An extension to the Wiener space of the arbitrary functions principle

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Abstract

The arbitrary functions principle says that the fractional part of nX converges stably to an independent random variable uniformly distributed on the unit interval, as soon as the random variable X possesses a density or a characteristic function vanishing at infinity. We prove a similar property for random variables defined on the Wiener space when the stochastic measure dB_s is crumpled on itself. *To cite this article: N. Bouleau, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

Résumé

Le principe des fonctions arbitraires dit que la partie fractionnaire de nX converge stablement vers une variable aléatoire indépendante uniformément répartie sur $[0, 1]$ dès que X a une densité ou seulement une fonction caractéristique tendant vers zéro à l'infini. Nous établissons une propriété analogue pour des variables aléatoires définies sur l'espace du mouvement brownien par repliement de la mesure stochastique dB_s sur elle-même. *Pour citer cet article : N. Bouleau, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

1. Introduction

Let us denote $\{x\}$ the fractional part of the real number x and $\stackrel{d}{\Longrightarrow}$ the weak convergence of random variables. Let (X, Y) be a pair of random variables with values in $\mathbb{R} \times \mathbb{R}^r$, we refer to the following property or its extensions as the arbitrary functions principle:

$$(\{nX\}, Y) \quad \stackrel{d}{\Longrightarrow} \quad (U, Y) \tag{1}$$

where U is uniformly distributed on $[0, 1]$ independent of Y.

This property is satisfied when X has a density or more generally a characteristic function vanishing at infinity. (cf [5] Chap. VIII §92 and §93, [2], [4]). It yields an approximation property of X by the random

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variable $X_n = X - \frac{1}{n}\{nX\} = \frac{[nX]}{n}$ where $[x]$ denotes the entire part of x:

Proposition 1. *Let X be a real random variable with density and Y a random variable with values in $\mathbb{R}^r$. Let $X_n = \frac{[nX]}{n}$*

a) For all $\varphi \in \mathcal{C}^1 \bigcap \mathrm{Lip}(\mathbb{R})$ and for all integrable random variable Z,

$$(n(\varphi(X_n) - \varphi(X)), Y) \quad \stackrel{d}{\Longrightarrow} \quad (-U\varphi'(X), Y)$$

$$n^2\mathbb{E}[(\varphi(X_n) - \varphi(X))^2 Z] \quad \rightarrow \quad \frac{1}{3}\mathbb{E}[\varphi'^2(X)Z]$$

where U is uniformly distributed on $[0,1]$ independent of (X, Y).

b) $\forall \psi \in L^1([0,1])$

$$(\psi(n(X_n - X)), Y) \quad \stackrel{d}{\Longrightarrow} \quad (\psi(-U), Y)$$

under any probability measure $\tilde{\mathbb{P}} \ll \mathbb{P}$.

We extend such results to random variables defined on the Wiener space.

2. Periodic isometries.

Let (B_t) be a standard d-dimensional Brownian motion and let m be the Wiener measure, law of B. Let $t \mapsto M_t$ be a bounded deterministic measurable map, periodic with unit period, into the space of orthogonal $d \times d$-matrices such that $\int_0^1 M_s ds = 0$ (e.g. a rotation in $\mathbb{R}^d$ of angle $2\pi t$). The transform $B_t \mapsto \int_0^t M_s dB_s$ defines an isometric endomorphism in $L^p(m), 1 \leq p \leq \infty$. Let be $M_n(s) = M(ns)$ and $T_n = T_{M_n}$. The transposed of the matrix N is denoted N^*.

Proposition 2. *Let be $X \in L^1(m)$. Let $\tilde{m}$ be a probability measure absolutely continuous w.r. to m. Under $\tilde{m}$ we have*

$$(T_n(X), B) \quad \stackrel{d}{\Longrightarrow} \quad (X(w), B).$$

The weak convergence acts on $\mathbb{R} \times \mathcal{C}([0,1])$ and $X(w)$ denotes a random variable with the same law as X had under m function of a Brownian motion W independent of B.

Proof. a) If $X = \exp\{i\int_0^1 \xi.dB + \frac{1}{2}\int_0^1 |\xi|^2 ds\}$ for some element $\xi \in L^2([0,1], \mathbb{R}^d)$, we have $T_n(X) = \exp\{i\int_0^1 \xi_s^* M_n(s) dB_s + \frac{1}{2}\int_0^1 |\xi|^2 ds\}$.

