Energy image density property and the lent particle method for Poisson measures

Nicolas BOULEAU and Laurent DENIS

Abstract

We introduce a new approach to absolute continuity of laws of Poisson functionals. It is based on the *energy image density* property for Dirichlet forms. The associated gradient is a local operator and gives rise to a nice formula called *the lent particle method* which consists in adding a particle and taking it back after some calculation.

AMS 2000 subject classifications: Primary 60G57, 60H05 ; secondary 60J45,60G51
Keywords: Poisson functionals, Dirichlet Forms, Energy Image Density, Lévy Processes, Gradient

1 Introduction

The aim of this article is to improve some tools provided by Dirichlet forms for studying the regularity of Poisson functionals. First, the *energy image density property* (EID) which guarantees the existence of a density for $\mathbb{R}^d$-valued random variables whose carré du champ matrix is almost surely regular. Second, the Lipschitz functional calculus for a *local gradient* satisfying the chain rule, which yields regularity results for functionals of Lévy processes.

For a local Dirichlet structure with carré du champ, the energy image density property is always true for real-valued functions in the domain of the form (Bouleau [5], Bouleau-Hirsch [10] Chap. I §7). It has been conjectured in 1986 (Bouleau-Hirsch [9] p251) that (EID) were true for any $\mathbb{R}^d$-valued function whose components are in the domain of the form for any local Dirichlet structure with carré du champ. This has been shown for the Wiener space equipped with the Ornstein-Uhlenbeck form and for some other structures by Bouleau-Hirsch (cf. [10] Chap. II §5 and Chap. V example 2.2.4) and also for the Poisson space by A. Coquio [12] when the intensity measure is the Lebesgue measure on an open set, but this conjecture being at present neither refuted nor proved in full generality, it has to be established in every particular setting. We will proceed in two steps : first (Part 2) we prove sufficient conditions for (EID) based mainly on a study of Shiqi Song [31] using a characterization of Albeverio-Röckner [2], then (Part 4) we show that the Dirichlet structure on the Poisson space obtained from a Dirichlet structure on the states space inherits from that one the (EID) property.

If we think a local Dirichlet structure with carré du champ $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ as a description of the Markovian movement of a particle on the space $(X, \mathcal{X})$ whose transition semi-group p_t is symmetric with respect to the measure ν and strongly continuous on $L^2(\nu)$, the construction of the Poisson measure allows to associate to this structure a structure on the Poisson space $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ which describes similarly the movement of a family of independent identical particles whose initial law is the Poisson measure with intensity ν. This construction is ancient and may be performed in several ways.

The simplest one, from the point of view of Dirichlet forms, is based on products and follows faithfully the probabilistic construction (Bouleau [6], Denis [14], Bouleau [7] Chap. VI §3). The cuts that this method introduces are harmless for the functional calculus with the carré du champ Γ, but it does not clearly show what happens for the generator and its domain.

Another way consists in using the transition semi-groups (Martin-Löf [20], Wu [33], partially Bichteler-Gravereaux-Jacod [4], Surgailis [32]). It is supposed that there exists a Markov process x_t with values in X whose transition semi-group π_t is a version of p_t (cf. Ma-Röckner [22] Chap. IV §3), the process starting at the point z is denoted by $x_t(z)$ and a probability space $(W, \mathcal{W}, \Pi)$ is considered where a family $(x_t(z))_{z\in X}$ of independent processes is realized. For a symmetric function F, the new semi-group P_t is directly defined by

$$(P_tF)(z_1, \ldots, z_n, \ldots) = \int F(x_t(z_1), \ldots, x_t(z_n), \ldots)\, d\Pi$$

Choosing as initial law the Poisson measure with intensity ν on $(X, \mathcal{X})$, it is possible to show the symmetry and the strong continuity of P_t. This method, based on a deep physical intuition, often used in the study of infinite systems of particles, needs a careful formalization in order to prevent any drawback from the fact that the mapping $X \ni z \mapsto x_t(z)$ is not measurable in general due to the independence. For extensions of this method see [19].

In any case, the formulas involving the carré du champ and the gradient require computations and key results on *the configuration space* from which the construction may be performed as starting point. From this point of view the works are based either on the chaos decomposition (Nualart-Vives [25]) and provide tools in analogy with the Malliavin calculus on Wiener space, but non-local (Picard [26], Ishikawa-Kunita [17], Picard [27]) or on the expression of the generator on a sufficiently rich class and Friedrichs' argument (cf. what may be called the German school in spite of its cosmopolitanism, especially [1] and [23]).

We will follow a way close to this last one. Several representations of the gradient are possible (Privault [28]) and we will propose here a new one with the advantages of both locality (chain rule) and simplicity on usual functionals. It provides a new method of computing the carré du champ Γ — the lent particle method — whose efficiency is displayed on some examples. With respect to the announcement [8] we have introduced a clearer new notation, the operator ε^- being shared from the integration by N. Applications to stochastic differential equations driven by Lévy processes will be gathered in an other article.

Contents

4. (EID) property on the upper space from (EID) property on the bottom space and the domain $\mathbb{D}_{loc}$

5. Two examples.

2 The Energy Image Density property (EID)

In this part we give sufficient conditions for a Dirichlet structure to fulfill (EID) property. These conditions concern finite dimensional cases and will be extended to the infinite dimensional setting of Poisson measures in Part 4.

For each positive integer d, we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-field on $\mathbb{R}^d$ and by λ^d the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and as usually when no confusion is possible, we shall denote it by dx. For f measurable $f_*\nu$ denotes the image of the measure ν by f.

For a σ-finite measure Λ on some measurable space, a Dirichlet form on $L^2(\Lambda)$ with carré du champ γ is said to satisfy (EID) if for any d and for any $\mathbb{R}^d$-valued function U whose components are in the domain of the form

$$U_*[(\det\gamma[U, U^t]) \cdot \Lambda] \ll \lambda^d$$

where det denotes the determinant.

2.1 A sufficient condition on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$

Given $r \in \mathbb{N}^*$, for any $\mathcal{B}(\mathbb{R}^r)$-measurable function $u : \mathbb{R}^r \to \mathbb{R}$, all $i \in \{1, \cdots, r\}$ and all $\bar{x} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r) \in \mathbb{R}^{r-1}$, we consider $u^{(i)}_{\bar{x}} : \mathbb{R} \to \mathbb{R}$ the function defined by

$$\forall s \in \mathbb{R}\ , u^{(i)}_{\bar{x}}(s) = u((\bar{x}, s)_i),$$

where $(\bar{x}, s)_i = (x_1, \cdots x_{i-1}, s, x_{i+1}, \cdots, x_r)$.
Conversely if $x = (x_1, \cdots, x_r)$ belongs to $\mathbb{R}^r$ we set $x^i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r)$.
Then following standard notation, for any $\mathcal{B}(\mathbb{R})$ measurable function $\rho : \mathbb{R} \to \mathbb{R}^+$, we denote by $R(\rho)$ the largest open set on which ρ^{-1} is locally integrable.
Finally, we are given $k : \mathbb{R}^r \to \mathbb{R}^+$ a Borel function and $\xi = (\xi_{ij})_{1 \leqslant i,j \leqslant r}$ an $\mathbb{R}^{r\times r}$-valued and symmetric Borel function.

