On numerical integration by the shift and application to Wiener space

Nicolas BOULEAU *

-=-

Since the advantages of quasi-Monte Carlo methods vanish when the dimension of the basic space increases, the question arises whether there are better methods than classical Monte Carlo in large or infinite dimensional basic spaces. We study here the use of the shift operator with the pointwise ergodic theorem whose implementation is particularly interesting. After recalling the theoretical results on the speed of convergence in a form useful for applications, we give sufficient criteria for the law of iterated logarithm in several cases and in particular in situations involving the Wiener space.

*CERMA, Ecole Nationale des Ponts et Chaussées 93167 Noisy-le-Grand cedex France

If a family of real random variables is naturally defined on a probability space which can be smoothly changed to be $([0,1]^s, \mathcal{B}([0,1]^s), dx)$ with small s, quasi-Monte Carlo methods are among the fastest ones for computing expectations, at least when the family is wide enough to exclude other specific methods. See for example [20], [21]. But the advantage of these methods vanishes when s increases (cf [22]). Practically, for the best low-discrepancy sequences available at present (cf [21], [12]), to compute expectations with an accuracy of 10^{-4} by unit of standard deviation, it is faster to come back to the classical Monte Carlo method as soon as the dimension s exceeds 20 (cf [1] [9]).

In large or infinite dimension (computation of expectations of stopping times for Markov chains, or of functionals of solutions of stochastic differential equations, etc.) the classical Monte Carlo method which is based on the law of large numbers, can nevertheless be supplanted by another method based on the pointwise ergodic theorem of Birkhoff and the shift operator. Particular features of the implementation of this method make it at present the most interesting way of integration in large or infinite dimension (cf [5]).

The aim of this study is to clarify the consequences of recent theoretical results for the numerical computation of expectation by the shift method, and in particular to yield sufficient criteria for the existence of speed of convergence of the type 'iterated logarithm' in several situations. We put particular accent to the case of Wiener space because it is the basic space of many situations useful in applications.

The content of the study is the following:

I. Law of iterated logarithm for the shift

II. Criteria of membership for the Gordin class

1. Case of the torus T^s
2. Case of the torus $T^{\mathbb{N}}$
3. Case of Wiener space
   a) The Wiener space as a product space
   b) Functionals of lipschitzian SDE's
   c) Multiple Wiener integrals
4. Other factorisations of the Wiener space

We give now some details on each of these parts.

The first part is concerned with the speed of convergence in the pointwise ergodic theorem for the shift on $T^{\mathbb{N}}$. In contrast to the case of the law of large numbers, there is no standard speed of convergence valid for every function in L^2. Nevertheless the successive improvements of the LIL (cf [16], [23], [15], and more recently [4]) have shown the importance of a sub-class of L^2 for which a form of the LIL is valid and which contains several useful examples (cf part II). We call this class the Gordin class by reference to [13] one of the first works where this decomposition in sum of martingale increments and a subsidiary harmless term is used. Our purpose is not to extend the general results (cf [4]) but to express useful consequences for applications. All the results are explicitely proved except the theorem of Heyde and Scott itself.

In the second part, we show first that functions in the Sobolev spaces $H^\alpha(T^s)$ are in the Gordin class for the shift of binary digits. Next for the torus $T^{\mathbb{N}}$ with the shift to the right, Dirichlet forms techniques are used to obtain a simple sufficient criterion for membership to the Gordin class. For the first factorisation of the Wiener space under study, the shift becomes the scaling $B_t \circ \tau = \frac{1}{\sqrt{2}} B_{2t}$. With this transform, Hölderian functions of solutions of Lipschitzian SDE's are shown to belong to the Gordin class. Somes examples are analysed which are related to multiple Wiener integrals. Other factorisations are discussed and especially the representation of Brownian motion on the Schauder basis of $\mathcal{C}([0,1])$ consisting of primitive functions of the Haar basis. The criterion obtained on $T^{\mathbb{N}}$ by Dirichlet forms method applies here as well.

We thank J.P. Conze and E. Lesigne for useful discussions and suggestions.

I. The law of iterated logarithm for the shift

We are interested by almost sure results. It is well known (see [14], [17]) that for every ergodic endomorphism τ on a Lebesgue space, and for every sequence, (α_n), $\alpha_n > 0$, $\alpha_n \to 0$, there is an f in L^2 such that

$$(\frac{1}{n}\sum_{k=0}^{n-1} f\circ\tau^k - \mathbb{E}f)/\alpha_n \to +\infty \qquad \text{a.s.}$$

Such a "slow" f is constructed by suitable application of the Rohlin-Halmos lemma which likewise furnishes a "fast" non constant $f \in L^2$ for which $(\frac{1}{n}\sum_{k=0}^{n-1} f\circ\tau^k - \mathbb{E}f)$ is $o(\frac{1}{n^{1-\epsilon}})$ (see also [18] pp.14-15)

Nevertheless such functions, by the nature of the Rohlin-Halmos lemma itself, are rather abstract examples, and do not prohibit an LIL from holding for a large class of functions containing the most common ones.

For later convenience, we assume the following framework:
$(E, \mathcal{E}, \mu)$ is a probability space and
$(\Omega, \mathcal{A}, \mathbb{P}) = (E^{\mathbf{Z}}, \mathcal{E}^{\otimes\mathbf{Z}}, \mu^{\otimes\mathbf{Z}})$.

The coordinates from Ω into E are denoted by X_n. We define the ergodic automorphism τ on $(\Omega, \mathcal{A}, \mathbb{P})$ by

$$X_n \circ \tau = X_{n-1} \qquad \forall n \in \mathbf{Z}.$$

We call τ the shift to the right. One puts

$$\mathcal{F}_m^n = \sigma(X_m, X_{m+1}, \ldots, X_n) \qquad \text{for } m \le n \in \mathbf{Z}$$

$$\mathcal{F}_{-\infty}^n = \sigma(X_k, k \le n)$$
$$\mathcal{F}_m^{+\infty} = \sigma(X_k, k \ge m)$$
$$\mathcal{F}_{-\infty}^{+\infty} = \sigma(X_k, k \in \mathbf{Z})$$

As stated in the introduction the following results can be proved in a more general setting, for other endomorphisms (see [16], [23], [15]) and for Banach-valued random variables (see [4]).

Let us consider on $L^1(\mathcal{F}_0^\infty, \mathbb{P})$ the Perron-Frobenius operator T defined by

$$Tf = \mathbb{E}[f|\mathcal{F}_1^\infty]\circ\tau \qquad f \in L^1(\mathcal{F}_0^\infty, \mathbb{P})$$

we then have:

Lemma 1 *For $f \in L^2(\mathcal{F}_0^\infty, \mathbb{P})$, $\mathbb{E}f = 0$, the following assumptions are equivalent:*
a) $\sum_{n=0}^N T^n f$ remains bounded in L^2,
b) $\sum_{n=0}^N T^n f$ converges weakly in L^2 when $N \uparrow \infty$,
c) $\sum_{n=0}^N T^n f$ converges in L^2 when $N \uparrow \infty$,
d) there exists $g \in L^2(\mathcal{F}_0^\infty, \mathbb{P})$ such that $f = (I - T)g$.