Putting $Z_t^n = \int_0^t \xi_s^* M_n(s) dB_s$ gives $\langle Z^n, Z^n\rangle_t = \int_0^t \xi_s^* M_n(s) M_n^*(s) \xi_s ds = \int_0^t |\xi|^2(s) ds$ which is a continuous function. Now by proposition 1, $\int_0^t \xi_s^* M_n(s) ds \to \int_0^t \xi_s^* ds \int_0^1 M_n(s) ds = 0$. which implies by Ascoli theorem $sup_t |\int_0^t \xi_s^* M_n(s) ds| \to 0$. The argument of H. Rootzén [6] applies and yields $(\int_0^\cdot \xi^* M_n dB, B) \stackrel{d}{\Longrightarrow} (\int_0^\cdot \xi.dW, B)$ giving the result in this case by continuity of the exponential function.

b) When $X \in L^1(m)$, we approximate X by X_k linear combination of exponentials of the preceding type and consider the caracteristic functions. The inequality

$$|\mathbb{E}[e^{iuT_n(X)} e^{i\int h.dB} - \mathbb{E}[e^{iuT_n(X_k)} e^{i\int h.dB}]| \leq |u|\mathbb{E}|T_n(X) - T_n(X_k)| = |u| \, \|X - X_k\|_{L^1}$$

gives the result.

c) This extends to the case $\tilde{m} \ll m$ by the properties of stable convergence. $\diamond$

3. Approximation of the Ornstein-Uhlenbeck structure.

From now on, we assume for simplicity that (B) is one-dimensional. Let θ be a periodic real function with unit period such that $\int_0^1 \theta(s)ds = 0$ and $\int_0^1 \theta^2(s)ds = 1$. We consider the transform R_n of the space $L^2_{\mathbb{C}}(m)$ defined by its action on the Wiener chaos:

If $X = \int_{s_1<\cdots<s_k} \hat{f}(s_1,\ldots,s_k)dB_{s_1}\ldots dB_{s_k}$ for $\hat{f}\in L^2_{sym}([0,1]^k,\mathbb{C})$,

$$R_n(X) = \int\limits_{s_1<\cdots<s_k} \hat{f}(s_1,\ldots,s_k)e^{i\frac{1}{n}\theta(ns_1)}dB_{s_1}\ldots e^{i\frac{1}{n}\theta(ns_k)}dB_{s_k}.$$

R_n is an isometry from $L^2_{\mathbb{C}}(m)$ into itself. From $n(e^{\frac{i}{n}\sum_{p=1}^k\theta(ns_p)}-1) = i\sum_{p=1}^k\theta(ns_p)\int_0^1 e^{\alpha\frac{i}{n}\sum_p\theta(ns_p)}d\alpha$ it follows that if X belongs to the k-th chaos

$$\|n(R_n(X)-X)\|^2_{L^2} \le k^2\|X\|^2_{L^2}\|\theta\|^2_\infty.$$

In other words, denoting A the Ornstein-Uhlenbeck operator, $X\in\mathcal{D}(A)$ implies

$$\|n(R_n(X)-X)\|_{L^2} \le 2\|AX\|_{L^2}\|\theta\|_\infty$$

and this leads to

Proposition 3. *If $X\in\mathcal{D}(A)$*

$$(-in(R_n(X)-X),B) \quad \stackrel{d}{\Longrightarrow} \quad (X^\#(\omega,w),B)$$

where W is an Brownian motion independent of B and $X^\# = \int_0^1 D_sX\,dW_s$.

Proof. If X belongs to the k-th chaos, expanding the exponential by its Taylor series gives

$$n(R_n(X)-X) = i\int\limits_{s_1<\cdots<s_k}\hat{f}(s_1,\ldots,s_k)\sum_{p=1}^k\theta(ns_p)dB_{s_1}\ldots dB_{s_k} + Q_n$$

with $\|Q_n\|^2 \le \frac{1}{4n}k^2\|\theta\|^2_\infty\|X\|^2$.

