AMELIORATION DE SIMULATIONS MONTE CARLO PAR DES FORMES DE DIRICHLET
IMPROVING MONTE CARLO SIMULATIONS BY DIRICHLET FORMS

Nicolas Bouleau

ENPC, ParisTech

Abstract. Equipping the probability space with a local Dirichlet form with square field operator $\Gamma$ and generator $A$ allows to improve Monte Carlo simulations of expectations and densities as soon as we are able to simulate a random variable $X$ together with $\Gamma[X]$ and $A[X]$. We give examples on the Wiener space, on the Poisson space and on the Monte Carlo space. When $X$ is real-valued we give an explicit formula yielding the density at the speed of the law of large numbers. To cite this article: N. Bouleau, C. R. Acad. Sci. Paris, Ser. I ... (2005).


1 Introduction

The efficiency of Dirichlet forms is known in order to obtain existence of densities under weak hypotheses (cf [3]). We show here that they are still usefull for the computation of such densities. Our framework is an error structure $(\Omega, A, \mathbb{P}, \mathbb{D}, \Gamma)$, i.e. a probability space equipped with a local Dirichlet form $(\mathcal{E}, \mathbb{D})$ admitting a square field operator $\Gamma$ (cf [2],[3]). The associated $L^2$-generator is denoted $(A, \mathcal{D}A)$.

We consider a random variable $X \in \mathcal{D}A$ such that $X$, $\Gamma[X]$ and $A[X]$ are simulatable.

Example 1. Wiener space.

Let us consider a stochastic differential equation (sde) defined on the Wiener space equipped with the Ornstein-Uhlenbeck error structure (cf [2],[3])

$$X_t = x_0 + \int_0^t \sigma(X_s, s)dB_s + \int_0^t r(X_s, s)ds$$

By the functional calculus for the operators $\Gamma$ and $A$, if the coefficients are smooth, the triplet
$$(X_t, \Gamma[X_i], A[X_i])$$ is a diffusion, solution to the equation

$$\begin{pmatrix} X_t \\ \Gamma[X_i] \\ A[X_i] \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix} + \int_0^t \begin{bmatrix} \sigma(X_s, s) & 0 & 0 \\ 0 & 2\sigma'(X_s, s) & 0 \\ -\frac{1}{2}\sigma(X_s, s) & \frac{1}{2}\sigma''(X_s, s) & \sigma'_x(X_s, s) \end{bmatrix} \begin{pmatrix} \Gamma[X_s] \\ A[X_s] \end{pmatrix} dB_s + \int_0^t \begin{bmatrix} r(X_s, s) & \sigma^2(X_s, s) & 0 \\ 0 & 2r'_x(X_s, s) + \sigma^2_x(X_s, s) & 0 \\ 0 & \frac{1}{2}r''(X_s, s) & r'_x(X_s, s) \end{bmatrix} \begin{pmatrix} \Gamma[X_s] \\ A[X_s] \end{pmatrix} ds$$

Denoting $Y_t$ the column vector $(X_t, \Gamma[X_i], A[X_i])$ this equation writes $Y_t = Y_0 + \int_0^t a(Y_s, s) dB_s + \int_0^t b(Y_s, s) ds$ and applying the Euler scheme with mesh $\frac{1}{n}$ on $[0, T]$ : $Y^n_T = Y_0 + \int_0^T a(Y^n_{\lceil sn \rceil}, \frac{\lceil sn \rceil}{n}) dB_s + \int_0^T b(Y^n_{\lceil sn \rceil}, \frac{\lceil sn \rceil}{n}) ds$. yields a process $Y^n_t = (X^n_t, (\Gamma[X^n])_t, (A[X^n])_t)$ for which it is easy to verify that $\Gamma[X^n] = (\Gamma[X])_t^n$ and $A[X^n] = (A[X])_t^n$.

By known results (cf [1] [4] [5]) in order to compute the density of $X_T$, we may approximate it by the solution $X^n_T$ of the Euler scheme. Thus, we have then to simulate $X^n_T$ in a situation where we are also able to simulate $\Gamma[X^n_T]$ and $A[X^n_T]$.

Example 2. Poisson space.

Let $(\mathbb{R}^d, B(\mathbb{R}^d), \mu, \sigma, \gamma)$ be an error structure on $\mathbb{R}^d$, $(a, D\mathcal{A})$ its generator. Let $N$ be a Poisson point process defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with state space $\mathbb{R}^d$ and intensity measure $\mu$. $(\Omega, \mathcal{A}, \mathbb{P})$ may be equipped with a so-called “white” error structure $(\Omega, \mathcal{A}, \mathbb{P}, D, \Gamma)$ (cf [2]) with the following properties : if $h \in D\mathcal{A}$ then $N(h) \in D\mathcal{A}$, $\Gamma[N(h)] = N(\gamma[h])$ and $A[N(h)] = N(a[h])$.

