SOME THOUGHTS UPON AXIOMATIZED LANGUAGES WITH EXTENSION TOOLS: A Focus on Probability Theory and Error Calculus with Dirichlet Forms

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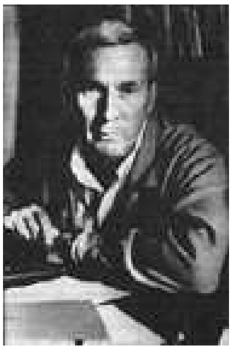
This year marks the centenary of the birth of Kolmogorov. It is a pleasure for me to acknowledge this occasion by giving a lecture in connection with his life's work. My purpose herein is certainly not to present a whole historical study of Kolmogorov's output, but rather provide some remarks on specific mathematical topics in which he played an active role. As you know, Kolmogorov produced some eight hundred publications encompassing all the main fields of mathematics: functional analysis, ergodic theory, turbulence, probability theory and statistics, and logic.

Andreï Nikolaïevitch Kolmogorov

He even delivred five seminal papers in the restricted domain of probability and stochastic processes foundations between 1931 to 1936, which make him one of the founders of the theory of continuous-time Markov processes or diffusions[1]. The subject I would like to discuss pertains to his famous "Grundbegriffe der Wahrscheinlichkeitsrechnung", which partially lies beyond his main body of mathematical work, in some respects it serves as *a manifesto* for how to tackle probability and probabilistic problems within the field of mathematics. I will be providing

[1]These significant articles are the following:
- Über die analytischen Methoden in der Wahrscheinlichkeitsrecgnung, 1931,
- Beitrage zur Masstheorie, 1933,
- Zur Theorie der stetigen zufälligen Prozessen, 1933,
- Grundbegriffe der Wahrscheinlichkeitsrechnung, 1993,
- Zur Theorie der Markoffschen Ketten, 1936.

some remarks on axiomatized languages that display the cases of both probability theory and of error calculus with Dirichlet forms. Based on these two examples, my aim is to emphasize the importance, in order for a language to be useful, of having an extension tool readily available.

I. A brief history of random sequences theory

In order to draw a comparison with Kolmogorov's axiomatic theory , it is helpful to explain what the "theory of random sequences" has become during the twentieth century. It did indeed serve an alternative way for incorporating probability into mathematics. Its purpose has been to describe a sequence of independent samples of a given quantity. In the simplest case, the theory pertains to samples of a random integer or even a single digit, so as to model the fair game of heads and tails, in the one-half / one-half perfectly symmetric case[2].

I.1 *The normal numbers of Borel (1909)*

It is now easy, and Emile Borel was already able to make the proof in 1909, that if we represent a real number over the unit interval $[0, 1]$ by its binary expansion

$$(a_0, a_1, \ldots) \in \{0,1\}^{\mathbb{N}} \qquad \longleftrightarrow \qquad x = \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}} \in [0, 1]$$

to the independent one-half / one-half distribution of the digits corresponds the Lebesgue measure on the interval $[0, 1]$.

Emile Borel

Emile Borel

As a consequence, for almost every real number $x \in [0, 1]$, the asymptotic frequency of any finite sequence is $\frac{1}{2}$ to the power of the sequence length. A real number fulfilling this property is said to be normal in the sense of Borel.

Now, proving that almost all real numbers are normal is just one step, another would be to exhibit such a number ! For the number π determining whether it is normal or not constitute a famous unsolved conjecture. Borel actually forwarded an effective, albeit sophisticated, construction of a normal number.

In 1933 however, Champernowne showed that the sequence obtained by writing the integer successively in dyadic representation is normal in the sense of Borel:

0 1 10 11 100 101 110 111 1000 1001 1010 1011 1100 1101 1110 1111 10000 ...

[2]This section is inspired by the very interesting study conducted by Claude Dellacherie entitled Nombres au hasard de Borel à Martin Löf" *Gazette des Mathématiciens* n^{0}11, 1978

This clearly displays that the concept of a normal number does not capture the idea of random sequence very well.

Already back in 1919, Von Mises had proposed an improvement toward the definition a random sequence, by means of a new concept of "collective"[3] which sought to describe a typical game of heads and tails. The idea is to ask for more than asymptotic averages and to think of a player gambling only at some random times depending on the evolution of the game : a sequence of digits is a "collective" if it satisfies the law of large numbers and if any subsequence obtained by a non-anticipative selection rule satisfies also the law of large numbers. This interesting approach, which portends the notion of "stopping time", does nevertheless have the disadvantage of being difficult to apply in practical terms. A. Wald, one of the founders of statistics and decision theory, proposed in 1937 the more precise notion of "collective relatively to a family of rules".


