AMELIORATION DE SIMULATIONS MONTE CARLO PAR DES FORMES DE DIRICHLET
IMPROVING MONTE CARLO SIMULATIONS BY DIRICHLETR FORMS

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Abstract. Equipping the probability space with a local Dirichlet form with square field operator Γ and generator A allows to improve Monte Carlo simulations of expectations and densities as soon as we are able to simulate a random variable X together with $\Gamma[X]$ and $A[X]$. We give examples on the Wiener space, on the Poisson space and on the Monte Carlo space. When X is real-valued we give an explicit formula yielding the density at the speed of the law of large numbers. *To cite this article: N. Bouleau, C. R. Acad. Sci. Paris, Ser. I ... (2005).*
Résumé. Nous montrons que, dans les situations où l'espace de probabilité est équipé d'une forme de Dirichlet locale avec carré du champ Γ et générateur A, la possibilité de simuler une variable aléatoire X ainsi que $\Gamma[X]$ et $A[X]$ permet d'accélérer le calcul de l'espérance de X et de sa densité. Nous donnons des exemples dans les cas de l'espace de Wiener, de l'espace de Poisson et de l'espace de Monte Carlo. Lorsque X est à valeurs réelles nous donnons une formule explicite permettant d'obtenir la densité à la vitesse de la loi des grands nombres. *Pour citer cet article : N. Bouleau, C. R. Acad. Sci. Paris, Ser. I ... (2005).*

1 Introduction

The efficiency of Dirichlet forms is known in order to obtain existence of densities under weak hypotheses (cf [3]). We show here that they are still usefull for the computation of such densities. Our framework is an error structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$, i.e. a probability space equipped with a local Dirichlet form $(\mathcal{E}, \mathbb{D})$ admitting a square field operator Γ (cf [2],[3]). The associated L^2-generator is denoted $(A, \mathcal{D}A)$.

We consider a random variable $X \in \mathcal{D}A$ such that X, $\Gamma[X]$ and $A[X]$ are simulatable.

Example 1. Wiener space.

Let us consider a stochastic differential equation (sde) defined on the Wiener space equipped with the Ornstein-Uhlenbeck error structure (cf [2],[3])

$$X_t = x_0 + \int_0^t \sigma(X_s, s)dB_s + \int_0^t r(X_s, s)ds$$

By the functional calculus for the operators Γ and A, if the coefficients are smooth, the triplet

$(X_t, \Gamma[X_t], A[X_t])$ is a diffusion, solution to the equation

$$\begin{pmatrix} X_t \\ \Gamma[X_t] \\ A[X_t] \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix} + \int_0^t \begin{bmatrix} \sigma(X_s,s) & 0 & 0 \\ 0 & 2\sigma'_x(X_s,s) & 0 \\ -\frac{1}{2}\sigma(X_s,s) & \frac{1}{2}\sigma''_{x^2}(X_s,s) & \sigma'_x(X_s,s) \end{bmatrix} \begin{pmatrix} 1 \\ \Gamma[X_s] \\ A[X_s] \end{pmatrix} dB_s$$

$$+ \int_0^t \begin{bmatrix} r(X_s,s) & 0 & 0 \\ \sigma^2(X_s,s) & 2r'_x(X_s,s) + \sigma'^2_x(X_s,s) & 0 \\ 0 & \frac{1}{2}r''_{x^2}(X_s,s) & r'_x(X_s,s) \end{bmatrix} \begin{pmatrix} 1 \\ \Gamma[X_s] \\ A[X_s] \end{pmatrix} ds$$

Denoting Y_t the column vector $(X_t, \Gamma[X_t], A[X_t])$ this equation writes $Y_t = Y_0 + \int_0^t a(Y_s,s)dB_s + \int_0^t b(Y_s,s)ds$ and applying the Euler scheme with mesh $\frac{1}{n}$ on [0,T] : $Y_t^n = Y_0 + \int_0^t a(Y^n_{\frac{[ns]}{n}}, \frac{[ns]}{n})dB_s + \int_0^t b(Y^n_{\frac{[ns]}{n}}, \frac{[ns]}{n})ds$. yields a process $Y_t^n = (X_t^n, (\Gamma[X])_t^n, (A[X])_t^n)^t$ for which it is easy to verify that $\Gamma[X_t^n] = (\Gamma[X])_t^n$ and $A[X_t^n] = (A[X])_t^n$.

By known results (cf [1] [4] [5]) in order to compute the density of X_T, we may approximate it by the solution X_T^n of the Euler scheme. Thus, we have then to simulate X_T^n in a situation where we are also able to simulate $\Gamma[X_T^n]$ and $A[X_T^n]$.

Example 2. Poisson space.