In order to simulate $N(\xi)$ we have only to draw a finite (poissonian) number of i.i.d. random variables with law $\mu$ so that we are indeed in a situation where $N(h)$, $\Gamma[N(h)]$, and $A[N(h)]$ are simulatable.

Example 3. Monte Carlo space.

Let $X = (X_0, U_1, \ldots, U_m, \ldots; V_0, V_1, \ldots, V_n, \ldots)$ be a random variable defined on the space $(([0, 1]^N, B([0, 1]^N), dx)^N) \times ([0, 1]^N, B([0, 1]^N), dx)^N$ where the $U_i$ are the coordinates of the first factor with respect to which $X$ is supposed to be regular, $V_j$ the ones of the second factor with respect to which $X$ is supposed to be irregular or discontinuous (rejection method, etc.).

Let us put on the $U_i$ the following error structure

$$([0, 1]^N, B([0, 1]^N), dx^N, D, \Gamma) = ([0, 1], B([0, 1]), dx, d, \gamma)^N$$

where $(d, \gamma)$ is the closure of the operator $\gamma[u](x) = x^2(1-x)^2u^2(x)$ for $u \in \mathcal{C}^1([0, 1])$.

Then under natural regularity assumptions, we have $\Gamma[X] = \sum_{i=0}^\infty F^n_i U_i^2 (1-U_i)^2$ and $A[X] = \sum_{i=0}^\infty \frac{1}{2} F^n_i U_i^2 (1-U_i)^2 + F^n_i U_i (1-U_i) (1-2U_i)$ so that $X$, $\Gamma[X]$ and $A[X]$ are simulatable.

2 Diminishing the bias

Let $(\Omega, \mathcal{A}, \mathbb{P}, D, \Gamma)$ be an error structure. For $X \in (D\mathcal{A})^d$, $\text{var}[X]$ denotes the covariance matrix of $X$, $A[X]$ the column vector with components $(A[X_1], \ldots, A[X_d])$, $\Sigma[X]$ is the matrix $\Gamma[X_1, X_j]$ and $\sqrt{\Sigma[X]}$ denotes the positive symmetric square root of $\Sigma[X]$. 
We follow the idea that the random variable \( X + \varepsilon A[X] + \sqrt{\varepsilon} \sqrt{\Gamma[X]} G \) where \( G \) is an exogenous independent reduced Gaussian variable, has almost the same law as \( X \). Starting from the fundamental relation of the functional calculus on \( A \), an integration by parts argument gives the following lemma.

**Lemma 2.1** Let \( X \in (DA)^d \). we suppose that \( X \) possesses a conditional density \( \eta(x, \gamma, a) \) given \( \Gamma[X] = \gamma \) and \( A[X] = a \) such that \( x \mapsto \eta(x, \gamma, a) \) be \( C^2 \) with bounded derivatives. Then \( \forall x \in \mathbb{R}^d \)

\[
\mathbb{E}[-(A[X])^t \nabla_x \eta(x, \Gamma[X], A[X]) + \frac{1}{2} \text{trace} \left( \Gamma[X] \text{Hess}_x \eta \right) (x, \Gamma[X], A[X])] = 0.
\]

**Theorem 2.2** Let \( g \) be the density of the normal law. Let \( X \) be as in the preceding lemma with density \( f \), the conditional density \( \eta(x, \gamma, a) \) being \( C^3 \) bounded with bounded derivatives. When \( \varepsilon \to 0 \), the quantity

\[
\frac{1}{\varepsilon^2} \left( \mathbb{E}[g(x - X - \varepsilon A[X], \varepsilon \Gamma[X])] - f(x) \right)
\]

has a finite limit equal to

\[
\frac{1}{2} \mathbb{E}[\langle (A[X])^t (\text{Hess}_x \eta) \rangle (x, \Gamma[X], A[X])] A[X] - \sum_{i,j,k=1}^d A[X_i] \Gamma[X_j, X_k] \eta''_{x, x, x_k} (x, \Gamma[X], A[X])].
\]

**Proof.** If we write \( \mathbb{E}[g(x - X - \varepsilon A[X], \varepsilon \Gamma[X])] = \int \mu(d\gamma, da) \int g(x - y - \varepsilon a, \varepsilon \gamma) \eta(y, \gamma, a) dy = \int \mu(d\gamma, da) \mathbb{E}\eta(x - \varepsilon a - \sqrt{\varepsilon} \sqrt{G}, \gamma, a) \) where \( G \) is an \( \mathbb{R}^d \)-valued reduced Gaussian variable, and if we expand with respect to \( \sqrt{\varepsilon} \) and take the expectation, terms in \( \varepsilon \) and \( \varepsilon \sqrt{\varepsilon} \) vanish because \( G \) and \( G^3 \) are centered and the term in \( \varepsilon \) vanishes also thanks to the lemma. This gives the result.