R. Von Mises


A. Wald


A. Church

Yet it would take the famous logician A. Church in 1940, with the first contribution from the field of logic into the debate, to propose an "absolute notion of collective" that uses the set of all *effective* non-anticipative rules as regards recursive functions theory. It thus appeared that the goal has been achieved by applying this new theory of effectiveness stemming from the recent works of the logicians in the 1930's (Gödel, Turing, Church).

Over this same period however, just prior to the Second World War, unsuspected new difficulties arose concerning the notion of "collective". In his work *Etude critique de la notion de collectif* (1939), Jean Ville demonstrated that random sequences possess some probabilistic properties that a "collective" may not always fulfill. A "collective" does not generally feature the right magnitude of fluctuations. In his argument Jean Ville uses the modern concept of mathematical *martingale* whose properties would be improved by J. L. Doob in particular during the 1950's. By transfering the term *martingale* from gambling to mathematics Ville added a spark to this notion and likely contributed to its subsequent importance.

We would have to wait until the 1960's to obtain a satisfactory answer to the question of random sequence. This answer came from mathematical logic and is owed to Martin Löf[4]. Roughly speaking, a random sequence successfully passes all effective statistical randomness tests. For a real number in $[0, 1]$, being random in the sense of Martin Löf signifies that it does not belong to any effective Lebesgue negligible set in $[0, 1]$. Such a number cannot be given by an algorithm, it is random in the sense of Church yet avoids Ville's critiques.

Although quite fascinating, the theory of random sequences remained useless for probabilists. The outstanding twentieth century development of probability theory, which began as a subsidiary field and became one of the primary domains of applied and even pure mathematics, is based on another approach : the construction of a *language* for handling probabilistic calculations.

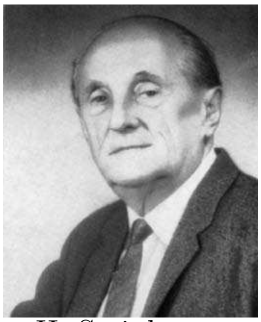
[3] Grundlagen der Wahrscheinlichkeitsrechnung" *Math. Zeitung* 5, 52-99, 1919.
[4] The definition of a random sequence" *Information and control* 9, 602-619, (1966).

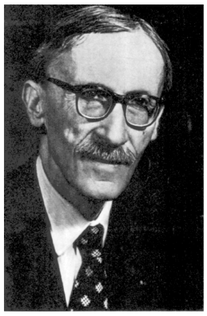
We would like to examine the reason behind this language's fruitfulness.

II. Axiomatization of Kolmogorov and σ-additivity

The paper entitled *Grundbegriffe der Wahrscheinlichkeitsrechnung* is an appeal to include probabilistic calculus into measure theory. Kolmogorov does not presume this idea is new, instead, he cites several authors who have already applied Lebesgue measure theory for probabilistic investigations, in particular Borel, Fréchet, Steinhaus, Lévy. He did proposes however new arguments, which proved to be highly valuable for subsequent research : the construction of probabilities on infinite dimensional spaces and the definition of conditional laws and conditional expectations using the Radon-Nikodym theorem.


M. Fréchet


H. Steinhaus


P. Lévy

He did not consider axiomatization as a pure formal system, but rather as a *language* that makes sense and that allows conducting thought and reasonning. In remarking that "every axiomatic theory admits, as is well known, an unlimited number of concrete interpretations"[5], he emphasizes the intuitive interpretation of his axiomatization. He went on to display a dictionary between random events and sets:

Theory of sets	*Random events*
1. A and B do not intersect, i.e. $AB = 0$	1. Events A and B are incompatible
2. $AB \dots N = 0$	2. Events $A, B, \dots, N$ are incompatible
3. $AB \dots N = X$	3. Event X is defined as the simultaneous occurrence of events $A, B, \dots, N$
4. $A \cup B \cup \dots \cup N = X$	4. Event X is defined as the occurence of at least one of the events $A, B, \dots, N$
5. The complementary set A^c	5. The non-occurence of event A
6. $A = 0$	6. Event A is impossible
7. $A = E$	7. Event A must occur
8. Disjoint decomposition of E $A_1 + A_2 + \dots + A_n = E$	8. Possible results $A_1, A_2, \dots, A_n$ of an experiment
9. B is a subset of A $B \subset A$	9. From the occurence of event B follows the inevitable occurence of A

For the axioms, the five first ones are elementary: *Let $\mathcal{F}$ a set of subsets of a set E.*

1. $\mathcal{F}$ is a field of sets

2. $\mathcal{F}$ contains the set E

3. To each set A in $\mathcal{F}$ is assigned a non negative real number $P(A)$, called the probability of event A

[5] We are in 1933 here and the works of Löwenheim and Skolem (1915-1920) are already known, which prove the existence of a countable model for any consistent theory.