Proof. b)⇒d): if $\sum_{n=0}^N T^n f$ converges weakly, by the Banach-Steinhaus theorem the limit g is an element of $L^2(\mathcal{F}_0^\infty, \mathbb{P})$. By composition with the bounded operator T, we obtain $g = f + Tg$.

d)$\Rightarrow$c): if $f = (I - T)g$, $g \in L^2$, it can be supposed $\mathbb{E}g = 0$. Then $\|T^N g\|_{L^2} \to 0$ when $N \uparrow \infty$. Indeed $\|T^N g\|_{L^2}^2 = \mathbb{E}[\mathbb{E}(g|\mathcal{F}_N^\infty)^2]$ and $\mathbb{E}(g|\mathcal{F}_N^\infty)$ is an inverse martingale which tends to zero in L^2.

Finally for a)$\Rightarrow$b), let us consider a subsequence N_k such that $\sum_{n=0}^{N_k} T^n f$ converges weakly in $L^2(\mathcal{F}_0^\infty)$ as $k \uparrow \infty$. Letting g be the limit, by composition with T we get

$$g - f + \lim_{k \uparrow \infty} T^{N_k+1} g = Tg$$

and the same argument as for c)$\Rightarrow$d) shows that $T^{N_k+1} f \to 0$ in L^2. $\square$

Remark. It is easy to see that these conditions are equivalent to the condition that $\sum_{n=0}^{N} \tau^n f$ converge for the topology $\sigma(L^2(\mathcal{F}_{-\infty}^{+\infty}), L^2(\mathcal{F}_0^{+\infty}))$.

We shall say that a function $f \in L^2(\mathcal{F}_0^{+\infty})$ belongs to **the Gordin class** (for which we write $f \in \mathcal{G}$) if $f - \mathbb{E}f$ satisfies the equivalent conditions of lemma 1.

Lemma 2 *The Gordin class is the class of the functions $f \in L^2(\mathcal{F}_0^{+\infty})$ admitting a decomposition*

$$f - \mathbb{E}f = \tilde{g} + h \circ \tau^{-1} - h \tag{1}$$

where $\tilde{g}, h \in f \in L^2(\mathcal{F}_0^{+\infty})$ with $\mathbb{E}(\tilde{g}|\mathcal{F}_1^\infty) = 0$ and $\mathbb{E}h = 0$. Such a decomposition, if it exists, is unique.

Proof. By lemma 1, if $f \in \mathcal{G}$ there is a $g \in L^2(\mathcal{F}_0^{+\infty})$ such that $f - \mathbb{E}f = g - Tg$. Putting $\tilde{g} = g - \mathbb{E}(g|\mathcal{F}_1^\infty)$ and $h = Tg = \mathbb{E}(g|\mathcal{F}_1^\infty) \circ \tau$, we get the decomposition (1).

Conversely, if f can be decomposed as (1), we have $T\tilde{g} = 0$ and $T(h \circ \tau^{-1}) = h$, hence $f - \mathbb{E}f = (I - T)g$ with $g = \tilde{g} + h \circ \tau^{-1}$. The uniqueness follows immediately. $\square$

The theorem of iterated logarithm is valid for functions in the Gordin class:

Theorem 3 *Let $f \in L^2(\mathcal{F}_0^{+\infty})$ be in the Gordin class, and $\tilde{g}$, h the elements of its decomposition (1). Then, putting $S_N = \sum_{n=0}^{N} (f - \mathbb{E}f) \circ \tau^n$, there holds*

a)

$$\lim_{N \uparrow \infty} \frac{1}{\sqrt{N}} \|S_N\|_{L^2} = \|\tilde{g}\|_{L^2}$$

b)

$$\limsup_{N \uparrow \infty} \frac{|S_N|}{\sqrt{2N \log \log N}} = \|\tilde{g}\|_{L^2}$$

Proof. Noting that

$$S_N = \sum_{n=0}^{N} \tilde{g} \circ \tau^n + h \circ \tau^{-1} - h \circ \tau^N$$

part a) comes from the following inequality, where the norms are taken in L^2:

$$\left| \|\frac{1}{\sqrt{N}} S_N\| - \|\frac{1}{\sqrt{N}} \sum_{n=0}^{N} \tilde{g} \circ \tau^n\| \right| \leq 2 \frac{\|h\|}{\sqrt{N}} \to_{N \uparrow \infty} 0$$

and from $\| \sum_{n=0}^{N} \tilde{g} \circ \tau^n \|^2 = (N+1)\|\tilde{g}\|^2$, which follows by orthogonality.

If $\tilde{g} = 0$ part b) is a consequence of the fact that, h being in L^2, $\frac{h \circ \tau^N}{\sqrt{N}} \to 0$ when $N \uparrow \infty$ by the pointwise ergodic theorem. Thus, when $\tilde{g} \neq 0$ it suffices to show that

$$\limsup_{N\uparrow\infty} \frac{|\sum_{n=0}^{N} \tilde{g}\circ\tau^n|}{\sqrt{2N\log\log N}} = \|\tilde{g}\|.$$

But this is given by the theorem of Heyde and Scott ([16] corollary 2). □

We shall now state sufficient conditions for membership to the Gordin class $\mathcal{G}$.

Without subscript, norms are L^2-norms.

Proposition 4 *Let $f \in L^2(\mathcal{F}_0^{+\infty})$ be such that*

$$\sum_{n=0}^{\infty} \|\mathbb{E}[f] - \mathbb{E}(f|\mathcal{F}_n^{\infty})\| < +\infty, \tag{2}$$

then $f \in \mathcal{G}$ and the $\tilde{g}$ of its decomposition satisfies

$$\|\tilde{g}\| \leq \sum_{n=0}^{\infty} \|\mathbb{E}[f] - \mathbb{E}(f|\mathcal{F}_n^{\infty})\|.$$

Proof. By the fact that

$$\|T^n(f-\mathbb{E}f)\| = \|\mathbb{E}f - \mathbb{E}(f|\mathcal{F}_n^{\infty})\|$$

the convergence of the series (2) implies that the series $\sum T^n(f-\mathbb{E}f)$ converges normally. Letting g be its sum, then $\tilde{g}$ is given by $g - \mathbb{E}(g|\mathcal{F}_1^{\infty})$ thus $\|\tilde{g}\| \leq \|g\|$. □