Let $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu, \mathbf{d}, \gamma)$ be an error structure on $\mathbb{R}^d$, $(a, \mathcal{D}a)$ its generator. Let N be a Poisson point process defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with state space $\mathbb{R}^d$ and intensity measure μ. $(\Omega, \mathcal{A}, \mathbb{P})$ may be equipped with a so-called "white" error structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ (cf [2]) with the following properties : if $h \in \mathcal{D}a$ then $N(h) \in \mathcal{D}A$, $\Gamma[N(h)] = N(\gamma[h])$ and $A[N(h)] = N(a[h])$.

In order to simulate $N(\xi)$ we have only to draw a finite (poissonian) number of i.i.d. random variables with law μ so that we are indeed in a situation where $N(h)$, $\Gamma[N(h)]$, and $A[N(h)]$ are simulatable.

Example 3. Monte Carlo space.

Let $X = F(U_0, U_1, \ldots, U_m, \ldots; V_0, V_1, \ldots, V_n, \ldots)$ be a random variable defined on the space $([0,1]^{\mathbb{N}}, \mathcal{B}([0,1]^{\mathbb{N}}), dx^{\mathbb{N}}) \times ([0,1]^{\mathbb{N}}, \mathcal{B}([0,1]^{\mathbb{N}}), dx^{\mathbb{N}})$ where the U_i are the coordinates of the first factor with respect to which X is supposed to be regular, V_j the ones of the second factor with respect to which X is supposed to be irregular or discontinuous (rejection method, etc.).

Let us put on the U_i the following error structure

$$([0,1]^{\mathbb{N}}, \mathcal{B}([0,1]^{\mathbb{N}}), dx^{\mathbb{N}}, \mathbb{D}, \Gamma) = ([0,1], \mathcal{B}([0,1]), dx, \mathbf{d}, \gamma)^{\mathbb{N}}$$

where $(\mathbf{d}, \gamma)$ is the closure of the operator $\gamma[u](x) = x^2(1-x)^2 u'^2(x)$ for $u \in \mathcal{C}^1([0,1])$.

Then under natural regularity assumptions, we have $\Gamma[X] = \sum_{i=0}^{\infty} F_i'^2 U_i^2 (1-U_i)^2$. and

$$A[X] = \sum_{i=0}^{\infty} (\frac{1}{2} F''_{ii} U_i^2 (1-U_i)^2 + F'_i U_i (1-U_i)(1-2U_i))$$

so that X, $\Gamma[X]$ and $A[X]$ are simulatable.

2 Diminishing the bias

Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ be an error structure. For $X \in (\mathcal{D}A)^d$, $\underline{\underline{\mathrm{var}}}[X]$ denotes the covariance matrix of X, $A[X]$ the column vector with components $(A[X_1], \ldots, A[X_d])$, $\underline{\underline{\Gamma}}[X]$ is the matrix $\Gamma[X_i, X_j]$ and $\sqrt{\underline{\underline{\Gamma}}[X]}$ denotes the positive symmetric square root of $\underline{\underline{\Gamma}}[X]$.

We follow the idea that the random variable $X + \varepsilon A[X] + \sqrt{\varepsilon}\sqrt{\underline{\underline{\Gamma}}[X]}\, G$ where G is an exogeneous independent reduced Gaussian variable, has almost the same law as X. Starting from the fundamental relation of the functional calculus on A, an integration by parts argument gives the following lemma.

Lemma 2.1 *Let $X \in (\mathcal{D}A)^d$. we suppose that X possesses a conditional density $\eta(x, \gamma, a)$ given $\underline{\underline{\Gamma}}[X]=\gamma$ et $A[X]=a$ such that $x \mapsto \eta(x, \gamma, a)$ be $\mathcal{C}^2$ with bounded derivatives. Then $\forall x \in \mathbb{R}^d$*

$$\mathbb{E}[-(A[X])^t \nabla_x \eta(x, \underline{\underline{\Gamma}}[X], A[X]) + \frac{1}{2}\text{trace}\left(\underline{\underline{\Gamma}}[X].\text{Hess}_x \eta\right)(x, \underline{\underline{\Gamma}}[X], A[X])] = 0.$$

Theorem 2.2 *Let g be the density of the normal law. Let X be as in the preceding lemma with density f, the conditional density $\eta(x, \gamma, a)$ being $\mathcal{C}^3$ bounded with bounded derivatives. When $\varepsilon \to 0$, the quantity*

$$\frac{1}{\varepsilon^2}\left(\mathbb{E}[g(x - X - \varepsilon A[X], \varepsilon\underline{\underline{\Gamma}}[X])] - f(x)\right)$$

has a finite limit equal to

$$\frac{1}{2}\mathbb{E}[(A[X])^t(\text{Hess}_x\eta)(x, \underline{\underline{\Gamma}}[X], A[X])A[X] - \sum_{i,j,k=1}^{d} A[X_i]\Gamma[X_j, X_k]\eta'''_{x_i x_j x_k}(x, \underline{\underline{\Gamma}}[X], A[X])].$$

Proof. If we write $\mathbb{E}[g(x - X - \varepsilon A[X], \varepsilon\underline{\underline{\Gamma}}[X])] = \int \mu(d\gamma, da) \int g(x - y - \varepsilon a, \varepsilon\gamma)\eta(y, \gamma, a)dy$ $= \int \mu(d\gamma, da)\mathbb{E}\eta(x - \varepsilon a - \sqrt{\varepsilon}\sqrt{\gamma}G, \gamma, a)$ where G is an $\mathbb{R}^d$-valued reduced Gaussian variable, and if we expand with respect to $\sqrt{\varepsilon}$ and take the expectation, terms in $\sqrt{\varepsilon}$ and $\varepsilon\sqrt{\varepsilon}$ vanish because G and G^3 are centered and the term in ε vanishes also thanks to the lemma. This gives the result.

