)_{i\in I}$.

Remark. If $S$ satisfies $(\mathbf{\Gamma})$, $S_{\mathrm{d\!I}}$ satisfies $(\mathbf{\Gamma})$ and its CCO is given by

$$\Gamma_{\mathrm{d\!I}}(v, v) = \mathbb{E}[\Gamma(v, v)|\sigma(\mathrm{d\!I})] \qquad \forall v \in \mathrm{d\!I}$$

Example. It is easily seen that the kernel of the form

$$K = \{f \in \mathbb{D} \ : \ \mathcal{E}(f, f) = 0\}$$

is a Dirichlet sub-space.

V.B. Conditional calculus.

We consider a D-structure $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ satisfying (**P**), (**M**), (**L**), (**Γ**).

Hypothesis (**G**). We shall say that $S$ admits a gradient if there exists a separable Hilbert space and a linear map $D$ from $\mathbb{D}$ into $L^2(m, H)$ such that

$$< Du, Du >_H = \Gamma(u,u) \qquad \forall u \in \mathbb{D}.$$

This is the case for the Wiener space, for some product spaces and some Classical Dirichlet space in the sense of Albeverio-Röckner.

From the functional calculus for $\Gamma$ we deduce

**Proposition 32** . *$D$ is continuous and satisfies*

*a)* $D(f \circ U) = f' \circ U.DU, \quad f \in \mathcal{L}(\mathbb{R}),\ U \in \mathbb{D}$

*b)* $D(F \circ \vec{U}) = \sum_i F' \circ \vec{U}.DU_i, \quad F \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d), \vec{U} \in \mathbb{D}^d$

Most of the features of the conditional calculus of Nualart-Zakaï extends to this situation :

Let $(X_i, i \in \mathbb{N})$ be a countable family in $\mathbb{D}$ nad let $\mathcal{H}$ be the following measurable field of sub-spaces of $H$

$$\mathcal{H} = (\mathcal{L}(DX_i, i \in \mathbb{N}))^{\perp}$$

For $F \in \mathbb{D}$, let us define

$$\begin{aligned} D^X(F) &= P^{\mathcal{H}}(DF) \\ \Gamma^X(F,F) &= < P^{\mathcal{H}}(DF), P^{\mathcal{H}}(DF) >_H \\ \mathcal{E}^X(F,F) &= \mathbb{E}[\Gamma^X(F,F)] \end{aligned}$$

**Proposition 33** . *a) $(D^X, \mathbb{D})$ is a closable operator in $L^2(m,H)$ iff the form $(\mathcal{E}^X, \mathbb{D})$ is closable.*

*b) This is the case if $P^{\mathcal{H}}h \in \mathbb{D}$ for all $h \in H$.*

*c) In this case the D-structure associated with $\mathcal{E}$ : $(\Omega, \mathcal{F}, m, \mathcal{E}^X, \mathbb{D}^X)$ satisfies* (**P**), (**M**), (**L**), (**Γ**) *and* (**G**) *with gradient operator $D^X$ and will be called the conditional structure knowing $X$.*

The main result in this theory is then :

**Theorem 34** . *Suppose the conditional structure knowing $X = (X_i, i \in \mathbb{N})$ exists. Let $\tau_X = \sigma(X_i, i \in \mathbb{N})$.*

*a) Let $F : \mathbb{R} \times \Omega \to \mathbb{R}$, $\mathcal{B}(\mathbb{R}) \otimes \tau_X$-measurable, s.t. $F(x,\omega)$ is Lipschitz in $x$, bounded as well as $F'_x$ then for all $U \in \mathbb{D}$ (even for $U \in \mathbb{D}^X$)*

$$(\omega \longrightarrow F(U(\omega), \omega)) \in \mathbb{D}^X$$

*and*

$$\Gamma^X(F(U(.),.), F(U(.),.))(\omega) = F_x'^2(U(\omega), \omega)\Gamma^X(U,U)(\omega) \quad m\text{−a.e.}$$

*b) For all $U \in \mathbb{D}$ (even for $U \in \mathbb{D}^X$), the image of the measure $\Gamma^X(U,U).m$ by the map $\omega \to (U(\omega), \omega)$from $(\Omega, \mathcal{F})$ into $(\mathbb{R}\times\Omega, \mathcal{B}(\mathbb{R})\otimes \tau_X)$ is absolutely continuous w.r. to $dx \times m$.*

*In particular if $\Gamma^X(U,U) > 0$ a.e., $U$ possesses a conditional law knowing the $\sigma$-field $\tau_X = \sigma(X_i, i \in \mathbb{N})$.*

There are two limitations for applying this theory in practice

1) The verification of the closability condition.

2) Most examples, especially from filtrage problems, do allow a direct treatment because the conditional law is absolutely continuous with respect to the Wiener measure: The ordinary criterion applies.

Example. Let $(\eta_t(w))_{t\in\mathbb{R}_+}$ and $(\xi_t(w))_{t\in\mathbb{R}_+}$ be two processes defined on a probability space $(W, \mathcal{A}, \mathbb{P})$.