Proposition 5 *Let $f \in L^2(\mathcal{F}_0^{+\infty})$, and let us consider the decomposition*

$$f = \mathbb{E}f + \sum_{n=0}^{\infty} f_n \tag{3}$$

with

$$f_0 = \mathbb{E}(f|\mathcal{F}_0^0) - \mathbb{E}(f)$$

and

$$f_n = \mathbb{E}(f|\mathcal{F}_0^n) - \mathbb{E}(f|\mathcal{F}_0^{n-1}) \qquad \text{for } n \geq 1$$

a) $f \in \mathcal{G}$ if and only if

$$\sup_N \sum_{j=o}^{\infty} \|\mathbb{E}(\sum_{n=0}^{N} f_{n+j}\circ\tau^n|\mathcal{F}_0^{\infty})\|^2 < +\infty.$$

b) This is satisfied if

$$\sum_{m\geq 0} \sqrt{\sum_{k\geq m} \|f_k\|^2} < +\infty$$

and then the $\tilde{g}$ associated with f in (1) is such that $\|\tilde{g}\| \leq \sum_{m\geq 0}\sqrt{\sum_{k\geq m}\|f_k\|^2}$.

c) This is also satified if

$$\sum_{m\geq 0} \sqrt{m}\|f_m\| < +\infty$$

and then the $\tilde{g}$ associated with f in (1) is such that $\|\tilde{g}\| \leq \sum_{m\geq 0}\sqrt{m}\|f_m\|$.

Proof. The existence of the decomposition (3) for any $f \in L^2(\mathcal{F}_0^{+\infty})$ comes from the fact that

$$\sum_{n=1}^{N} f_n = \mathbb{E}(f|\mathcal{F}_0^N) - \mathbb{E}(f|\mathcal{F}_0^0)$$

is a martingale which converges in L^2.

Let $f \in L^2(\mathcal{F}_0^{+\infty})$, and put $\tilde{f} = f - \mathbb{E}f$. Using the fact that for $n \geq 0$, $T^n\tilde{f} = \mathbb{E}[\tilde{f} \circ \tau^n|\mathcal{F}_0^\infty]$ we get

$$T^n\tilde{f} = \sum_{k\geq n} \mathbb{E}[f_k \circ \tau^n|\mathcal{F}_0^\infty]$$

because for $k < n$, $f_k \circ \tau^n$is $\mathcal{F}_{-\infty}^{-1}$-measurable. It follows that

$$\sum_{n=0}^{N} T^n\tilde{f} = \sum_{n=0}^{N}\sum_{k\geq n} \mathbb{E}[f_k \circ \tau^n|\mathcal{F}_0^\infty] \tag{4}$$

$$= \sum_{j=0}^{\infty} \mathbb{E}(\sum_{n=0}^{N} f_{n+j} \circ \tau^n|\mathcal{F}_0^\infty).$$

But the random variables

$$Z_j^N = \mathbb{E}(\sum_{n=0}^{N} f_{n+j} \circ \tau^n|\mathcal{F}_0^\infty)$$

form an orthogonal sequence and therefore

$$\|\sum_{n=0}^{N} T^n\tilde{f}\|^2 = \sum_{j=0}^{\infty} \|Z_j^N\|^2$$

and part a) follows from lemma 1.

From the equality (4) we have also

$$\|\sum_{n=0}^{N} T^n\tilde{f}\| \leq \sum_{n=0}^{N} \|\sum_{k\geq n} \mathbb{E}[f_k \circ \tau^n|\mathcal{F}_0^\infty]\|.$$

For every fixed n the sequence $(\mathbb{E}(f_k \circ \tau^n\|\mathcal{F}_0^\infty))_{k\geq n}$ is orthogonal, and so

$$\|\sum_{k\geq n} \mathbb{E}[f_k \circ \tau^n|\mathcal{F}_0^\infty]\| = \left(\sum_{k\geq n} \|\mathbb{E}[f_k \circ \tau^n|\mathcal{F}_0^\infty]\|^2\right)^{1/2}$$

$$\leq \left(\sum_{k\geq n} \|f_k\|^2\right)^{1/2}$$

which gives part b).

Taking once more the equality (4) rewritten as

$$\sum_{n=0}^{N} T^n\tilde{f} = \sum_{m=0}^{\infty} \mathbb{E}(f_m + f_m \circ \tau + \cdots + f_m \circ \tau^{m\wedge N}|\mathcal{F}_0^N)$$

gives

$$\|\sum_{n=0}^{N} T^n\tilde{f}\| \leq \sum_{m=0}^{\infty} \|f_m + f_m \circ \tau + \cdots + f_m \circ \tau^{m\wedge N}\|$$

$$\leq \sum_{m=0}^{\infty} \sqrt{m}\|f_m\|,$$

by the fact that the sequence $f_m \circ \tau^{m\wedge N}$, $f_m \circ \tau^{m\wedge N-1}, \ldots, f_m$ is orthogonal. Part c) follows by the same arguments. □

If T is an almost finite stopping time of the σ-fields $(\mathcal{F}_0^n)_{n\geq 0}$ and if f is an $\mathcal{F}_0^T$-measurable random variable, f can be written as

$$f = \sum_{k\geq 0} f\, 1_{\{T=k\}}$$

with $f\, 1_{\{T=k\}}$ $\mathcal{F}_0^k$-measurable. This is a particular case of the following situation:

Proposition 6 *Let $f \in L^2(\mathcal{F}_0^\infty)$ admit the following representation converging in L^2:*

$$f = \sum_{k=0}^{\infty} f_k \quad \textit{with } f_k \ \mathcal{F}_0^k\textit{-measurable.}$$

If the condition

$$\sum_{k=0}^{\infty} k\|f_k - \mathbb{E}f_k\| < +\infty \tag{5}$$

is fulfilled, then $f \in \mathcal{G}$, and the associated $\tilde{g}$ satisfies

$$\|\tilde{g}\| \leq \sum_{k=0}^{\infty} \sqrt{k+1}\|f_k - \mathbb{E}f_k\|. \tag{6}$$

Proof. By the fact that $T^n(f_k - \mathbb{E}f_k) = 0$ for $n > k$,

$$T^n(f - \mathbb{E}f) = \sum_{k\geq n} T^n(f_k - \mathbb{E}f_k)$$

and therefore under condition (5) the series $\sum_n T^n(f - \mathbb{E}f)$ is normally convergent and $f \in \mathcal{G}$.

