A remark on random and equidistributed sequences

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If $\xi = (\xi_n)$ is an equidistributed sequence on the s-dimensional cube, the average of the values of a function f on the sequence ξ converges to the integral of f, for f Riemann integrable. We study here the quite natural fact that a slight random perturbation of the sequence ξ allows to integrate more irregular functions. The notions used are defined in [1] and [2].

1 Strongly μ-distributed sequences

Let μ be a probability measure on $[0,1]^s$, $s \in \mathbb{N}^*$, equipped with its natural topology. Except what envolves the convolution product, the sequel could be extended to any l.c.d. space.

Let (X_n) be a sequence of random variables with values in $[0,1]^s$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. "Almost surely", (a.s.), means $\mathbb{P}$-almost surely. If, with probability 1, the sequence $(X_n(\omega))$ is μ-distributed on $[0,1]^s$, i.e. if almost surely

$$(1) \qquad \forall f \in \mathcal{C}\left([0,1]^s, \mathbb{R}\right) \ \frac{1}{N} \sum_{n=0}^{N-1} f(X_n(\omega)) \to \int f d\mu$$

then (X_n) will be said to be almost surely μ-distributed.

Now, the example of i.i.d. random variables (X_n) with common law μ, shows that some random sequences have the following stronger property:

Definition 1 *A random sequence (X_n) is said to be strongly μ-distributed if, for every bounded μ-measurable function f from $[0,1]^s$ into $\mathbb{R}$,*

$$(2) \qquad \left(\frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \to \int f d\mu \ a.s.\right).$$

That a strongly μ-distributed sequence (X_n) be almost surely μ-distributed comes from the fact that if D is a countable dense subset of $\mathcal{C}([0,1]^s, \mathbb{R})$ such a sequence satisfies

$$\left(\forall f \in D \qquad \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \to \int f d\mu\right) a.s.$$

which implies easily

$$\left(\frac{1}{N} \sum_{n=0}^{N-1} \varepsilon_{X_n(\omega)} \to \mu \text{ narrowly} \right) a.s.$$

Proposition 1 *Let (X_n) be a sequence of random variables with values in $[0,1]^s$. If for every open set A in $[0,1]^s$ it holds*

$$\left(\frac{1}{N} \sum_{n=0}^{N-1} 1_A(X_n) \to \mu(A) \ a.s. \right) \tag{3}$$

then (X_n) is strongly μ-distributed.

We shall use the following lemma.

Lemma 1 *Property (3) for open sets implies property (2) for continuous f's.*

Proof. Let f be continuous. We may suppose $0 < f < 1$. Then

$$f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n} \right[\right) = G_{k+1} \backslash G_k$$

for an increasing sequence of open sets G_k. Approximating uniformly up to ε the function f by

$$f_K = \sum_{k=1}^{K} \alpha_k 1_{A_k} \text{ with } A_k = G_{k+1} \backslash G_k$$

the fact that

$$\left(\frac{1}{N} \sum_{n=0}^{N-1} f_K(X_n) \to \int f_K d\mu \quad a.s. \right)$$

gives

$$\int f d\mu - 2\varepsilon \leq \underline{\lim} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \leq \overline{\lim} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \leq \int f d\mu + 2\varepsilon$$

which shows the lemma. □

For the proposition, consider a bounded μ-measurable function f. By the Lusin property, for every sequence ε_k decreasing to zero, there exists a sequence of open sets G_k and a sequence of continuous functions f_k such that

$$\forall k. \qquad \mu(G_k) \leq \varepsilon_k, \qquad f = f_k \text{ outside } G_k, \qquad \| f_k \|_\infty \leq \| f_k \|_\infty .$$

Let us write

$$\frac{1}{N} \sum_{n=0}^{N-1} f(X_n) = \frac{1}{N} \sum_{n=0}^{N-1} f_k(X_n) + \frac{1}{N} \sum_{n=0}^{N-1} (f - f_k)(X_n).$$