4. $P(E)$ equals 1
5. If A and B have no elements in common, then $P(A+B) = P(A) + P(A)$

Kolmogorov underscores the importance of the sixth axiom : "In all future investigations we shall assume that besides axioms 1 through 5, another axiom holds true as well :
6. For a decreasing sequence of events

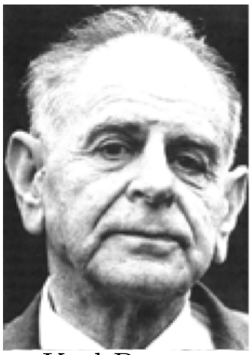
$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

in $\mathcal{F}$ for which $\cap_n A_n = 0$ the following relation holds $\lim_n P(A_n) = 0$*".*

This axiom of σ-additivity implies the probability P to be a measure in the sense of Lebesgue and Borel, which then embeds probability theory into measure theory :

probability	$\longleftrightarrow$	measure
event	$\longleftrightarrow$	measurable set
random variable	$\longleftrightarrow$	measurable function
expectation	$\longleftrightarrow$	integral
independence	$\longleftrightarrow$	product of measurable spaces
conditional expectation	$\longleftrightarrow$	Radon-Nikodym derivative

Let's remark that as late as 1938, the philosopher Karl Popper, whose main education stemmed from the field of psychology, was not convinced of the interest in placing probability theory within the framework of measure theory. Even in 1955, he still seemed proud to emphasize that a theory with only the first five axioms is more general. He wrote "Kolmogorov's system can be taken, however, as one of the interpretation of mine"[6].


Karl Popper

We know clearly now, thanks to the development of stochastic analysis over the twentieth century, that σ-additivity is the key tool making this language expansive. It allows defining the probability of events or the expectation of functions that are not given by simple closed formulae, but rather by limits. This fact is of absolutely prime importance since several mathematical objects are defined by limits and *the methods for defining these converging sequences of objects are not a priori restricted.*

This paves the way to the study of stochastic processes : if we know the probabilistic properties of a finite number of coordinates X_n on a product space, without the σ-additivity we cannot conclude anything about functions depending upon an infinite number of X_n's.

[6]K. Popper, *The logic of Scientific Discovery*, Hutchinson, 1972, p319.

Thanks to σ-additivity, connections with functional analysis may be developed, thereby giving rise to probabilistic interpretations. For example, potential theory is connected with Markov processes theory and martingales theory. Let's recall that J. L. Doob proved his extension of Fatou's lemma at the boundary from conical limits to non-tangential limits, first using a probabilistic argument and then, one year later, by means of a purely analytical approach.

III. Error calculus with Dirichlet forms

I would now like to present a more recent theory, in some repect a "cousin" to probability theory, which also possesses a means of extension providing it with remarkable power and fruitfulness. I have in mind the theory of Dirichlet forms with its interpretation in terms of errors. I shall begin with the ideas of Gauss about errors which are the elementary bases of the theory.

III.1. *Gauss formulae for the propagation of errors*

The ideas of Gauss were forwarded at the beginning of the XIXth century, at a time when several mathematicians were concerned with measurements errors, especially in the field of celestial mechanics. First of all, Legendre (*Nouvelles méthodes pour la dtermination des orbites des planètes*, 1805) proposed the least squares principle to choose the best value of a quantity obtained by several different measures.


F. Gauss in 1803


Legendre


Laplace

Secondly, Gauss himself (*Theoria motus coelestium*, 1809) elaborated the famous argument proving (with some implicit hypotheses) that once it has been assumed the arithmetic average is the best value to retain from among several results of quantity measurements, then, the probability law of the error is necessarily the normal law. This argument has been made more rigorous by Poincaré at the end of the century. Thirdly, Laplace (Théorie analytique des probabilités, 1811) demonstrated how the least squares method is usefull for solving linear systems when the number of equations does not agree with the number of unknowns.