Let us put $g(f_k) = \sum_{n\geq 0} T^n(f_k - \mathbb{E}f_k)$ and $g(f) = \sum_{n\geq 0} T^n(f - \mathbb{E}f)$. Under condition (5) we have thus

$$g(f) = \sum_{k\geq 0} g(f_k),$$

the series converging normally. Therefore

$$g(f) - \mathbb{E}(g(f)|\mathcal{F}_0^1) = \sum_{k\geq 0} [g(f_k) - \mathbb{E}(g(f_k)|\mathcal{F}_0^1)],$$

the series again converging normally. But by lemma 7 below and proposition 5 applied to f_k we have

$$\|g(f_k) - \mathbb{E}(g(f_k)|\mathcal{F}_0^1)\| \leq \sqrt{k+1}\|f_k - \mathbb{E}f_k\|$$

so that

$$\|g(f) - \mathbb{E}(g(f)|\mathcal{F}_0^1)\| \leq \sum_k \sqrt{k+1}\|f_k - \mathbb{E}f_k\|$$

which proves the proposition. □

Lemma 7 *If $f \in L^2$ depends only on d consecutive coordinates then*

$$\limsup_N \frac{|f + f \circ \tau + \cdots + f \circ \tau^{N-1} - N\mathbb{E}f|}{\sqrt{2N \log\log N}} \leq \sqrt{d}\|f - \mathbb{E}f\|$$

Proof. This is a simple application of the LIL of Hartman-Wintner for independent variables. Let us put $N-1 = pd + q$ with $0 \leq q < d$, and let us suppose f to be centred. Then

$$\sum_{i=0}^{N-1} f \circ \tau^i = \sum_{k=0}^{d-1}\sum_{j=0}^{p-1} f \circ \tau^{jd+k} + \sum_{n=pd}^{pd+q} f \circ \tau^n. \tag{7}$$

By the fact that for every fixed k

$$\limsup_p \frac{|\sum_{j=0}^{p-1} f \circ \tau^{jd+k}|}{\sqrt{2p \log\log p}} = \|f\|$$

we have

$$\begin{aligned}\limsup_N \frac{|\sum_{k=0}^{d-1}\sum_{j=0}^{p-1} f \circ \tau^{jd+k}|}{\sqrt{2N \log\log N}} &\leq d\|f\| \lim \frac{\sqrt{2p \log\log p}}{\sqrt{2N \log\log N}} \\ &\leq \sqrt{d}\|f\|.\end{aligned}$$

Now the second term of (7) gives

$$\frac{|\sum_{n=pd}^{pd+q} f \circ \tau^n|}{\sqrt{2N \log\log N}} \leq \frac{\sum_{j=0}^{d-1} |f| \circ \tau^{pd+j}}{\sqrt{2N \log\log N}}$$

which vanishes almost surely as $N \uparrow \infty$ by the ergodic theorem because $f \in L^2$. The lemma follows from these estimates. $\square$

Remark. For $f \in L^2(\mathcal{F}_0^\infty)$ admitting the representation $f = \sum_k f_k$ converging in L^2 with f_k $\mathcal{F}_0^k$-measurable, we dont know whether the sole hypothesis $\sum_k \sqrt{k+1}\|f_k - \mathbb{E}f_k\| < +\infty$ suffices to imply

$$\limsup_N \frac{|\sum_{n=0}^N (f - \mathbb{E}f) \circ \tau^n|}{\sqrt{2N \log\log N}} \leq \sum_k \sqrt{k+1}\|f_k - \mathbb{E}f_k\|.$$

The following result, whose statement is simple, is a rather rough consequence of the preceding proposition.

Proposition 8 *Let T be an a.s. finite stopping time of $(\mathcal{F}_0^n)_{n\geq 0}$, and $f \in L^2(\mathcal{F}_0^T)$.*
If there is an $\alpha > 1$ such that

$$\mathbb{E}[f^2 T^3 \log^\alpha T] < +\infty$$

then

$$\limsup_N \frac{|\sum_{n=0}^N (f - \mathbb{E}f) \circ \tau^n|}{\sqrt{2N \log\log N}} \leq \frac{\sqrt{6}}{\pi}\|f(T+1)^{3/2}\|.$$

Proof. Putting $f_k = f\, 1_{\{T=k\}}$ we have

$$\sum_{k\leq 1} k\|f_k - \mathbb{E}f_k\| \leq \sum_k k\|f_k\| = \sum_k \frac{1}{\sqrt{k \log^\alpha k}}\|\sqrt{k^3 \log^\alpha k}\, f_k\|$$

which by the Cauchy-Schwarz inequality, is bounded by

$$\leq \sqrt{\sum_k \frac{1}{k \log^\alpha k}} \sqrt{\sum_k k^3 \log^\alpha k f_k} \leq c\sqrt{\mathbb{E}[T^3 \log^\alpha T f^2]}.$$

Therefore the inequality (5) is satisfied and similarly

$$\sum_{k=0}^{\infty} \sqrt{k+1} \|f_k - \mathbb{E} f_k\| \leq \frac{\sqrt{6}}{\pi} \|f(T+1)^{3/2}\|.$$

□

Remark. It is worth noting that if $f \in \mathcal{G}$ it can of course happen that

$$\|\tilde{g}\| < \|f - \mathbb{E} f\|.$$

This is the case if $f - \mathbb{E} f = h \circ \tau^{-1} - h$ with $h \in L^2$ and other examples are easily constructed by the Gordin decomposition. This can occur even when f depends only on a finite number of coordinates. In this case integration by the shift method runs (asymptotically) faster than by classical Monte Carlo.

Nevertheless, the principal interest of the shift method does not come from this phenomenon but from certain facilities afforded by its implementation (see [5]).

II. Criteria of membership to the Gordin class

II.1. The case of the torus T^s

Let us consider the following transform of $T^s \times T^s$

$$((x_1, \ldots, x_s), (y_1, \ldots, y_s)) \xrightarrow{\tau} (([2x_1], \ldots, [2x_s])), (\frac{2x_1 - [2x_1] + y_1}{2}, \ldots, (\frac{2x_s - [2x_s] + y_s}{2}))$$

where $[x]$ is the fractional part of $x \in \mathbb{R}$, which is easily seen to correspond to the bilateral Bernoulli shift by binary expansion of real numbers.

We have for this transformation and for $f \in L^2(T^s, dy_1 \cdots dy_s)$:

$$Tf(y) = \sum_{n \in \{0,1\}^s} \frac{1}{2^s} f(\frac{n}{2} + \frac{y}{2})$$

$$T^n f(y) = \sum_{k \in \{0, \ldots, 2^n - 1\}^s} \frac{1}{2^{ns}} f(\frac{k}{2^n} + \frac{y}{2^n}).$$

Using then the Fourier representation of f

$$f(y) = \sum_{m \in \mathbf{Z}^s} a_m \; e^{2i\pi <m,y>}$$

one easily obtains that if $\mathbb{E} f = 0$

$$\sum_{n=0}^{N} T^n f(y) = \sum_{q \in \mathbf{Z}^s,\; q \neq 0} \; \sum_{n=0}^{N} a_{2^n q} \; e^{2i\pi <q,y>}$$

and we get:

Proposition 9 *The function $f \in L^2(T^s)$ belongs to $\mathcal{G}$ if and only if*

$$\sup_N \sum_{q\in\mathbf{Z}^s,\, q\neq 0} |\sum_{n=0}^{N} a_{2^n q}|^2 < +\infty$$

in which case $\forall q \neq 0$, $\sum_{n=0}^{N} a_{2^n q} \to_{N\uparrow\infty} b_q$ with $\sum_q |b_q|^2 < +\infty$ and the $\tilde{g}$ of f in (1) satisfies $\|\tilde{g}\|^2 \leq \sum_q |b_q|^2$.