By the lemma the following inequalities hold a.s. :

$$\overline{\lim}\frac{1}{N}\sum_{n=0}^{N-1} f(X_n) \quad \leq \quad \int f_k d\mu \quad + \quad \| f \|_\infty \mu(G_k)$$

$$\underline{\lim}\frac{1}{N}\sum_{n=0}^{N-1} f(X_n) \quad \geq \quad \int f_k d\mu \quad - \quad \| f \|_\infty \mu(G_k),$$

hense a.s. $\lim \frac{1}{N}\sum_{n=0}^{N} f(X_n) = \int f d\mu$. □

Remark. The property that (2) be satisfied for continuous functions f is weaker than the property (3) for open sets. It is indeed equivalent to the almost sure μ-distribution of (X_n). And there exist almost surely μ-distributed sequences which are not strongly μ-distributed : consider a deterministic equidistributed sequence $\xi = (\xi_n)$, take $X_n = \xi_n$, and take $f(x) = 1_{\{x:\exists n \xi_n = x\}}$

Proposition 2 *Let $\xi = (\xi_n)$ be an equidistributed sequence on $[0,1]^s$ and V_n be a sequence of i.i.d. random variables with absolutely continuous common law ν on $\mathbb{R}^s$. Then the sequence $X_n = \{\xi_n + V_n\}$ is strongly λ_s-distributed on the cube. (Here $\{x\}$ means the fractional part of x component by component, and λ_s is the Lebesgue measure in dimension s).*

Proof. Let us define absolutely continuous probability measures μ_n on the cube by

$$\mu_n(A) = \int_{\mathbb{R}^s} 1_A(\{\xi_n + x\})d\nu(x)$$

for every λ_s-measurable subset A of $[0,1]^s$.

Lemma 2 *For every f Lebesgue measurable and bounded on the cube,*

$$\lim_{N\uparrow\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu_n(f) = \lambda_s(f)$$

Proof. The function

$$g(y) = \int_{\mathbb{R}^s} f(\{y + x\})d\nu(x)$$

is continuous and bounded on $\mathbb{R}^s$ as convolution product of a function in $L^\infty(\mathbb{R}^s)$ by a function in $L^1(\mathbb{R}^s)$. Hence

$$\frac{1}{N}\sum_{n=0}^{N-1} g(\xi_n) \to \int_{[0,1]^s} g(y)dy = \int_{[0,1]^s} f(y)dy$$

□

Now, to achieve the proof of the proposition, we use a classical argument.

Let $\tilde{S}_N = \frac{1}{N}\sum_{n=0}^{N-1} (f(X_n) - \mu_n(f))$

a) It holds $\tilde{S}_{N^2} \to 0 \quad a.s.$

Indeed,

$$\begin{aligned}
\mathbb{P}(\exists N \geq m_1 : |\tilde{S}_{N^2}| > \varepsilon) &\leq \sum_{N=m_1}^{\infty} \frac{1}{\varepsilon^2} \frac{1}{N^4} \sum_{n=1}^{N^2} [\mu_n(f^2) - (\mu_n(f))^2] \\
&\leq \frac{1}{\varepsilon^2} \sum_{m_1}^{\infty} \frac{1}{N^2} \| f \|_\infty^2 \\
\text{thus} \qquad \overline{\lim}_N |\tilde{S}_{N^2}| &\leq \varepsilon \quad \text{a.s.}
\end{aligned}$$

b) It holds $\tilde{S}_N \to 0$ a.s.

Indeed, if $M^2 \leq N \leq (M+1)^2$

$$|\tilde{S}_N - \tilde{S}_{M^2}| \leq \left(\frac{|N - M^2|}{M^2} + N \frac{2M}{NM^2} \right) \| f \|_\infty \leq \frac{4}{M} \| f \|_\infty \to 0.$$

So by the lemma

$$\frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \to \int f d\lambda_s.$$

$\square$

2 Extension

The preceding proposition fails if ν is no more absolutely continuous, as seen when ν is a Dirac mass. But if ν does not charge polar sets it is possible to integrate with (X_n) quasi-continuous functions and even a little more. For conveniency we work now with the torus $T^s = (\mathbb{R}/\mathbf{Z})^s$ instead of the cube.