We make the following assumptions which generalize Hamza's condition (cf. Fukushima-Oshima-Takeda [16] Chap. 3 §3.1 (3°), p105):

Hypotheses (HG):

1. For any $i \in \{1, \cdots, r\}$ and λ^{r-1}-almost all $\bar{x} \in \{y \in \mathbb{R}^{r-1} : \int_{\mathbb{R}} k^{(i)}_y(s)\, ds > 0\}$, $k^{(i)}_{\bar{x}} = 0$, λ^1-a.e. on $\mathbb{R} \setminus R(k^{(i)}_{\bar{x}})$.

2. There exists an open set $O \subset \mathbb{R}^r$ such that $\lambda^r(\mathbb{R}^r \setminus O) = 0$ and ξ is locally elliptic on O in the sense that for any compact subset K, in O, there exists a positive constant c_K such that
$$\forall x \in K, \forall c \in \mathbb{R}^r\ \sum_{i,j=1}^{r} \xi_{ij}(x) c_i c_j \geqslant c_K |c|^2.$$

Following Albeverio-Röckner, Theorems 3.2 and 5.3 in [2] and also Röckner-Wielens Section 4 in [29], we consider $\mathbf{d}$ the set of $\mathcal{B}(\mathbb{R}^r)$-measurable functions u in $L^2(kdx)$, such that for any $i \in \{1, \cdots, r\}$, and λ^{r-1}-almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u_{\bar{x}}^{(i)}$ has an absolute continuous version $\tilde{u}_{\bar{x}}^{(i)}$ on $R(k_{\bar{x}}^{(i)})$ (defined λ^1-a.e.) and such that $\sum_{i,j} \xi_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \in L^1(kdx)$, where

$$\frac{\partial u}{\partial x_i} = \frac{d\tilde{u}_{\bar{x}}^{(i)}}{ds}.$$

Sometimes, we will simply denote $\frac{\partial}{\partial x_i}$ by ∂_i.
And we consider the following bilinear form on $\mathbf{d}$:

$$\forall u, v \in \mathbf{d},\ e[u, v] = \frac{1}{2} \int_{\mathbb{R}^r} \sum_{i,j} \xi_{ij}(x) \partial_i u(x) \partial_j v(x) k(x)\, dx.$$

As usual we shall simply denote $e[u, u]$ by $e[u]$. We have

Proposition 1. *$(\mathbf{d}, e)$ is a local Dirichlet form on $L^2(kdx)$ which admits a carré du champ operator γ given by*

$$\forall u, v \in \mathbf{d},\ \gamma[u, v] = \sum_{i,j} \xi_{ij} \partial_i u \partial_j v.$$

Proof. All is clear excepted the fact that e is a closed form on $\mathbf{d}$. To prove it, let us consider a sequence $(u_n)_{n\in\mathbb{N}^*}$ of elements in $\mathbf{d}$ which converges to u in $L^2(kdx)$ and such that $\lim_{n,m\to+\infty} e[u_n - u_m] = 0$. Let $W \subset O$, an open subset which satisfies $\bar{W} \subset O$ and such that $\bar{W}$ is compact.

Let $\mathbf{d}_W$ be the set of $\mathcal{B}(\mathbb{R}^r)$-measurable functions u in $L^2(\mathbf{1}_W \times k\, dx)$, such that for any $i \in \{1, \cdots, r\}$, and λ^{r-1}-almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u_{\bar{x}}^{(i)}$ has an absolute continuous version $\tilde{u}_{\bar{x}}^{(i)}$ on $R((\mathbf{1}_W \times k)_{\bar{x}}^{(i)})$ and such that $\sum_{i,j} \xi_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \in L^1(\mathbf{1}_W \times k\, dx)$, equipped with the bilinear form

$$\forall u, v \in \mathbf{d}_W, e_W[u, v] = \frac{1}{2} \int_W \sum_i \partial_i u(x) \partial_i v(x) k(x)\, dx = \frac{1}{2} \int_W \nabla u(x) \cdot \nabla v(x) k(x)\, dx.$$

One can easily verify, since W is an open set, that for all $\bar{x} \in \mathbb{R}^{r-1}$

$$S_{\bar{x}}^i(W) \cap R(k_{\bar{x}}^{(i)}) \subset R((\mathbf{1}_W \times k)_{\bar{x}}^{(i)}), \tag{1}$$

where $S_{\bar{x}}^i(W)$ is the open set $\{s \in \mathbb{R} :\ (\bar{x}, s)_i \in W\}$.
Then it is clear that the function $\mathbf{1}_W \times k$ satisfies property 1. of (HG) and as a consequence of Theorems 3.2 and 5.3 in [2], $(\mathbf{d}_W, e_W)$ is a Dirichlet form on $L^2(\mathbf{1}_W \times kdx)$.
We have for all $n, m \in \mathbb{N}$

$$e_W(u_n - u_m) = \frac{1}{2} \int_W |\nabla u_n(x) - \nabla u_m(x)|^2\, k(x) dx \leqslant \frac{1}{c_{\bar{W}}} e(u_n - u_m),$$

as $(\mathbf{d}, e_W)$ is a closed form, we conclude that u belongs to $\mathbf{d}_W$.
Consider now an exhaustive sequence (W_m), of relatively compact open sets in O such that for all $m \in \mathbb{N}$, $\bar{W}_m \subset W_{m+1} \subset O$. We have that for all m, u belongs to $\mathbf{d}_{W_m}$ hence by

Theorems 3.2 and 5.3 in [2], for all $i \in \{1, \cdots, r\}$, and λ^{r-1}-almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u_{\bar{x}}^{(i)}$ has an absolute continuous version on $\bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)_{\bar{x}}^{(i)})$. Using relation (1), we have

$$S_{\bar{x}}^i(O) \cap R(k_{\bar{x}}^{(i)}) = \bigcup_{m=1}^{+\infty} S_{\bar{x}}^i(W_m) \cap R(k_{\bar{x}}^{(i)}) \subset \bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)_{\bar{x}}^{(i)}).$$

As $\lambda^r(\mathbb{R}^r \setminus O) = 0$, we get that for almost all $\bar{x} \in \mathbb{R}^{r-1}$, $\bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)_{\bar{x}}^{(i)}) = R(k_{\bar{x}}^{(i)})$ λ^1-a.e. Moreover, by a diagonal extraction, we have that a subsequence of (∇u_n) converges kdx-a.e. to ∇u, so by Fatou's Lemma, we conclude that $u \in \mathbf{d}$ and then $\lim_{n \to +\infty} e[u_n - u] = 0$, which is the desired result. $\square$

For any $d \in \mathbb{N}^*$, if $u = (u_1, \cdots, u_d)$ belongs to $\mathbf{d}^d$, we shall denote by $\gamma[u]$ the matrix $(\gamma[u_i, u_j])_{1 \leqslant i,j \leqslant d}$.