If the law of $\xi$ knowing $\eta$ (i.e. knowing $\tau = \sigma(\eta_s, s \in \mathbb{R}_+))$ is absolutely continuous w.r. to the Wiener measure, then a sufficient condition for a random variable $F : \Omega \to \mathbb{R}^d$ of the form

$$F = f(\eta, \xi)$$

to possess a conditional density knowing $\tau$ is that for $\mathbb{P}$-a.e. $w$, setting $F_w(\omega) = f(\eta(w), \omega)$,

1) $F_w \in \mathbb{D}$ $(= D_{2,1}$ here)

2) $\det \Gamma(F_w, F_w^*)(\omega) > 0 \qquad dm(\omega)$-a.e..

V.C. A glance to stationary processes.

Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a D-structure satisfying $(\mathbf{P})$, $(\mathbf{M})$, $(\mathbf{L})$, $(\mathbf{\Gamma})$.

A map $X \ : \ t \to X_t$ from $\mathbb{R}$ into $\mathbb{D}$ will be called a D-stationary process if its marginal D-laws are invariant under translations of time.

Let $F \in \mathcal{L} \cap \mathcal{C}\mathbb{R}^n)$, the relation

$$\mathcal{T}_t[F(X_{t_1}, \dots, X_{t_n})] = F(X_{t_1+t}, \dots, X_{t_n+t})$$

defines a group of isometries which extends to the Dirchlet sub-space generated by $X \ : \ \mathbb{D}_X = D(X_t, t \in \mathbb{R})$

It is easy to see that $(\mathcal{T}_t)$ is strongly continuous on $\mathbb{D}_x$ if and only if $t \to X_t$ is continuous from $\mathbb{R}$ into $\mathbb{D}$.

If this is satisfied, we get a spectral representation : $\mathcal{T}_t = e^{itA}$, $A$ self-adjoint on $\mathbb{D}_X$ and if $E(d\lambda)$ is the resolution of the identity associated with $A$ :

$$X_t = \int e^{i\lambda t} E(d\lambda) X_0 \qquad \text{in } \mathbb{D}_X$$

Let $\Gamma_X$ be the CCO of the sub-structure $(\Omega, \sigma(X), m, \mathcal{E}_X, \mathbb{D}_X)$ which is given by

$$\Gamma_X(U, V) = \mathbb{E}[\Gamma(U, V) | \sigma(X)].$$

Let us suppose moreover that $\Gamma(X_s, X_t)$ be deterministic $\forall s, t$.

(This happens often without any Gaussian hypothesis : for example for product spaces

$$\bigotimes_{n=0}^{\infty} (\mathbb{R}, \mathcal{B}(\mathbb{R}), h_n dx, \int \nabla^2 \, h_n dx, \mathbb{D}_n)$$

the $h_n$'s satisfying the Hamza condition and $\int x^2 h_n(x) dx < \infty$ if for all $t$, $X_t$ belongs to the closure in $\mathbb{D}$ of linear combinations of coordinates.)

Then $\Gamma(X_s, X_t) = \Gamma(X_0, X_{t-s})$ hence by Bochner theorem

$$\Gamma(X_{t+h}, X_t) = \gamma(h) = \int e^{i\lambda h} d\mu(\lambda)$$

for a finite positive measure $\mu$ since $\gamma$ is continuous.

It follows that

. $X_t$ has a density as soon as $\mu \neq 0$

. $(X_{t_1}, \ldots, X_{t_n})$ has a density if the functions

$$e^{it_1 \bullet}, \ldots, e^{it_n \bullet}$$

are linearly independent in $L^2(\mathbb{R}, \mu)$.

Let $\nu$ be the usual spectral measure of $X$, from the two spectral representations it follows that the space $\overline{\mathcal{L}(X_t, t \in \mathbb{R})}^{\mathbb{D}_X}$ is isomorphic to $L^2(\sigma(X), \mu + \nu)$.

Hence if $\nu << \mu$ with $\frac{d\nu}{d\mu}$ bounded, $\Gamma(Y, Y)^{1/2}$ is on $\mathcal{L}(X_t, t \in \mathbb{R})$ a norm equivalent to $\|Y\|_{\mathbb{D}_X}$.

If we project $X_{t+h}$ on $\overline{\mathcal{L}(X_s, s \leq t)}^{\mathbb{D}_X}$ for this Hilbert scalar product we get

$$\widehat{X_{t+h}} \in \overline{\mathcal{L}(X_s, s \leq t)}^{\mathbb{D}_X}$$

and $\widehat{X_{t+h}}$ is also the best estimate of $X_{t+h}$ in the whole space $\mathbb{D}(X_s, s \leq t)$ in the sense of the Dirichlet form $\mathcal{E}$, because

$$\begin{aligned} &\mathcal{E}(X_{t+h} - \widehat{X_{t+h}}, X_s) = 0 \quad \forall s \leq t \\ \Longrightarrow \quad &\mathcal{E}(X_{t+h} - \widehat{X_{t+h}}, F(X_{s_1}, \ldots, X_{s_n})) = 0 \qquad \forall F \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^n) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s_i \leq t \;\; i = 1, \ldots, n \end{aligned}$$

This situation is similar to the Gaussian case for the filtrage of Wiener.