About the variance, we obtain

**Proposition 2.3** Let \( X \) satisfying the assumptions of the lemma and such that \((\det \Gamma[X])^{-\frac{1}{2}} \in L^1\), then

\[
\lim_{\varepsilon \to 0} \varepsilon^{d/2} \mathbb{E}g^2(x - X - \varepsilon A[X], \varepsilon \Gamma[X]) = \lim_{\varepsilon \to 0} \varepsilon^{d/2} \text{var}g(x - X - \varepsilon A[X], \varepsilon \Gamma[X]) = \mathbb{E} \left[ \eta(x, \Gamma[X], A[X]) \right] (4\pi)^{d/2} \sqrt{\det \Gamma[X]}.
\]

The quantity \( \mathbb{E}g(x - X - \varepsilon A[X], \varepsilon \Gamma[X]) \) is obtained by simulation with the law of large numbers, so that the approximation \( \hat{f} \) of the density \( f \) of \( X \) is

\[
\hat{f}(x) = \frac{1}{N} \sum_{n=1}^N g(x - X_n - \varepsilon (A[X])_n, \varepsilon (\Gamma[X])_n)
\]

where the indices \( n \) denote independent drawings. The preceding results show that, with respect to the usual kernel method, the speed, in the sense of the \( L^2 \)-norm, is the same as if the dimension was divided by 2.

### 3 Direct formulae

In the case where \( X \) is real-valued, if in addition to \( X, A[X], \Gamma[X] \) we are able to simulate \( \Gamma[X, \frac{1}{4}] \), it is possible to obtain the density of \( X \) at the speed of the law of large numbers thanks
to the following formulae:

**Theorem 3.1**  
a) If $X \in \mathcal{D}A$ with $\Gamma[X] \in \mathbb{D}$ and $\Gamma[X] > 0$ a.s. then $X$ has a density $f$ which possesses an l.s.c. version $\tilde{f}$ given by

$$\tilde{f}(x) = \lim_{\varepsilon \to 0} \frac{1}{2} \mathbb{E}\left(\text{sign}(x - X)(\Gamma[X, \frac{1}{\varepsilon + \Gamma[X]}] + \frac{2A[X]}{\varepsilon + \Gamma[X]})\right).$$

b) If in addition $\frac{1}{|X|} \in \mathbb{D}$, then $X$ has a density $f$ which is absolutely continuous and given by

$$f(x) = \frac{1}{2} \mathbb{E}\left(\text{sign}(x - X)(\Gamma[X, \frac{1}{\Gamma[X]}] + \frac{2A[X]}{\Gamma[X]})\right).$$

The proof is based on the relation

$$\mathbb{E}[\varphi''(X)\frac{\Gamma[X]}{\varepsilon + \Gamma[X]}] = -\mathbb{E}[\varphi'(X)(\frac{1}{\varepsilon + \Gamma[X]} + \frac{2A[X]}{\varepsilon + \Gamma[X]})],$$

valid for any $C^2$-function $\varphi$ with bounded derivatives which comes from the functional calculus using the general relation $\mathcal{E}[u, v] = -<A[u], v> \forall u \in \mathcal{D}A \forall v \in \mathbb{D}$, and then applying it with $\varphi = \sqrt{\lambda^2 + (y - x)^2}$ in order to get the monotone convergence result.

Under the hypotheses of theorem 3.1, as soon as $G \in \mathbb{D} \cap L^\infty$, there are similar formulae for conditional expectations $\mathbb{E}[G|X = x]$

$$f(x)\mathbb{E}[G|X = x] = \frac{1}{2} \mathbb{E}\left(\text{sign}(x - X)(\Gamma[X, \frac{G}{\Gamma[X]}] + \frac{2GA[X]}{\Gamma[X]})\right).$$

Let us finally remark that in these formulae, the factor on the right of $\text{sign}(x - X)$ is centered and a variance optimisation may be performed thanks to an arbitrary deterministic function as done in [4] where direct formulae similar to those of section 3 are given in the case of the Wiener space involving Skorokhod integrals instead of Dirichlet forms.

**References**