Theorem 2. *(EID) property : the structure* $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r), k\,dx, \mathbf{d}, \gamma)$ *satisfies*

$$\forall d \in \mathbb{N}^* \;\; \forall u \in \mathbf{d}^d \quad u_*[(\det \gamma[u]) \cdot kdx] \ll \lambda^d.$$

Proof. Let us mention that a proof was given by S. Song in [31] Theorem 16, in the more general case of *classical Dirichlet forms*. Following his ideas, we present here a shorter proof.
The proof is based on the *co-area formula* stated by H. Federer in [15], Theorems 3.2.5 and 3.2.12.
We first introduce the subset $A \subset \mathbb{R}^r$:

$$A = \{x \in \mathbb{R}^r : \; x_i \in R(k_{x^i}^{(i)}) \; i = 1, \cdots, r\}.$$

As a consequence of property 1. of (HG), $\int_{A^c} k(x)dx = 0$.
Let $u = (u_1, \cdots, u_d) \in \mathbf{d}^d$. We follow the notation and definitions introduced by Bouleau-Hirsch in [10], Chap. II Section 5.1.
Thanks to Theorem 3.2 in [2] and Stepanoff's Theorem (see Theorem 3.1.9 in [15] or Remark 5.1.2 Chap. II in [10]), it is clear that for almost all $a \in A$, the *approximate derivatives* $\mathrm{ap}\frac{\partial u}{\partial x_i}$ exist for $i = 1, \cdots, r$ and if we set: $Ju = \left[\det\left((\sum_{k=1}^r \partial_k u_i \partial_k u_j)_{1 \leqslant i,j \leqslant d}\right)\right]^{1/2}$, this is equal kdx a.e. to the determinant of the *approximate Jacobian matrix* of u. Then, by Theorem 3.1.4 in [15], u is *approximately differentiable* at almost all points a in A.
We denote by $\mathcal{H}^{r-d}$ the $(r-d)$-dimensional Hausdorff measure on $\mathbb{R}^r$.
As a consequence of Theorems 3.1.8, 3.1.16 and Lemma 3.1.7 in [15], for all $n \in \mathbb{N}^*$, there exists a map $u^n : \mathbb{R}^r \to \mathbb{R}^d$ of class $\mathcal{C}^1$ such that

$$\lambda^r(A \setminus \{x : u(x) = u^n(x)\}) \leqslant \frac{1}{n}$$

and

$$\forall a \in \{x : u(x) = u^n(x)\}, \; \mathrm{ap}\frac{\partial u}{\partial x_i}(a) = \mathrm{ap}\frac{\partial u^n}{\partial x_i}(a), \; i = 1, \cdots, r.$$

Assume first that $d \leqslant r$. Let B be a Borelian set in $\mathbb{R}^d$ such that $\lambda^r(B) = 0$. Thanks to the co-area formula we have

$$\begin{aligned}
\int_{\mathbb{R}^r} \mathbf{1}_B(u(x)) Ju(x) k(x)\, dx &= \int_A \mathbf{1}_B(u(x)) Ju(x) k(x)\, dx \\
&= \lim_{n\to+\infty} \int_{A\cap\{u=u^n\}} \mathbf{1}_B(u(x)) Ju(x) k(x)\, dx \\
&= \lim_{n\to+\infty} \int_{A\cap\{u=u^n\}} \mathbf{1}_B(u^n(x)) Ju^n(x) k(x)\, dx \\
&= \lim_{n\to+\infty} \int_{\mathbb{R}^r} \left(\int_{(u^n)^{-1}(y)} \mathbf{1}_{A\cap\{u=u^n\}}(x) \mathbf{1}_B(u^n(x)) k(x) d\mathcal{H}^{r-d}(x) \right) d\lambda^r(y) \\
&= \lim_{n\to+\infty} \int_{\mathbb{R}^r} \mathbf{1}_B(y) \left(\int_{(u^n)^{-1}(y)} \mathbf{1}_{A\cap\{u=u^n\}}(x) k(x) d\mathcal{H}^{r-d}(x) \right) d\lambda^r(y) \\
&= 0
\end{aligned}$$

So that, $u_*(Ju \cdot kdx) \ll \lambda_d$.
We have

$$Ju = \left[\det\left(Du \cdot (Du)^t\right)\right]^{1/2} \text{ and } \gamma(u) = Du \cdot \xi \cdot Du^t,$$

where Du is the $d \times r$ matrix: $\left(\dfrac{\partial u_i}{\partial x_k}\right)_{1 \leqslant i \leqslant d; 1 \leqslant k \leqslant r}$.
As $\xi(x)$ is symmetric and positive definite on O and $\lambda^r(\mathbb{R}^r \setminus O) = 0$, we have

$$\{x \in A;\ Ju(x) > 0\} = \{x \in A;\ \det(\gamma(u)(x)) > 0\}\ a.e.,$$

and this ends the proof in this case.
Now, if $d > r$, $\det(\gamma(u)) = 0$ and the result is trivial. □

2.2 The case of a product structure

We consider a sequence of functions ξ^i and k_i, $i \in \mathbb{N}^*$, k_i being non-negative Borel functions such that $\int_{\mathbb{R}^r} k_i(x)\, dx = 1$. We assume that for all $i \in \mathbb{N}^*$, ξ^i and k_i satisfy hypotheses (HG) so that, we can construct, as for k in the previous subsection, the Dirichlet form $(\mathbf{d}_i, e_i)$ on $L^2(\mathbb{R}^r, k_i dx)$ associated to the carré du champ operator γ_i given by:

$$\forall u, v \in \mathbf{d}_i,\ \gamma_i[u,v] = \sum_{k,l} \xi^i_{kl} \partial_k u \partial_l v.$$

We now consider the product Dirichlet form $(\tilde{\mathbf{d}}, \tilde{e}) = \prod_{i=1}^{+\infty} (\mathbf{d}_i, e_i)$ defined on the product space $\left((\mathbb{R}^r)^{\mathbb{N}^*}, (\mathcal{B}(R^r))^{\mathbb{N}^*}\right)$ equipped with the product probability $\Lambda = \prod_{i=1}^{+\infty} k_i dx$. We denote by $(X_n)_{n\in\mathbb{N}^*}$ the coordinates maps on $(\mathbb{R}^r)^{\mathbb{N}^*}$.
Let us recall that $U = F(X_1, X_2, \cdots, X_n, \cdots)$ belongs to $\tilde{\mathbf{d}}$ if and only if :

1. U belongs to $L^2\left((\mathbb{R}^r)^{\mathbb{N}^*}, (\mathcal{B}(\mathbb{R}^r))^{\mathbb{N}^*}, \Lambda\right)$.
2. For all $k \in \mathbb{N}^*$ and Λ-almost all $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots)$ in $(\mathbb{R}^r)^{\mathbb{N}^*}$, $F(x_1, \cdots, x_{k-1}, \cdot, x_{k+1}, \cdots)$ belongs to $\mathbf{d}_k$.