F. Gauss in 1828


H. Poincaré

Within this same context, a few years later, Gauss became interested in the propagation of errors through calculations (*Theoria combinationis*, 1821) and stated the following problem :

Given a quantity $U = F(V_1, V_2, V_3, \ldots)$ function of the erroneous quantities $V_1, V_2, V_3, \ldots$, compute the potential quadratic error to expect on U with the quadratic errors $\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots$ on $V_1, V_2, V_3, \ldots$ being known and assumed small and independent.

His answer is the following formula :

$$\sigma_U^2 = (\frac{\partial F}{\partial V_1})^2\sigma_1^2 + (\frac{\partial F}{\partial V_2})^2\sigma_2^2 + (\frac{\partial F}{\partial V_3})^2\sigma_3^2 + \cdots \tag{1}$$

He also provides the covariance between an error on U and an error on another function of the V_i's.

Formula (1) displays a property which makes it much to be preferred in several respects to other formulae encountered in textbooks throughout the XIXth and XXth centuries. It features a coherence property. With a formula such as

$$\sigma_U = |\frac{\partial F}{\partial V_1}|\sigma_1 + |\frac{\partial F}{\partial V_2}|\sigma_1 + |\frac{\partial F}{\partial V_3}|\sigma_3 + \cdots \tag{2}$$

errors may depend on the way in which the function F is written. Already in dimension 2, we can note that if the indentity map were written as the composition of an injective linear map with its inverse, errors would be increased, which is hardly acceptable.

This difficulty does not arise in Gauss' calculus. Introducing the differential operator

$$L = \frac{1}{2}\sigma_1^2\frac{\partial^2}{\partial V_1^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2}{\partial V_2^2} + \cdots$$

and supposing the functions to be smooth, we remark that formula (1) can be written as

$$\sigma_U^2 = L(F^2) - 2FLF$$

and coherence follows from the transport of a differential operator by an application. If u and v are regular injective mappings, then, in denoting the operator $\varphi \longrightarrow L(\varphi \circ u) \circ u^{-1}$ by $\theta_u L$, we obtain $\theta_{v \circ u} L = \theta_v(\theta_u L)$.

The errors on $V_1, V_2, V_3, \ldots$ are not necessarily supposed to be independent nor constant and may depend on $V_1, V_2, V_3, \ldots$ Considering a field of positive symmetric matrices $\sigma_{ij}(v_1, v_2, \ldots))$ on $\mathbb{R}^n$ representing the conditional variances and covariances of errors on $V_1, V_2, V_3, \ldots$ given the values $v_1, v_2, v_3, \ldots$ of $V_1, V_2, V_3, \ldots$, then the error on $U = F(V_1, V_2, V_3, \ldots)$ given the values $v_1, v_2, v_3, \ldots$ of $V_1, V_2, V_3, \ldots$ is

$$\sigma_U^2 = \sum_{ij} \frac{\partial F}{\partial V_1}(v_1, v_2, v_3, \ldots)\frac{\partial F}{\partial V_2}(v_1, v_2, v_3, \ldots)\sigma_{ij}(v_1, v_2, v_3, \ldots)$$

which depends solely on F as mapping. This is the general form of the error calculus à la Gauss.

III.2 *Error propagation through calculations : the error calculus based on Dirichlet forms*

The error calculus of Gauss contains the limitation of supposing that both the function F and the random variables $V_1, V_2, V_3, \ldots$ are explicitely known. In probabilistic modelling however, we are often confronted by a situation in which all the random variables, functions and covariances matrices are given by limits. For such situations, a means of extension thereby becomes essential.

Let the quantities be defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The quadratic error on a random variable X is itself random, let us denote it $\Gamma[X]$. Intuitively speaking we still assume that the errors are infinitely small, even though this assumption does not appear in the notation. It is as though an infinitely small unit were available for measuring errors fixed throughout the entire problem. The extension tool lies in the following : we assume that if $X_n \to X$ in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ and if the error $\Gamma[X_m - X_n]$ on $X_m - X_n$ can be made as small as we wish in $L^1(\Omega, \mathcal{A}, \mathbb{P})$ for m, n large enough, then the error $\Gamma[X_n - X]$ on $X_n - X$ goes to zero in L^1.

This idea can be interpreted as a reinforced coherence principle, it means that the error on X is attached to X and furthermore, if the sequence of pairs (X_n, error on X_n) converges suitably, it converges necessarily to a pair (X, error on X).