Corollary 10 *Let $f \in L^2(T^s)$ be such that there are $c_n \geq 0$ with $\sum_{n=0}^{\infty} c_n < +\infty$ and $|a_{2^n m}| \leq c_n |a_m|$ $\forall m \in \mathbf{Z}^s \backslash \{0\}$. Then $f \in \mathcal{G}$ and*

$$\|\tilde{g}\| \leq \|f - \mathbb{E}f\| \sum_{n=0}^{\infty} c_n.$$

Example. Letting f belong to the Sobolev space $H^\alpha(T^s)$ defined by $\sum_{p\in\mathbf{Z}^s} |a_p|^2 (\sum_{i=0}^{s} p_i^2)^\alpha < +\infty$ for some $\alpha > 0$. Then $f \in \mathcal{G}$ and $\|\tilde{g}\| \leq c(\sum_p |a_p|^2 (\sum_{i=0}^{s} p_i^2)^\alpha)^{1/2}$.

II.2. Case of the infinite dimensional torus

We consider here the Bernoulli shift (to the right) on $([0,1]^s, \mathcal{B}([0,1]^s), dx)^{\mathbf{Z}}$.

The property of membership to the Gordin class is strongly related to the dependence of f on the size of the derivatives of f (when they exist) with respect to the faraway coordinates. This is particularly simple to espress by means of Dirichlet forms:

Let us consider a Dirichlet form $(\mathrm{d}, \varepsilon)$ on $L^2([0,1]^s, dx)$ possessing a carré du champ operator γ (cf [7]) and let us consider the product Dirichlet structure (cf [8]):

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \mathcal{E}) = ([0,1]^s, \mathcal{B}([0,1]^s), dx, \mathrm{d}, \varepsilon)^{\otimes \mathbf{Z}}$$

This structure has a carré du champ Γ given by

$$\Gamma(f,f) = \sum_{i\in\mathbf{Z}} \gamma_i(f,f) \quad \forall f \in \mathbb{D}$$

where γ_i operates on the i-th coordinate. We consider the shift τ given by

$$X_n \circ \tau = X_{n-1}$$

where $(X_n)_{n\in\mathbf{Z}}$ are the coordinates. For $F \in L^2(\Omega, \mathcal{F}_0^\infty)$ there holds

$$Tf(x_0, x_1, \ldots, x_n, \ldots) = \int_{x\in[0,1]^s} f(x, x_0, x_1, \ldots) dx.$$

We make the following assumption (8): Let $L_0^2 = \{f \in L^2 \; \mathbb{E}f = 0\}$

$$(8) \qquad \left\{ \begin{array}{l} \text{There exists } K > 0 \text{ such that} \\ \forall f \in \mathbb{D} \cap L_0^2 \quad \|f\|_{L^2}^2 \leq K\mathcal{E}(f,f) \end{array} \right.$$

Then the space $\mathbb{D} \cap L_0^2$ is a Hilbert space for the norm $\sqrt{\mathcal{E}(f,f)}$ which is invariant by τ. Let $\mathbb{D}_0 = \{f \in \mathbb{D},\ \mathbb{E}f = 0,\ f \text{ is } \mathcal{F}_0^\infty\text{-measurable}\}$ which is closed in $\mathbb{D}$

Proposition 11 *Under hypothesis (8), let $f \in \mathbb{D}_0$ be such that*

$$\sum_{k=0}^{\infty} (\sum_{i=k}^{\infty} \mathbb{E}[\gamma_i(f,f)])^{1/2} < +\infty.$$

Then $f \in \mathcal{G}$.

Proof. This is straightforward by the fact that

$$\mathcal{E}(T^n f, T^n f) \leq \frac{1}{2}\mathbb{E}[\sum_{i=n}^{\infty} \gamma_i(f,f)]$$

□

Corollary 12 *Let $f \in L^2(\Omega, \mathcal{F}_0^\infty)$ be such that for every $n \in N$,*

$$[0,1]^s \ni x_n \longrightarrow f(x_0, \ldots, x_n \ldots) \in \mathbb{R}$$

possesses a derivative in the sense of distributions in $L^2(dx_n)$ $(dx_0 \cdots dx_{n-1}dx_{n+1}\cdots)$-almost surely.

Then if

$$\sum_{i=2}^{\infty} i^2(\log^\alpha i)\mathbb{E}[f_i'^2] < +\infty$$

for an $\alpha > 1$, then $f \in \mathcal{G}$.

Proof. The preceding proposition is here applied to the case $(\mathrm{d}\mathrm{I}, \varepsilon) = (H^1([0,1]^s, dx,\ \int \nabla^2,\ dx)$

a) Let us prove first that the hypothesis (8) is fulfilled. For this we use the fact that this hypothesis is satisfied on the Wiener space equipped by the Ornstein-Uhlenbeck semi-group, as it is easily seen by the spectral representation on the chaos. This is equivalent to saying that (8) is satisfied on the Gaussian structure

$$\left(\mathbb{R}^s,\ \mathcal{B}(\mathbb{R}^s),\ N_s(0,1),\ \int \nabla^2,\ H^1(\mathbb{R}^s, N_s(0,1))\right)^{\otimes \mathbf{Z}}$$

with the constant $K = 1$. The property is therefore true for every image structure of this structure (cf [8]) and the result comes then from the following easy fact :

Let be $\varphi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$; then

$$\int_{\mathbb{R}} (f \circ \varphi)'^2 dN(0,1) \leq \frac{1}{2\pi} \int_0^1 f'^2(x)dx.$$

b) It remains only to prove that

$$\sum_{1=2}^{\infty} i^2(\log^\alpha i) a_i^2 < +\infty \Rightarrow \sum_{k=0}^{\infty}(\sum_{i=k}^{\infty} a_i^2)^{1/2} < +\infty.$$

which a consequence of the Cauchy-Schwarz inequality. □

II.3. The case of Wiener space

3.a) The Wiener space as a product space

Let us consider the space $W = \{f \in \mathcal{C}([0,1], \mathbb{R}^d),\ f(0) = 0\}$ equipped with its Borelian σ-field $\mathcal{B}$ and with the Wiener measure m.

On the space $(\Omega, \mathcal{A}, \mathbb{P}) = \prod_{n=-\infty}^{+\infty}(W_n, \mathcal{B}_n, m_n)$ where $(W_n, \mathcal{B}_n, m_n)$ are copies of $(W, \mathcal{B}, m)$ we define a Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ in the following manner: Letting X_n be the coordinate map from Ω into W_n, for $t \in]\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, $k \in \mathbf{Z}$ we put

$$B_t = \sum_{n=k+1}^{\infty} \frac{X_n(1)}{2^{\frac{n+1}{2}}} + \frac{X_k(\frac{t-1/2^{k+1}}{1/2^{k+1}})}{2^{\frac{k+1}{2}}}.$$

The process thus defined is Gaussian centred with independent increments, tends to zero as t goes to zero and its covariance is easily computed to be $s \wedge t$ times the identity matrix; it is therefore a standard $\mathbb{R}^d$-valued Brownian motion.