3. $\tilde{e}(U) = \sum_k \int_{(\mathbb{R}^r)^{\mathbb{N}^*}} e_k(F(X_1(x), \cdots, X_{k-1}(x), \cdot, X_{k+1}(x), \cdots))\, \Lambda(dx) < +\infty.$

Where as usual, the form e_k acts only on the k-th coordinate.
It is also well known that $(\tilde{\mathbf{d}}, \tilde{e})$ admits a carré du champ $\tilde{\gamma}$ given by

$$\tilde{\gamma}[U] = \sum_k \gamma_k[F(X_1, \cdots, X_{k-1}, \cdot, X_{k+1}, \cdots)](X_k).$$

To prove that (EID) is satisfied by this structure, we first prove that it is satisfied for a finite product. So, for all $n \in \mathbb{N}^*$, we consider $(\tilde{\mathbf{d}}_n, \tilde{e}_n) = \prod_{i=1}^n (\mathbf{d}_i, e_i)$ defined on the product space $((\mathbb{R}^r)^n, (\mathcal{B}(R^r))^n)$ equipped with the product probability $\Lambda_n = \prod_{i=1}^n k_i dx$. By restriction, we keep the same notation as the one introduced for the infinite product. We know that this structure admits a carré du champ operator $\tilde{\gamma}_n$ given by $\tilde{\gamma}_n = \sum_{i=1}^n \gamma_i$.

Lemma 3. *For all $n \in \mathbb{N}^*$, the Dirichlet structure $(\tilde{\mathbf{d}}_n, \tilde{e}_n)$ satifies (EID):*

$$\forall d \in \mathbb{N}^* \ \forall U \in (\tilde{\mathbf{d}}_n)^d \ \ U_*[(\det \tilde{\gamma}_n[U]) \cdot \Lambda_n] \ll \lambda^d.$$

Proof. The proof consists in remarking that this is nothing but a particular case of Theorem 2 on $\mathbb{R}^{nd}$, ξ being replaced by Ξ, the diagonal matrix of the ξ^i, and the density being the product density. □

As a consequence of Chapter V Proposition 2.2.3. in Bouleau-Hirsch [10], we have

Theorem 4. *The Dirichlet structure $(\tilde{\mathbf{d}}, \tilde{e})$ satisfies (EID):*

$$\forall d \in \mathbb{N}^* \ \forall U \in \tilde{\mathbf{d}}^d \ \ U_*[(\det \tilde{\gamma}[U]) \cdot \Lambda] \ll \lambda^d.$$

2.3 The case of structures obtained by injective images

The following result could be extended to more general images (see Bouleau-Hirsch [10] Chapter V §1.3 p 196 *et seq.*). We give the statement in the most useful form for Poisson measures and processes with independent increments.

Let $(\mathbb{R}^p\backslash\{0\}, \mathcal{B}(\mathbb{R}^p\backslash\{0\}), \nu, \mathbf{d}, \gamma)$ be a Dirichlet structure on $\mathbb{R}^p\backslash\{0\}$ satisfying (EID). Thus ν is σ-finite, γ is the carré du champ operator and the Dirichlet form is $e[u] = 1/2 \int \gamma[u] d\nu$.

Let $U : \mathbb{R}^p\backslash\{0\} \mapsto \mathbb{R}^q\backslash\{0\}$ be an injective map such that $U \in \mathbf{d}^q$. Then $U_*\nu$ is σ-finite. If we put

$$\begin{aligned} \mathbf{d}_U &= \{\varphi \in L^2(U_*\nu) : \varphi \circ U \in \mathbf{d}\} \\ e_U[\varphi] &= e[\varphi \circ U] \\ \gamma_U[\varphi] &= \frac{d\, U_*(\gamma[\varphi \circ U].\nu)}{d\, U_*\nu} \end{aligned}$$

we have

Proposition 5. *The term $(\mathbb{R}^q\backslash\{0\}, \mathcal{B}(\mathbb{R}^q\backslash\{0\}), U_*\nu, \mathbf{d}_U, \gamma_U)$ is a Dirichlet structure satisfying (EID).*

Proof. a) That $(\mathbb{R}^q\backslash\{0\}, \mathcal{B}(\mathbb{R}^q\backslash\{0\}), U_*\nu, \mathbf{d}_U, \gamma_U)$ be a Dirichlet structure is general and does not use the injectivity of U (cf. the case ν finite in Bouleau-Hirsch [10] Chap. V §1 p. 186 *et seq.*).
b) By the injectivity of U, we see that for $\varphi \in \mathbf{d}_U$

$$(\gamma_U[\varphi]) \circ U = \gamma[\varphi \circ U] \quad \nu\text{-a.s.}$$

so that if $f \in (\mathbf{d}_U)^r$

$$f_*[\det \gamma_U[f] \cdot U_*\nu] = (f \circ U)_*[\det \gamma[f \circ U] \cdot \nu]$$

which proves (EID) for the image structure. □

Remark 1. *Applying this result yields examples of Dirichlet structures on $\mathbb{R}^n$ satisfying (EID) whose measures are carried by a (Lipschitzian) curve in $\mathbb{R}^n$ or, under some hypotheses, a countable union of such curves, and therefore whithout density.*

3 Dirichlet structure on the Poisson space related to a Dirichlet structure on the states space

Let $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ be a local symmetric Dirichlet structure which admits a carré du champ operator i.e. $(X, \mathcal{X}, \nu)$ is a measured space called *the bottom space*, ν is σ-finite and the bilinear form

$$e[f, g] = \frac{1}{2}\int \gamma[f, g]\, d\nu,$$

is a local Dirichlet form with domain $\mathbf{d} \subset L^2(\nu)$ and carré du champ operator γ (see Bouleau-Hirsch [10], Chap. I). We assume that for all $x \in X$, $\{x\}$ belongs to $\mathcal{X}$ and that ν is diffuse ($\nu(\{x\}) = 0\ \forall x$). The generator associated to this Dirichlet structure is denoted by a, its domain is $\mathcal{D}(a) \subset \mathbf{d}$ and it generates the Markovian strongly continuous semigroup $(p_t)_{t\,\geqslant\,0}$ on $L^2(\nu)$.

Our aim is to study, thanks to Dirichlet forms methods, functionals of a Poisson measure N, associated to $(X, \mathcal{X}, \nu)$. It is defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where Ω is the configuration space, the set of measures which are countable sum of Dirac measures on X, $\mathcal{A}$ is the sigma-field generated by N and $\mathbb{P}$ is the law of N (see Neveu [24]). The probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is called the *upper space*.