Then using that $\int_{s_1<\cdots<s_p<\cdots<s_k} h(s_1,\ldots,s_k)\theta(ns_p)dB_{s_1}\ldots dB_{s_p}\ldots dB_{s_k}$ converges stably to $\int_{s_1<\cdots<s_p<\cdots<s_k} h(s_1,\ldots,s_k)dB_{s_1}\ldots dW_{s_p}\ldots dB_{s_k}$ one gets

$$-in(R_n(X)-X) \quad \stackrel{s}{\Longrightarrow} \quad \begin{array}{l} \displaystyle\int\limits_{t<s_2<\cdots<s_k}\hat{f}(t,s_2,\ldots,s_k)dW_tdB_{s_2}\ldots dB_{s_k} \\ \displaystyle +\int\limits_{s_1<t<\cdots<s_k}\hat{f}(s_1,t,\ldots,s_k)dB_{s_1}dW_t\ldots dB_{s_k} \\ +\cdots \\ \displaystyle +\int\limits_{s_1<\cdots<s_{k-1}<t}\hat{f}(s_1,\ldots,s_{k-1},t)dB_{s_1}\ldots dB_{s_{k-1}}dW_t \end{array}$$

which equals $\int D_s(X)dW_s = X^\#$.

The general case in obtained by approximation of X by X_k for the $\mathbb{D}^{2,2}$ norm and the same argument as in the proof of proposition 2 by the caracteristic functions gives the result. $\diamond$

By the properties of stable convergence, the weak convergence of prop. 3 also holds under $\tilde{m}\ll m$. By similar computations we obtain

Proposition 4. $\forall X \in \mathcal{D}(A)$
$$n^2\mathbb{E}[|R_n(X) - X|^2] \to 2\mathcal{E}[X]$$
where $\mathcal{E}$ is the Dirichlet form associated with the Ornstein-Uhlenbeck operator.

Following the same lines, it is possible to show that the theoretical $\overline{A}$ and practical $\underline{A}$ bias operators (cf. [1]) defined on the algebra $\mathcal{L}\{e^{\int \xi dB}\ ;\ \xi \in \mathcal{C}^1\}$ by

$$n^2\mathbb{E}[(R_n(X) - X)Y] =< \overline{A}X, Y >_{L^2(m)}$$
$$n^2\mathbb{E}[(X - R_n(X))R_n(Y)] =< \underline{A}X, Y >_{L^2(m)}$$

are defined and equal to A.

Comment. The preceding properties are very similar to the results concerning the weak asymptotic error for the resolution of SDEs by the Euler scheme, involving also an "extra"-Brownian motion (cf. [3]).

Nevertheless these results do not use the arbitrary functions principle because a convergence like $(n\int_0^{\cdot}(s-\frac{[ns]}{n})dB_s, B) \stackrel{d}{\Longrightarrow} (\frac{1}{\sqrt{12}}W+\frac{1}{2}B, B)$ is hidden by a dominating phenomenon $(\sqrt{n}\int_0^{\cdot}(B_s-B_{\frac{[ns]}{n}}dB_s, B)$ $\stackrel{d}{\Longrightarrow} (\frac{1}{\sqrt{2}}\tilde{W}, B)$ due to the fact that when a sequence of variables in the second (or higher order) chaos converges stably to a Gaussian variable, this one appears to be independent of the fisrt chaos and therefore of B.

The arbitrary functions principle is slightly different, it is a crumpling of the random orthogonal measure dB_s on itself. This operates even on the first chaos. Concerning the solution of SDEs by the Euler scheme, it is in force for SDEs of the form

$$\begin{cases} X_t^1 = x_0^1 + \int_0^t f^{11}(X_s^2)dB_s + \int_0^t f^{12}(X_s^1, X_s^2)ds \\ X_t^2 = x_0^2 + \int_0^t f^{22}(X_s^1, X_s^2)ds \end{cases}$$

where X^1 is with values in $\mathbb{R}^{k_1}$, X^2 in $\mathbb{R}^{k_2}$, B in $\mathbb{R}^d$ and f^{ij} are matrices with suitable dimensions which are encountered for the description of mechanical systems under noisy sollicitations when the noise depend only on the position of the system and the time. In such equations, integration by parts reduces the stochastic integrals to ordinary integrals and it may be shown that solved by the Euler scheme they present a weak asymptotic error in $\frac{1}{n}$ instead of $\frac{1}{\sqrt{n}}$ as usual.

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