The transform τ defined on Ω by

$$X_n \circ \tau = X_{n-1} \quad n \in \mathbf{Z}$$

is a scaling

$$B_t \circ \tau = \frac{1}{\sqrt{2}} B_{2t}$$

and the results of section I apply with

$$\begin{aligned} \mathcal{F}_0^\infty &= \sigma(B_s,\ s \leq 1) \\ \mathcal{F}_1^\infty &= \sigma(B_s,\ s \leq \tfrac{1}{2}) \\ \mathcal{F}_0^k &= \sigma(B_s - B_{\frac{1}{2^{k+1}}},\ s \in]\tfrac{1}{2^{k+1}}, 1]) \end{aligned}$$

We shall put $\mathcal{B}_t = \sigma(B_s,\ s < t)$.

3.b) Functionals of Lipschitzian SDE's

Let us consider maps

$$\sigma : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^{m \times d}, \quad b : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m$$

satisfying the Lipschitz hypotheses:

$\exists C > 0$ such that $\forall s \in [0,1]$

$$|\sigma(x,s) - \sigma(y,s)| + |b(x,s) - b(y,s)| < C|x - y|$$

$$|\sigma(x,s)| + |b(x,s)| \leq C(1 + |x|)$$

where $|\,.\,|$ is one of the equivalent norms on Euclidean spaces.

Let X_t^x be the solution of the the SDE:

$$X_t^x = x + \int_0^t \sigma(X_s^x, s)dB_s + \int_0^t b(X_s^x, s)ds \quad x \in \mathbb{R}^m$$

Proposition 13 *Let be $f = h(X_t^x)$ for $t \leq 1$, with $h : \mathbb{R}^m \to \mathbb{R}$ Hölderian of exponent $\lambda \in]0,1]$. Then $f \in \mathcal{G}$.*

Proof. Let A be the Hölder constant of h:

$$|h(x) - h(y)| \leq A|x - y|^\lambda$$

and let $(P_t)_{t \leq 0}$ be the semi-group of the diffusion associated with the flow X_t^x. By classical estimates (cf [19] chapter 2) we have

$$|P_u h(x) - P_u h(y)| = |\mathbb{E}h(X_u^x) - \mathbb{E}h(X_u^y)| \leq A\mathbb{E}|X_u^x - X_u^y|^\lambda$$

$$\leq K|x - y|^\lambda \quad \forall u \in [0,1]$$

for some constant K depending on the dimensions m, d and on the constants C and A.

If φ is Hölder with exponent λ, we have

$$\operatorname{var}[\varphi(X_s^x)] = \mathbb{E}|\varphi(X_s^x) - \mathbb{E}\varphi(X_s^x)|^2 \leq \mathbb{E}|\varphi(X_s^x) - \varphi(E(X_s^x))|^2$$

$$\leq c_1 \mathbb{E}|X_s^x - \mathbb{E}X_s^x|^{2\lambda} \leq c_2(1+|x|^{2\lambda})s^\lambda$$

(cf [19] theorem 2.1)

Now, let us remark that

$$T^n f = \mathbb{E}[h(X_t^x)|\mathcal{B}_{\frac{1}{2^n}}] \circ \tau^n = P_{t-\frac{1}{2^n}} h(X^x_{\frac{1}{2^n}}) \circ \tau^n.$$

Hence by the preceding estimates we get

$$\|T^n(f-\mathbb{E}f)\|^2 = \mathrm{var}[T^n f] = \mathrm{var}[P_{t-\frac{1}{2^n}} h(X^x_{\frac{1}{2^n}})]$$

$$\leq c(1+|x|^{2\lambda})\frac{1}{2^{n\lambda}}$$

and the series $\sum \|T^n(f-\mathbb{E}f)\|$ converges geometrically. □

Proposition 14 *Let $\mu(ds,dx)$ be a measure on $[0,1]\times\mathbb{R}^m$ such that*

$$\int_{[0,1]\times\mathbb{R}^m} (1+|x|^\lambda)|\mu(ds,dx)| < +\infty$$

with $\lambda \in]0,1]$, and let g be a Hölderian function of exponent λ. Then the functional

$$f = \int_{[0,1]\times\mathbb{R}^m} g(X_s^x)\mu(ds,dx)$$

belongs to $\mathcal{G}$.

Proof. We have

$$T^n f = \int_0^{\frac{1}{2^n}} \int_{\mathbb{R}^m} g(X_s^x)\mu(ds,dx)\circ\tau^n + \int_{\frac{1}{2^n}}^1 \int_{\mathbb{R}^m} P_{s-\frac{1}{2^n}} g(X_{\frac{1}{2^n}})\mu(ds,dx)\circ\tau^n$$

and hence

$$\|T^n(f-\mathbb{E}f)\| \leq \int_0^{\frac{1}{2^n}} \int_{\mathbb{R}^m} \|g(X_s^x)-g(\mathbb{E}X_s^x)\| |d\mu| + \int_{\frac{1}{2^n}}^1 \int_{\mathbb{R}^m} (var[P_{s-\frac{1}{2^n}} g(X_{\frac{1}{2^n}})])^{\frac{1}{2}} |d\mu|$$

and therefore by the estimates used in the preceding proof:

$$\|T^n(f-\mathbb{E}f)\| \leq \int_0^{\frac{1}{2^n}} \int_{\mathbb{R}^m} A(1+|x|^\lambda)s^{\frac{\lambda}{2}}|\mu(ds,dx)| + \int_{\frac{1}{2^n}}^1 \int_{\mathbb{R}^m} B(1+|x|)^\lambda)\frac{1}{2^{\frac{n\lambda}{2}}}|\mu(ds,dx)|.$$

By hypothesis the second term is bounded by $C\frac{1}{2^{\frac{n\lambda}{2}}}$. For the first one, let us remark that

$$\sum_{n=0}^{\infty} 1_{[0,\frac{1}{2^n}]}(s)s^{\frac{\lambda}{2}}$$

is bounded on $s\in[0,1]$ from which it follows

$$\sum \|T^n(f-\mathbb{E}f)\| < +\infty$$

□

3.c) Multiple Wiener integrals

The case of multiple Wiener integrals is important on one hand because their family is in some sense the universal diffusion process (cf [2] [3]) and on the other hand because most of them are quite irregular and such that every Borelian version is discontinuous at every point in the Wiener space. Such functionals are not Riemann integrable and have to be approximated by more regular functionals before simulation (cf [6]).

Here we give some examples to illustrate which irregularity at the origin can have functions in the Gordin class for scaling.