3.1 Density lemmas

Let $(F, \mathcal{F}, \mu)$ be a probability space such that for all $x \in F$, $\{x\}$ belongs to $\mathcal{F}$ and μ is diffuse. Let $n \in \mathbb{N}^*$, we denote by $x_1, x_2, \cdots, x_n$ the coordinates maps on $(F^n, \mathcal{F}^{\otimes n}, \mu^{\times n})$ and we consider the random measure $m = \sum_{i=1}^n \varepsilon_{x_i}$.

Lemma 6. *Let $\mathcal{S}$ be the symmetric sub-sigma-field in $\mathcal{F}^{\otimes n}$ and $p \in [1, +\infty[$. Sets $\{m(g_1)\cdots m(g_n) :\ g_i \in L^\infty(\mu)\ \forall i = 1, \cdots, n\}$ and $\{e^{m(g)} :\ g \in L^\infty(\mu)\}$ are both total in $L^p(F^n, \mathcal{S}, \mu^{\times n})$ and the set $\{e^{im(g)} :\ g \in L^\infty(\mu)\}$ is total in $L^p(F^n, \mathcal{S}, \mu^{\times n}; \mathbb{C})$.*

Proof. Because μ is diffuse, the set $\{g_1(x_1)\cdots g_n(x_n)\ :\ \ g_i\in L^\infty(\mu),\, g_i$ with disjoint supports $\forall i=1,\cdots,n\}$ is total in $L^p(\mu^{\times n})$. Let $G(x_1,\cdots,x_n)$ be a linear combination of such functions. If $F(x_1,\cdots,x_n)$ is symmetric and belongs to $L^p(\mu^{\times n})$ then the distance in $L^p(\mu^{\times n})$ between $F(x_1,\cdots,x_n)$ and $G(x_{\sigma(1)},\cdots,x_{\sigma(n)})$ for $\sigma\in\mathbf{S}$ the set of permutations on $\{1,\cdots,n\}$, does not depend on σ and as a consequence is larger than the distance between $F(x_1,\cdots,x_n)$ and the barycenter $\frac{1}{n!}\sum_{\sigma\in\mathbf{S}}G(x_{\sigma(1)},\cdots,x_{\sigma(n)})$. So, the set $\{\frac{1}{n!}\sum_{\sigma\in\mathbf{S}}G(x_{\sigma(1)},\cdots,x_{\sigma(n)}\ :\ \ g_i\in L^\infty(\mu)\,,g_i$ with disjoint supports $\forall i=1,\cdots,n\}$ is total in $L^p(F^n,\mathcal{S},\mu^{\times n})$. We conclude by using the following property : if f_i $i=1,\cdots,n$, are $\mathcal{F}$-measurable functions with disjoint supports then: $m(f_1)\cdots m(f_n)=\sum_{\sigma\in\mathbf{S}}f_1(x_{\sigma(1)})\cdots f_n(x_{\sigma(n)})$. □

Lemma 7. *Let N_1 be a random Poisson measure on $(F,\mathcal{F},\mu_1)$ where μ_1, the intensity of N_1, is a finite and diffuse measure, defined on some probability space $(\Omega_1,\mathcal{A}_1,\mathbb{P}_1)$ where $\mathcal{A}_1=\sigma(N_1)$. Then, for any $p\in[1,+\infty[$, the set $\{e^{-N_1(f)}\ :\ \ f\geqslant 0, f\in L^\infty(\mu_1)\}$ is total in $L^p(\Omega_1,\mathcal{A}_1,\mathbb{P}_1)$ and $\{e^{iN_1(f)}\ :\ \ f\in L^\infty(\mu_1)\}$ is total in $L^p(\Omega_1,\mathcal{A}_1,\mathbb{P}_1;\mathbb{C})$.*

Proof. Let us put $P=N_1(F)$, it is an integer valued random variable. As $\{e^{i\lambda P}\ :\ \lambda\in\mathbb{R}\}$ is total in $L^p(\mathbb{N},\mathcal{P}(\mathbb{N}),\mathbb{P}_P)$ where $\mathbb{P}_P$ is the law of P, for any $n\in\mathbb{N}^*$ and any $g\in L^\infty(\mu_1)$, one can approximate in $L^p(\Omega_1,\mathcal{A}_1,\mathbb{P}_1;\mathbb{C})$ the random variable $\mathbf{1}_{\{P=n\}}e^{iN_1(g)}$ by a sequence of variables of the form $\sum_{k=1}^K a_k e^{i\lambda_k P}e^{iN_1(g)}$ with $a_k,\lambda_k\in\mathbb{R}$, $k=1\cdots K$. But, as a consequence of the previous lemma, we know that $\{\mathbf{1}_{\{P=n\}}e^{iN_1(f)}\ :\ \ f\in L^\infty(\mu_1)\}$ is total in $L^p(\{P=n\},\mathcal{A}_1|_{\{P=n\}},\mathbb{P}_1|_{\{P=n\}};\mathbb{C})$, which provides the result. □

We now give the main lemma, with the notation introduced at the beginning of this section.

Lemma 8. *For $p\in[1,\infty[$, the set $\{e^{-N(f)}\ :\ \ f\geqslant 0, f\in L^1(\nu)\cap L^\infty(\nu)\}$ is total in $L^p(\Omega,\mathcal{A},\mathbb{P})$ and $\{e^{iN(f)}\ :\ f\in L^1(\nu)\cap L^\infty(\nu)\}$ is total in $L^p(\Omega,\mathcal{A},\mathbb{P};\mathbb{C})$.*

Proof. Assume that ν is non finite. Let $(F_k)_{k\in\mathbb{N}}$ be a partition of Ω such that for all k, $\nu(F_k)$ be finite. By restriction of N to each set F_k, we construct a sequence of independent Poisson measures (N_k) such that $N=\sum_k N_k$. As any variable in L^p is the limit of variables which depend only on a finite number of N_k, we conclude thanks to the previous lemma. □

3.2 Construction using the Friedrichs' argument

3.2.1. Basic formulas and pre-generator

We set $\tilde{N}=N-\nu$ then the identity $\mathbb{E}[(\tilde{N}(f))^2]=\int f^2\,d\nu$, for $f\in L^1(\nu)\cap L^2(\nu)$ can be extended uniquely to $f\in L^2(\nu)$ and this permits to define $\tilde{N}(f)$ for $f\in L^2(\nu)$. The Laplace characteristic functional

$$\mathbb{E}[e^{i\tilde{N}(f)}]=e^{-\int(1-e^{if}+if)\,d\nu}\quad f\in L^2(\nu)\tag{2}$$

yields:

Proposition 9. *For all $f\in\mathbf{d}$ and all $h\in\mathcal{D}(a)$,*

$$\mathbb{E}\left[e^{i\tilde{N}(f)}\left(\tilde{N}(a[h])+\frac{i}{2}N(\gamma[f,h])\right)\right]=0.\tag{3}$$