About the variance, we obtain

Proposition 2.3 *Let X satisfying the assumptions of the lemma and such that $(\text{det}\underline{\underline{\Gamma}}[X])^{-\frac{1}{2}} \in L^1$*, then

$$\lim_{\varepsilon\to 0} \varepsilon^{d/2}\mathbb{E}g^2(x-X-\varepsilon A[X], \varepsilon\underline{\underline{\Gamma}}[X]) = \lim_{\varepsilon\to 0} \varepsilon^{d/2}\text{var}g(x-X-\varepsilon A[X], \varepsilon\underline{\underline{\Gamma}}[X]) = \mathbb{E}\left[\frac{\eta(x, \underline{\underline{\Gamma}}[X], A[X])}{(4\pi)^{d/2}\sqrt{\text{det}\underline{\underline{\Gamma}}[X]}}\right].$$

The quantity $\mathbb{E}g(x - X - \varepsilon A[X], \varepsilon\underline{\underline{\Gamma}}[X])$ is obtained by simulation with the law of large numbers, so that the approximation $\hat{f}$ of the density f of X is

$$\hat{f}(x) = \frac{1}{N}\sum_{n=1}^{N} g(x - X_n - \varepsilon(A[X])_n, \varepsilon(\underline{\underline{\Gamma}}[X])_n)$$

where the indices n denote independent drawings. The preceding results show that, with respect to the usual kernel method, the speed, in the sense of the L^2-norm, is the same as if the dimension was divided by 2.

3 Direct formulae

In the case where X is real-valued, if in addition to X, $A[X]$, $\Gamma[X]$ we are able to simulate $\Gamma[X, \frac{1}{X}]$, it is possible to obtain the dentity of X at the speed of the law of large numbers thanks to the following formulae :

Theorem 3.1 *a) If $X \in \mathcal{D}A$ with $\Gamma[X] \in \mathbb{D}$ and $\Gamma[X] > 0$ a.s. then X has a density f which possesses an l.s.c. version $\tilde{f}$ given by*

$$\tilde{f}(x) = \lim_{\varepsilon \downarrow 0} \uparrow \frac{1}{2}\mathbb{E}\left(\mathrm{sign}(x-X)(\Gamma[X, \frac{1}{\varepsilon + \Gamma[X]}] + \frac{2A[X]}{\varepsilon + \Gamma[X]})\right).$$

b) If in addition $\frac{1}{\Gamma[x]} \in \mathbb{D}$, then X has a density f which is absolutely continuous and given by

$$f(x) = \frac{1}{2}\mathbb{E}\left(\mathrm{sign}(x-X)(\Gamma[X, \frac{1}{\Gamma[X]}] + \frac{2A[X]}{\Gamma[X]})\right).$$

The proof is based on the relation

$$\mathbb{E}[\varphi''(X)\frac{\Gamma[X]}{\varepsilon + \Gamma[X]}] = -\mathbb{E}[\varphi'(X)(\Gamma[X, \frac{1}{\varepsilon + \Gamma[X]}] + \frac{2A[X]}{\varepsilon + \Gamma[X]})].$$

valid for any $\mathcal{C}^2$-function φ with bounded derivatives which comes from the functional calculus using the general relation $\mathcal{E}[u,v] = - < A[u], v > \forall u \in \mathcal{D}A \; \forall v \in \mathbb{D}$, and then applying it with $\varphi = \sqrt{\lambda^2 + (y-x)^2}$ in order to get the monotone convergence result.

Under the hypotheses of theorem 3.1, as soon as $G \in \mathbb{D} \cap L^\infty$, there are similar formulae for conditional expectations $\mathbb{E}[G|X = x]$:

$$f(x)\mathbb{E}[G|X = x] = \frac{1}{2}\mathbb{E}\left(\mathrm{sign}(x-X)(\Gamma[X, \frac{G}{\Gamma[X]}] + \frac{2GA[X]}{\Gamma[X]})\right).$$

Let us finally remark that in these formulae, the factor on the right of $\mathrm{sign}(x - X)$ is centered and a variance optimisation may be performed thanks to an arbitrary deterministic function as done in [4] where direct formulae similar to those of section 3 are given in the case of the Wiener space involving Skorokhod integrals instead of Dirichlet forms.

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