Proposition 3 *Let $\xi = (\xi_n)$ be an equidistributed sequence on T^s and V_n be a sequence of i.i.d. random variables with values in T^s with common law ν which does not charge sets of Newton capacity zero. Let $X_n = \xi_n + V_n$. Then*

$$\left(\frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \to \int_{T^s} f d\lambda_s \quad a.s. \right) \tag{4}$$

for every f from T^s into $\mathbb{R}$ with the following property :

$$\begin{cases} \forall \varepsilon > 0 \ \exists u, v & \textit{quasi-continuous and bounded such that} \\ u \leq f \leq v & \textit{and } \int_{T^s} (v-u) d\lambda_s \leq \varepsilon \end{cases} \tag{5}$$

<u>Proof</u>

a) Let us suppose first that the probability ν be a measure of finite energy integral for the classical Dirichlet structure on T^s and that f be a quasi-continuous version of an element of $H^1(T^s)$. Then approximating f in H^1 by functions f_n in H^1 and continuous, and denoting τ_x the translation by x, give

$$| < \tau_x f - \tau_x f_n, \nu > | \leq C_\nu \parallel \tau_x f - \tau_x f_n \parallel_{H^1} = C_\nu \parallel f - f_n \parallel_{H^1}$$

and that implies that the convolution product $f * \nu$ is a continuous function on T^s.

b) Let us suppose now f be quasi-continuous and bounded and the probability measure ν do not charge polar sets. Then there exist probability measures of finite energy integrals ν_p such that $\nu = \sum_p \alpha_p \nu_p$, $\alpha_p > 0$, $\Sigma \alpha_p = 1$.

Let $C_p = \parallel U_1 \nu_p \parallel_{H^1}$ be the energy-norms of the $\nu_p{}'s$.

Let $\varepsilon > 0$. For each p let us choose a continuous function f_p and an open set G_p such that

$$\parallel f_p \parallel_\infty \leq \parallel f \parallel_\infty, \quad f = f_p \text{ outside } G_p, \quad Cap(G_p) \leq \frac{\varepsilon}{2^p . C_p}.$$

We have

$$f * \nu = \sum_{p=0}^{\infty} \alpha_p (f_p * \nu_p + (f - f_p) * \nu_p)$$

but

$$|(f - f_p) * \nu_p| \leq 2 \parallel f \parallel_\infty e_1(G_p) * \nu_p \leq 2 \parallel f \parallel_\infty Cap(G_p).C_p = 2 \parallel f \parallel_\infty \frac{\varepsilon}{2^p}$$

where $e_1(G_p)$ is the equilibrium 1-potential of G_p.

It follows that $f * \nu$ is uniform limit of continuous functions and therefore is continuous.

c) Under the hypotheses of part b), putting

$$\mu_n = \tau_{\xi_n} \nu$$

it holds therefore by the equidistribution of (ξ_n)

$$\frac{1}{N} \sum_{n=0}^{N-1} \mu_n(f) \to \int_{T_s} f d\lambda_s$$

then the same argument that in the proof of proposition 2 shows that

$$\frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \to \int_{T_s} f d\lambda_s \ a.s.$$

d) At last, if f satisfies property (5), we have

$$\int_{T^s} u d\lambda_s \leq \underline{\lim} \frac{1}{N} \sum_{n=0}^{N-1} u(X_n) \leq \underline{\lim} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \leq$$

$$\leq \overline{\lim} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \leq \overline{\lim} \frac{1}{N} \sum_{n=0}^{N-1} v(X_n) = \int_{T_s} v d\lambda_s$$

almost surely. Hence

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) = \int f d\lambda_s \quad a.s.$$

$\square$

The functions satisfying (5) can be shown to be the functions which are bounded and finely continuous at λ_s-almost every point of T^s.

REFERENCES

[1] L. Kuipers and H.N. Niederreiter, Uniform distribution of sequences, Wiley (1974)

[2] M. Fukushima, Dirichlet forms and Markov processes, North-Holland (1980)