Proof. Derivating in 0 the map $t \to \mathbb{E}\left[e^{i\tilde{N}(f+ta[h])}\right]$, we have thanks to (2),

$$\mathbb{E}[e^{i\tilde{N}(f)+\int(1-e^{if}+if)d\nu}\tilde{N}(a[h])] = \int (e^{if}-1)a[h]\,d\nu, \tag{4}$$

then using the fact that function $x \mapsto e^{ix}-1$ is Lipschitz and vanishes in 0 and the functional calculus related to a local Dirichlet form (see Bouleau-Hirsch [10] Section I.6) we get that the member on the right hand side in (4) is equal to

$$-\frac{1}{2}\int \gamma[e^{if}-1,h]\,d\nu = -\frac{i}{2}\int e^{if}\gamma[f,h]\,d\nu.$$

We conclude by applying once more (4) with $\gamma[f,h]$ instead of $a[h]$. $\square$

The linear combinations of variables of the form $e^{i\tilde{N}(f)}$ with $f \in \mathcal{D}(a)\bigcap L^1(\nu)$ are dense in $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{C})$ thanks to Lemma 8. This is a natural choice for test functions, but, for technical reason, we need in addition that $\gamma[f]$ belongs to $L^2(\nu)$. So we suppose :
Bottom core hypothesis (BC). The bottom structure is such that there exists a subspace H of $\mathcal{D}(a)\bigcap L^1(\nu)$ such that $\forall f \in H,\ \gamma[f] \in L^2(\nu)$, and the space $\mathcal{D}_0$ of linear combinations of $e^{i\tilde{N}(f)},\ f \in H$, is dense in $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{C})$.

This hypothesis will be fulfilled in all cases on $\mathbb{R}^r$ where $\mathcal{D}(a)$ contains the $\mathcal{C}^\infty$ functions with compact support and γ operates on them.

If $U = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$ belongs to $\mathcal{D}_0$, we put

$$A_0[U] = \sum_p \lambda_p e^{i\tilde{N}(f_p)}(i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p])). \tag{5}$$

This is a natural choice as candidate for the pregenerator of the upper structure, since, as easily seen using (5), it induces the relation $\Gamma[N(f)] = N(\gamma[f])$ between the carré du champ operators of the upper and the bottom structures, which is satisfied in the case $\nu(X) < \infty$.

One has to note that for the moment, A_0 is not uniquely determined since *a priori* $A_0[U]$ depends on the expression of U which is possibly non unique.

Proposition 10. *Let $U, V \in \mathcal{D}_0$, $U = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$ and $V = \sum_q \mu_q e^{i\tilde{N}(g_q)}$. One has*

$$-\mathbb{E}[A_0[U]\overline{V}] = \frac{1}{2}\mathbb{E}\left[\sum_{p,q}\lambda_p\overline{\mu_q}e^{i\tilde{N}(f_p-g_q)}N(\gamma[f_p,g_q])\right] \tag{6}$$

which is also equal to

$$\frac{1}{2}\mathbb{E}[\sum_{p,q}F'_p\overline{G'_q}N(\gamma[f_p,g_q])], \tag{7}$$

where F and G are such that $U = F(\tilde{N}(f_1),\cdots,\tilde{N}(f_n))$ and $V = G(\tilde{N}(g_1),\cdots,\tilde{N}(g_m))$ and $F'_p = \frac{\partial F}{\partial x_p}(\tilde{N}(f_1),\cdots,\tilde{N}(f_n))$, $G'_q = \frac{\partial G}{\partial x_q}(\tilde{N}(g_1),\cdots,\tilde{N}(g_m))$.

Proof. We have

$$-\mathbb{E}[A_0[U]\overline{V}] = -\mathbb{E}\left[\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p-g_q)}(i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p]))\right].$$

Thanks to Proposition 9,

$$\begin{aligned}-\mathbb{E}\left[\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p-g_q)} i\tilde{N}(a[f_p])\right] &= -\frac{1}{2}\mathbb{E}\left[\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p-g_q)} N(\gamma[f_p, f_p - g_q])\right] \\ &= \frac{1}{2}\mathbb{E}\left[\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p-g_q)} N(\gamma[f_p, g_q])\right] - \frac{1}{2}\mathbb{E}\left[\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p-g_q)} N(\gamma[f_p])\right]\end{aligned}$$

which gives the statement. □

It remains to prove that A_0 is uniquely determined and so is an operator acting on $\mathcal{D}_0$. To this end, thanks to the previous proposition, we just have to prove that the quantity $\sum_{p,q} F'_p \overline{G'_q} N(\gamma[f_p, g_q])$ does not depend on the choice of representations for U and V. In the same spirit as Ma-Röckner (see [23]), the introduction of a gradient will yield this non-dependence. Let us mention that the gradient we introduce is different from the one considered by these authors and is based on a notion that we present now.

3.2.2. Particle-wise product of a Poisson measure and a probability

We are still considering N the random Poisson measure on $(X, \mathcal{X}, \nu)$ and we are given an auxiliary probability space $(R, \mathcal{R}, \rho)$. We construct a random Poisson measure $N \odot \rho$ on $(X \times R, \mathcal{X} \otimes \mathcal{R}, \nu \times \rho)$ such that if $N = \sum_i \varepsilon_{x_i}$ then $N \odot \rho = \sum_i \varepsilon_{(x_i, r_i)}$ where (r_i) is a sequence of i.i.d. random variables independent of N whose common law is ρ. Such a random Poisson measure $N \odot \rho$ is sometimes called a *marked* Poisson measure.
The construction of $N \odot \rho$ follows line by line the one of N. Let us recall it. We first study the case where ν is finite and we consider the probability space

$$(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_{\nu(X)}) \times (X, \mathcal{X}, \frac{\nu}{\nu(X)})^{\mathbb{N}^*},$$

where $P_{\nu(X)}$ denotes the Poisson law with intensity $\nu(X)$ and we put

$$N = \sum_{i=1}^{Y} \varepsilon_{x_i}, \qquad \text{(with the convention } \sum_{1}^{0} = 0)$$

where $Y, x_1, \cdots, x_n, \cdots$ denote the coordinates maps. We introduce the probability space

$$(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}}) = (R, \mathcal{R}, \rho)^{\mathbb{N}^*},$$

and the coordinates are denoted by $r_1, \cdots, r_n, \cdots$. On the probability space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_{\nu(X)}) \times (X, \mathcal{X}, \frac{\nu}{\nu(X)})^{\mathbb{N}^*} \times (\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$, we define the random measure $N \odot \rho = \sum_{i=1}^{Y} \varepsilon_{(x_i, r_i)}$. It is a Poisson random measure on $X \times R$ with intensity measure $\nu \times \rho$. For $f \in L^1(\nu \times \rho)$

$$\hat{\mathbb{E}}[\int_{X \times R} f dN \odot \rho] = \int_X (\int_R f(x, r) d\rho(r)) N(dx) \quad \mathbb{P} - a.e. \tag{8}$$

and if $f \in L^2(\nu \times \rho)$

$$\hat{\mathbb{E}}[(\int_{X\times R} f dN \odot \rho)^2] = (\int_X \int_R f d\rho dN)^2 - \int_X (\int_R f d\rho)^2 dN + \int_X \int_R f^2 d\rho dN, \tag{9}$$

where $\hat{\mathbb{E}}$ stands for the expectation under the probability $\hat{\mathbb{P}}$.