The axiomatization of these idea involves the notion of closed quadratic differential form or Dirichlet form :

An error structure is a term

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$$

where $(\Omega, \mathcal{A}, \mathbb{P})$ *is a probability space, satisfying the following properties*

1) $\mathbb{D}$ *is a dense subvector space of* $L^2(\Omega, \mathcal{A}, \mathbb{P})$

2) Γ *is a positive symmetric bilinear map from* $\mathbb{D} \times \mathbb{D}$ *into* $L^1(\mathbb{P})$ *fulfilling the functional calculus of class* $\mathcal{C}^1 \cap Lip$*, which means that if* $u \in \mathbb{D}^m$, $v \in \mathbb{D}^n$*, for* F *and* G *of class* $\mathcal{C}^1$ *and Lipschitz from* $\mathbb{R}^m$ *[resp.* $\mathbb{R}^n$*] into* $\mathbb{R}$*, one has* $F \circ u \in \mathbb{D}$, $G \circ v \in \mathbb{D}$ *and*

$$\Gamma[F \circ u, G \circ v] = \sum_{ij} F_i' \circ u \;\; G_i' \circ v \;\; \Gamma[u_i, v_j] \qquad \mathbb{P} - a.s.$$

3) the bilinear form $\mathcal{E}[f, g] = \mathbb{E}[\Gamma[f, g]]$ *is closed, i.e.* $\mathbb{D}$ *is complete under the norm*

$$\|.\|_{\mathbb{D}} = (\|.\|_{L^2}^2 + \mathcal{E}[.])^{1/2}.$$

(then the form $\mathcal{E}$ is a Dirichlet form.)

The main benefit of the extension tool is that error theory based on Dirichlet forms extends to the infinite dimension, which allows for error calculus on stochastic processes (especially on Brownian motion but also on the Poisson space), provides several new results on stochastic differential equations, and gives applications to fluctuations in physics and to sensitivity analysis in finance[7].

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[7]See the books of Malliavin, Fukushima, Ikeda-Watanabe, Bismut, Bichteler-Gravereau-Jacod, Watanabe, Strook, Bouleau-Hirsch, Ma-Röckner, Nualart, Øksendal & al., Ustunel-Zakai, etc. and the papers of several hundred of researchers.
Regarding the interpretation in terms of error propagation, see N. Bouleau, *Error Calculus for Finance and Physics, the Language of Dirichlet Forms*, De Gruyter, 235p, 2003.

IV. Languages with extension tools and Richard's paradox

In comparing Kolmogorov's axiomatic theory of probability with the random sequences theory, we have emphasized for the former

- the presence of a language (syntax and semantics)
- a powerful extension tool yielding, in some sense, risky results.

This may be placed in analogy with the language of Analysis that handles real numbers. We know, indeed, the existence of $2^{\aleph_0}$ real numbers, although only $\aleph_0$ will ever be indicated with precision. This is the situation highlighted by Richard's paradox (1905).

Jules Antoine Richard (1862-1956)

The paradox can be stated as follows :

Let's write all of the pairs using the 28 characters (the 26 letters, the space and the comma to separate words) in alphabetic order; then the triples, and so forth, all finite sequences. Every definition of a real number will appear in the list.

Let's cross out all the sequences which are not definitions of real numbers.

Let u_1 be the real number defined by the first remaining definition;

u_2 the one defined by the following definition;

u_3 the one defined by the third one;

and so forth.

We thus obtain all the real numbers defined by finitely many words, written in a particular order. The number a *given by the definition "the number without entire part, each decimal of which immediately follows the decimal of same rank of the number of same rank in the sequence (u_n), the zero being considered as following the numeral nine" should be in the list, but cannot be equal to any number u_n.*

Mathematical logic is capable, of course, of overcoming the apparent contradiction in this paradox. Nevertheless, a true phenomenon has indeed been highlighted : there are $2^{\aleph_0}$ real numbers, we dont know how large this cardinal $2^{\aleph_0}$ actually is, and only $\aleph_0$ real numbers will ever be precisely defined.

In such a situation, we have opted for Analysis a language with an extension tool : the Cauchy criterion. This strategy allows handling real numbers defined by limits regardless of the construction of the used convergent sequence. This tool has then been carried from the real case to the functional case by the notions of Hilbert space and Banach space which are certainly ones of the most powerful concepts of XXth century Analysis.