Let

$$F = \int_{0<t_1<\cdots<t_m<1} h(t_1,\ldots,t_m)\; dB^{i_1}_{t_1} dB^{i_2}_{t_2}\cdots dB^{i_m}_{t_m}$$

where $i_k \in \{1,2,\ldots,d\}$ for $k=1,\ldots,m$ with

$$\int_{0<t_1<\cdots<t_m<1} h^2(t_1,\ldots,t_m)\; dt_1 dt_2\cdots dt_m < +\infty.$$

One has easily

$$T^nF = \int_{0<t_1<\cdots<t_m<1} \frac{1}{2^{\frac{nm}{2}}} h(\frac{t_1}{2^n},\ldots,\frac{t_m}{2^n})\; dB^{i_1}_{t_1} dB^{i_2}_{t_2}\cdots dB^{i_m}_{t_m}.$$

Therefore F belongs to the Gordin class if and only if

$$\sup_N \int_{0<t_1<\cdots<t_m<1} (\sum_{n=0}^{N} \frac{1}{2^{\frac{nm}{2}}} h(\frac{t_1}{2^n},\ldots,\frac{t_m}{2^n}))^2\; dt_1 dt_2\cdots dt_m < +\infty$$

Example 1. Let us take $m=1$, $h(x)=\frac{1}{x^\alpha}$, $\alpha<\frac{1}{2}$. It is easily seen that

$$F = \int_0^1 \frac{1}{t^\alpha}\, dB_t \;\in \mathcal{G} \;\; \forall \alpha < \frac{1}{2}.$$

Example 2. Let us take

$$h(x) = \frac{1}{\sqrt{x}(-\log x)^\beta} \qquad \text{with} \qquad \beta > \frac{1}{2}.$$

Then

$$F = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{t}(-\log t)^\beta}\, dB_t$$

is in the Gordin class if $\beta > 1$, but $F \notin \mathcal{G}$ if $\beta \in]\frac{1}{2},1]$ although $h\in L^2[0,1]$ in that case.
Example 3. If we take

$$h(x) = \frac{1}{\sqrt{x}} \frac{\sin(\pi \log_2 x)}{\log x}$$

the functional $F=\int_0^{\frac{1}{2}} h(t)\; dB_t$ gives an example of a functional in $\mathcal{G}$ such that

$$\sum_n \|T^nF\| = +\infty$$

and such that $\int_0 \frac{1}{2}|h(t)|\; dB_t \notin \mathcal{G}$.
Example 4. Let us consider a real Brownian motion ($d=1$), and a function F square integrable with the following Wiener chaos expansion:

$$F = F_0 + \sum_m F_m = F_0 + \sum_{m=1}^{\infty} \int_{0<t_1<\cdots<t_m<1} h(t_1,\ldots,t_m)\; dB_{t_1} dB_{t_2}\cdots dB_{t_m}$$

and let us suppose $|h_m(t_1,\dots,t_m)| \le a_m \frac{1}{t_1^{\alpha_1^m}\cdots t_m^{\alpha_m^m}}$ with $\alpha_i^m < \frac{1}{2}$ $\forall i = 1\dots,m$. We get, by the fact that the chaos are invariant by T,

$$\|\sum_{n=0}^{N} T^n(F-F_0)\|^2 = \sum_{m=1}^{\infty} \|\sum_{n=0}^{N} T^n F_m\|^2$$

$$\le \sum_{m=1}^{\infty} \frac{a_m^2}{[1-2^{(\alpha_1^m+\cdots+\alpha_m^m-\frac{m}{2})}]^2 \prod_{i=1}^m (i - 2\sum_{k=1}^i \alpha_k^m)}$$

so that, if all α_i^m's are equal to $\alpha < \frac{1}{2}$, $F \in \mathcal{G}$ as soon as the series

$$\sum_{m=1}^{\infty} \frac{a_m^2}{m!(1-2\alpha)^m}$$

converges.

II.4. Other factorisations of the Wiener space.

4.a)

Let $(\chi_n(t))_{n\ge0}$ be an orthonormal basis of $L^2[0,1]$ and let $\varphi_n(t) = \int_0^t \chi_n(s)\,ds$. Let $(g_n)_{n\ge0}$ be a sequence of independent standard Gaussian variables built as the coordinates of $(\Omega,\mathcal{A},\mathbb{P}) = (\mathbb{R}^{\mathbb{N}},\mathcal{B}(\mathbb{R}^{\mathbb{N}}),N(0,1)^{\otimes\mathbb{N}})$.

The series

(9)
$$\sum_n g_n\varphi_n$$

converges in $\mathcal{C}([0,1])$ a.s. and in $L^p((\Omega,\mathcal{A},\mathbb{P}),\mathcal{C}([0,1]))$ $p\in[1,\infty[$ and its sum is a Brownian motion under $\mathbb{P}$.

Indeed, if on the Wiener space we put $\tilde\chi_n(\omega) = \int_0^1 \chi_n(s)\,dB_s$ and $\mathcal{F}_n = \sigma(\tilde\chi_k,\ k\le n)$, we obtain, denoting by B the identity map from $\mathcal{C}[0,1]$ into itself,

(10)
$$\mathbb{E}[B|\mathcal{F}_n] = \sum_{k=0}^{n} \tilde\chi_k\varphi_k$$

as can be seen by applying a continuous linear functional μ on $\mathcal{C}[0,1]$ to both sides of (10) and by remarking that $(\mu,\tilde\chi_0,\dots,\tilde\chi_n)$ is a Gaussian array. By the convergence properties of vector martingales, we have therefore

(11)
$$B = \sum_{k=0}^{\infty} \tilde\chi_k\varphi_k$$

a.s. and in L^p. Since the family of partial sums of the series (9) has the same law as the sums of (11) the assertion is proved.

Such a representation of the Brownian motion

$$B = \sum_{k=0}^{\infty} \tilde\chi_k\varphi_k$$

allows us to define the shift, and the associated Gordin class clearly depends on the basis (χ_n) which is chosen.

4.b)

The case of Haar functions is particularly interesting. Let us put $\chi = 1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},1[}$ and

$$\chi_{m,k}(t) = 2^{\frac{m}{2}}\chi(2^m t - k) \tag{12}$$

$$\varphi_{m,k}(t) = \int_0^t \chi_{m,k}(s)\, ds \tag{13}$$

for $t \in \mathbb{R}_+$, $m \in \mathbf{Z}$, $k \in \mathbb{N}$.

The functions $(\chi_{m,k})_{m\in\mathbf{Z},\, k\in\mathbb{N}}$ form an orthonormal basis of $L^2(\mathbb{R}_+)$ and if $g_{m,k}$ are standard independent Gaussian variables, the Brownian motion can be represented by

$$B(\omega, t) = \sum_{m=-\infty}^{+\infty} (\sum_{k=0}^{\infty} \varphi_{m,k}(t) g_{m,k}(\omega))$$

and the scaling studied in paragraph II.3 is the mapping which transforms the sequence

$$(g_{m,k}(\omega))_{m,k}$$

into the sequence

$$(g_{m-1,k}(\omega))_{m,k}.$$

The space generated by the functions $(\chi_{m,k})_{m\geq 0,\, 0\leq k<2^m}$ is the subspace of $L^2[0,1]$ orthogonal to the constants, and the process

$$Z_t = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \varphi_{m,k}(t) g_{m,k} \tag{14}$$

is a standard Brownian bridge vanishing at zero and one. The representation (14) is unique and converges in $\mathcal{C}_{00}[0,1] = \{f \in \mathcal{C}[0,1]\ f(0) = f(1) = 0\}$. The functions $\varphi_{m,k}$ form a Schauder basis of this space. If $f \in \mathcal{C}_{00}[0,1]$ with

$$f(t) = \sum_{m=0}^{\infty} \sum_{h=0}^{2^m-1} \varphi_{m,k}(t) a_{m,k}(f) \tag{15}$$

there holds

$$a_{m,k}(f) = [2f(\frac{k}{2^m} + \frac{1}{2^{m+1}}) - f(\frac{k}{2^m}) - f(\frac{k+1}{2^m})]2^{\frac{m}{2}}.$$

The Banach spaces of Hölderian functions of exponent $\alpha \in]0,1[$ of $\mathcal{C}_{00}[0,1]$ can be interpreted in terms of spaces ℓ^∞ and c_0 on the sequences $(2^{m\alpha} a_{m,k})_{m,k}$ (cf [10]).

To approach a continuous function by a partial sum of the series (15) is convenient practically, and if we change the notations by putting $a_{2^m+k} = a_{m,k}$ $m \geq 0$, $k = 0, \ldots, 2^m - 1$ the simple shift on the a_n i.e., the transform

$$F(a_1, \ldots, a_n, \ldots) \longrightarrow F \circ \tau = F(a_0, a_1, \ldots, a_{n+1}, \ldots)$$

(which does not correspond to a scaling) is quite thrifty in random drawings. By proposition 11, a sufficient condition for a function F to be in the Gordin class for this transform is that it be in L^2 and possess partial derivatives such that

$$\sum_{k=0}^{\infty} (\sum_{i=k}^{\infty} \mathbb{E}(F_i'^2))^{\frac{1}{2}} < +\infty$$

where the expectation is taken on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), N(0,1)^{\otimes \mathbb{N}})$.
Example. For fixed $t \in [0,1]$, let us consider the functional

$$F(\omega) = \sum_{n \geq 0} \frac{1}{n+1} \sqrt{\varphi_n(t)} \left(\int_0^1 \chi_n(s)\, dZ_s \right)^2$$

which, with the preceding notations F can be written

$$F = \sum_{n \geq 0} \frac{1}{n+1} \sqrt{\varphi_n(t)}\; a_n^2.$$

Now F belongs to L^2 by the fact that the series

$$\sum_{n \geq 0} \frac{1}{(n+1)^2} \varphi_n(t)$$

converges and we have $F_i' = \frac{2}{i+1} \sqrt{\varphi_i(t)} a_i$ so that

$$\sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \mathbb{E} F_i'^2 = \sum_{k=0}^{\infty} \frac{4}{k+1} \varphi_k(t) < +\infty$$

because $(\frac{1}{k+1}) \in \ell^2$. And thus $F \in \mathcal{G}$.

References

[1] M.B. Alaya. On the simulation of expectations of random variables depending on a stopping time, Stoc. Anal. and Appl. (to appear)

[2] R. Azencott. Formule de Taylor stochastique et développement asymptotique d'intégrales de Feynmann.
237-285, in Sem. Prob. XVI supp. Géométrie diff. stoch. Lect. Notes in M. 921 Springer (1982)

[3] G. Ben Arous. Flots et séries de Taylor stochastiques.
Probab. th. rel. fields, 81, 29-77, (1989)

[4] E. Berger. An almost sure invariance principle for stationary ergodic sequences of Banach space valued random variables.
Prob. Th. Rel. F. 84, 161-201, (1990)

[5] N. Bouleau. On effective computation of expectations in large or infinite dimension.
J. of Computational and App. Math. 31,23-34, (1990)

[6] N. Bouleau. Irregular and simulatable functionals on Wiener space, in *Probabilités Numériques*, p39-53, INRIA, (1991)

[7] N. Bouleau, F. Hirsch. Formes de Dirichlet générales et densité de variables aléatoires sur l'espace de Wiener.
J. Funct. Analysis, vol 69, 227-259, (1986)

[8] N. Bouleau, F. Hirsch. Algèbre des structures de Dirichlet.
C. R. Acad. Sc. Paris t310, sI, 15-18, (1990)

[9] N. Bouleau, G. Pagès, J. Xiao. Extension des Méthodes de Monte Carlo accélérées.
Contrat DRET, CERMA Ecole Nationale des Ponts et Chaussées, Paris (1990)

[10] Z. Ciesielski. On the isomorphism of spaces H_α and m.
Bull. Acad. Polonaise des Sc., s. des Sc. math. astr. phys., VIII, n4, 217-222, (1960)

[11] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai. *Ergodic Theory.*
Springer (1982)

[12] H. Faure.Discrépance de suites associées à un système de numération (en dimension s).
Acta Arythm. 41, 337-351, (1982)

[13] M.I. Gordin. The central limit theorem for stationary sequences.
Soviet Math. Dokl. vol. 10, n5, 1174-1175, (1969)

[14] G. Halász. Remarks on the remainder in Birkhoff's ergodic theorem.
Acta Math. Acad. Hungar. 28,389-395, (1976)

[15] P. Hall, C.C. Heyde. *Martingale limit theory and applications.*
Acad. Press (1980)

[16] C.C. Heyde, D.J. Scott. Invariance principles for the law of iterated logarithm for martingales and processes with stationary increments.
Ann. Prob. vol 1, n3, 428-436, (1973)

[17] U. Krengel.On the speed of convergence in the ergodic theorem.
Monatsh. Math. 86, 3-6, (1978)

[18] U. Krengel. *Ergodic theorems.*
de Gruyter (1985)

[19] H. Kunita. Stochastic differential equations and stochastic flows of diffeomorphisms.
in Ecole d'été de St Flour XII, 143-303, Springer lect. notes in M. 1087, (1982)

[20] H. Niederreiter.Quasi-Monte Carlo methods and pseudo-random numbers.
Bull. Amer. Math. Soc. 84, 957-1041, (1978)

[21] H. Niederreiter.Low-discrepancy and low-dispersion sequences.
J. of Number Th. 30, n1, 51-70, (1988)

[22] P.K. Sarkar, M.A. Prasad. A comparative study of pseudo and quasi random sequences for solution of integral equations .
J. of computational Physics 68, 66-88, (1987)

[23] D.J. Scott. Central limit theorems for martingales and processes with stationary increments using a Skorokod representation approach.
Adv. Appl. Prob. 5,119-137, (1973)