If ν is σ-finite, we extend this construction by a standard product argument. Eventually in all cases, we have constructed N on $(\Omega, \mathcal{A}, \mathbb{P})$ and $N \odot \rho$ on $(\Omega, \mathcal{A}, \mathbb{P}) \times (\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$, it is a random Poisson measure on $X \times R$ with intensity measure $\nu \times \rho$.
We now are able to generalize identities (8) and (9):

Proposition 11. *Let F be an $\mathcal{A} \otimes \mathcal{X} \otimes \mathcal{R}$ measurable function such that $\mathbb{E} \int_{X\times R} F^2 \, d\nu d\rho$ and $\mathbb{E} \int_R (\int_X |F| d\nu)^2 d\rho$ are both finite then the following relation holds*

$$\hat{\mathbb{E}}[(\int_{X\times R} F dN \odot \rho)^2] = (\int_X \int_R F d\rho dN)^2 - \int_X (\int_R F d\rho)^2 dN + \int_X \int_R F^2 d\rho dN, \tag{10}$$

Proof. Approximating first F by a sequence of elementary functions and then introducing a partition (B_k) of subsets of X of finite ν-measure, this identity is seen to be a consequence of (9). $\square$

We denote by $\mathbb{P}_N$ the measure $\mathbb{P}_N = \mathbb{P}(dw) N_w(dx)$ on $(\Omega \times X, \mathcal{A} \otimes \mathcal{X})$. Let us remark that $\mathbb{P}_N$ and $\mathbb{P} \times \nu$ are singular because ν is diffuse.
We will use the following consequence of the previous proposition :

Corollary 12. *Let F be an $\mathcal{A} \otimes \mathcal{X} \otimes \mathcal{R}$ measurable function. If F belongs to $L^2(\Omega \times X \times R, \mathbb{P}_N \times \rho)$ and $\int F(w, x, r) \rho(dr) = 0$ for $\mathbb{P}_N$-almost all (w, x), then $\int F dN \odot \rho$ is well-defined and belongs to $L^2(\mathbb{P} \times \hat{\mathbb{P}})$, moreover*

$$\hat{\mathbb{E}}[(\int_{X\times R} F dN \odot \rho)^2] = \int F^2 dN d\rho \quad \mathbb{P}\text{-a.e.} \tag{11}$$

Proof. If F satisfies hypotheses of Proposition 11 then the result is clear. The general case is obtained by approximation. $\square$

3.2.3. Gradient and welldefinedness

From now on, we assume that the Hilbert space $\mathbf{d}$ is separable so that (see Bouleau-Hirsch [10], ex.5.9 p. 242) the bottom Dirichlet structure admits a gradient operator in the sense that there exist a separable Hilbert space H and a continuous linear map D from $\mathbf{d}$ into $L^2(X, \nu; H)$ such that

- $\forall u \in \mathbf{d}$, $\|D[u]\|_H^2 = \gamma[u]$.
- If $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz then
$$\forall u \in \mathbf{d}, \ D[F \circ u] = (F' \circ u) Du.$$
- If F is $\mathcal{C}^1$ (continuously differentiable) and Lipschitz from $\mathbb{R}^d$ into $\mathbb{R}$ (with $d \in \mathbb{N}$) then
$$\forall u = (u_1, \cdots, u_d) \in \mathbf{d}^d, \ D[F \circ u] = \sum_{i=1}^d (F_i' \circ u) D[u_i].$$

As only the Hilbertian structure plays a role, we can choose for H the space $L^2(R,\mathcal{R},\rho)$ where $(R,\mathcal{R},\rho)$ is a probability space such that the dimension of the vector space $L^2(R,\mathcal{R},\rho)$ is infinite. As usual, we identify $L^2(\nu;H)$ and $L^2(X\times R,\mathcal{X}\otimes\mathcal{R},\nu\times\rho)$ and we denote the gradient D by $\flat$:

$$\forall u\in\mathbf{d},\ Du=u^\flat\in L^2(X\times R,\mathcal{X}\otimes\mathcal{R},\nu\times\rho).$$

Without loss of generality, we assume moreover that operator $\flat$ takes its values in the orthogonal space of 1 in $L^2(R,\mathcal{R},\rho)$, in other words we take for H the orthogonal of 1. So that we have

$$\forall u\in\mathbf{d},\ \int u^\flat d\rho=0\ \ \nu\text{-}a.e. \tag{12}$$

Let us emphasize that hypothesis (12) although restriction-free, is a key property here (as in many applications to error calculus cf [7] Chap. V p225 et seq.) Thanks to Corollary 12, it is the feature which will avoid non-local finite difference calculation on the upper space. Finally, although not necessary, we assume for simplicity that constants belong to $\mathbf{d}_{loc}$ (see Bouleau-Hirsch [10] Chap. I Definition 7.1.3.)

$$1\in\mathbf{d}_{loc}\text{ which implies }\ \gamma[1]=0\text{ and }1^\flat=0. \tag{13}$$

We now introduce the creation and annihilation operators ε^+ and ε^- well-known in quantum mechanics (see Meyer [21], Nualart-Vives [25], Picard [26] etc.) in the following way:

$$\begin{aligned}\forall x,w\in\Omega,\ \varepsilon_x^+(w)&=w\mathbf{1}_{\{x\in supp\,w\}}+(w+\varepsilon_x)\mathbf{1}_{\{x\notin supp\,w\}}\\ \forall x,w\in\Omega,\ \varepsilon_x^-(w)&=w\mathbf{1}_{\{x\notin supp\,w\}}+(w-\varepsilon_x)\mathbf{1}_{\{x\in supp\,w\}}.\end{aligned}$$

One can verify that for all $w\in\Omega$,

$$\varepsilon_x^+(w)=w\text{ and }\varepsilon_x^-(w)=w-\varepsilon_x\text{ for }N_w\text{-almost all }x \tag{14}$$

and

$$\varepsilon_x^+(w)=w+\varepsilon_x\text{ and }\varepsilon_x^-(w)=w\text{ for }\nu\text{-almost all }x \tag{15}$$

We extend this operator to the functionals by setting:

$$\varepsilon^+H(w,x)=H(\varepsilon_x^+w,x)\quad\text{ and }\quad\varepsilon^-H(w,x)=H(\varepsilon_x^-w,x).$$

The next lemma shows that the image of $\mathbb{P}\times\nu$ by ε^+ is nothing but $\mathbb{P}_N$ whose image by ε^- is $\mathbb{P}\times\nu$:

Lemma 13. *Let H be $\mathcal{A}\otimes\mathcal{X}$-measurable and non negative, then*

$$\mathbb{E}\int\varepsilon^+Hd\nu=\mathbb{E}\int HdN\quad\text{ and }\quad\mathbb{E}\int\varepsilon^-HdN=\mathbb{E}\int Hd\nu.$$

Proof. Let us assume first that $H=e^{-N(f)}g$ where f and g are non negative and belong to $L^1(\nu)\cap L^2(\nu)$. We have:

$$\mathbb{E}\int\varepsilon^+Hd\nu=\mathbb{E}\int e^{-N(f)}e^{-f(x)}g(x)d\nu(x),$$

and by standard calculations based on the properties of the Laplace functional we obtain that

$$\mathbb{E}\int e^{-N(f)}e^{-f(x)}g(x)d\nu(x)=\mathbb{E}[e^{-N(f)}N(g)]=\mathbb{E}\int HdN.$$

We conclude using a monotone class argument and similarly for the second equation. $\square$

Let us also remark that if $F \in L^2(\mathbb{P}_N \times \rho)$ satisfies $\int F d\rho = 0$ $\mathbb{P}_N$-a.e. then if we put $\varepsilon^+ F(w,x,r) = F(\varepsilon_x^+(w), x, r)$ we have

$$\int \varepsilon^+ F dN \odot \rho = \int F dN \odot \rho \ \ \mathbb{P}\text{-}a.e. \tag{16}$$

Indeed $\int (\varepsilon^+ F - F)^2 dN d\rho = 0$ $\mathbb{P}$-a.e. because $\varepsilon_x^+(w) = w$ for N_w-almost all x.

Definition 14. *For all $F \in \mathcal{D}_0$, we put*

$$F^\sharp = \int \varepsilon^-((\varepsilon^+ F)^\flat)\, d\, N \odot \rho.$$

Thanks to hypothesis (13) we have the following representation of $F^\sharp$:

$$F^\sharp(w, \hat{w}) = \int_{X \times R} \varepsilon^-((F(\varepsilon_\cdot^+(w)) - F(w))^\flat)(x,r)\ N \odot \rho(dxdr).$$

Let us also remark that Definition 14 makes sense because for all $F \in \mathcal{D}_0$ and $\mathbb{P}$-almost all $w \in \Omega$, the map $y \mapsto F(\varepsilon_y^+(w)) - F(w)$ belongs to $\mathbf{d}$. To see this, take $F = e^{i\tilde{N}(f)}$ with $f \in \mathcal{D}(a) \bigcap L^1(\nu)$, then

$$F(\varepsilon_y^+(w)) - F(w) = e^{i\tilde{N}(f)}(e^{if(y)} - 1),$$

and we know that $e^{if} - 1 \in \mathbf{d}$. We now proceed and obtain

$$(e^{i\tilde{N}(f)})^\sharp = \int \varepsilon^-(e^{i\tilde{N}(f)}(e^{if} - 1)^\flat)\ dN \odot \rho = \int \varepsilon^-(e^{i\tilde{N}(f) + if}(if)^\flat)\ dN \odot \rho$$

and eventually

$$(e^{i\tilde{N}(f)})^\sharp = \int e^{i\tilde{N}(f)}(if)^\flat\ dN \odot \rho.$$

So, if $F, G \in \mathcal{D}_0$, $F = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$, $G = \sum_q \mu_q e^{i\tilde{N}(g_q)}$, as $\int f_p^\flat d\rho = \int g_q^\flat d\rho = 0$ and thanks to Corollary 12, we have

$$\hat{\mathbb{E}}[F^\sharp \overline{G^\sharp}] = \sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p - g_q)} \int (if_p)^\flat \overline{(ig_q)^\flat} dN d\rho,$$

and so

$$\hat{\mathbb{E}}[F^\sharp \overline{G^\sharp}] = \sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p - g_q)} N(\gamma(f_p, g_q)) \tag{17}$$

But, by Definition 14, it is clear that $F^\sharp$ does not depend on the representation of F in $\mathcal{D}_0$ so as a consequence of the previous identity $\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p - g_q)} N(\gamma(f_p, g_q))$ depends only on F and G and thanks to (6), we conclude that A_0 is well-defined and is a linear operator from $\mathcal{D}_0$ into $L^2(\mathbb{P})$.

3.2.4. Upper structure and first properties

As a consequence of Proposition 10, it is clear that A_0 is symmetric, non positive on $\mathcal{D}_0$ therefore (see Bouleau-Hirsch [10] p.4) it is closable and we can consider its Friedrichs extension $(A, \mathcal{D}(A))$ which generates a closed Hermitian form $\mathcal{E}$ with domain $\mathbb{D} \supset \mathcal{D}(A)$ such that

$$\forall U \in \mathcal{D}(A)\ \forall V \in \mathbb{D},\ \mathcal{E}(U,V) = -\mathbb{E}[A[U]\overline{V}].$$

Moreover, thanks to Proposition 10, it is clear that contractions operate, so (see Bouleau-Hirsch [10] ex. 3.6 p.16) $(\mathbb{D}, \mathcal{E})$ is a local Dirichlet form which admits a carré du champ operator Γ. The upper structure that we have obtained $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies the following properties :

- $\forall f \in \mathbf{d},\ \tilde{N}(f) \in \mathbb{D}$ and
$$\Gamma[\tilde{N}(f)] = N(\gamma[f]), \tag{18}$$
moreover the map $f \mapsto \tilde{N}(f)$ is an isometry from $\mathbf{d}$ into $\mathbb{D}$.

- $\forall f \in \mathcal{D}(a),\ e^{i\tilde{N}(f)} \in \mathcal{D}(A),$ and
$$A[e^{i\tilde{N}(f)}] = e^{i\tilde{N}(f)}(i\tilde{N}(a[f]) - \frac{1}{2}N(\gamma[f])). \tag{19}$$

- The operator $\sharp$ (defined on $\mathcal{D}_0$) admits an extension on $\mathbb{D}$, still denoted $\sharp$, it is a gradient associated to Γ and for all $f \in \mathbf{d}$:
$$(\tilde{N}(f))^\sharp = \int_{X\times R} f^\flat\, dN \odot \rho. \tag{20}$$

As a gradient for the Dirichlet structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$, $\sharp$ is a closed operator from $L^2(\mathbb{P})$ into $L^2(\mathbb{P} \times \hat{\mathbb{P}})$. It satisfies the chain rule and operates on the functionals of the form $\Phi(\tilde{N}(f))$, Φ Lipschitz $f \in \mathbf{d}$, or more generally $\Psi(\tilde{N}(f_1), \cdots, \tilde{N}(F_n))$ with Ψ Lipschitz and $\mathcal{C}^1$ and $f_1, \cdots, f_n$ in $\mathbf{d}$.
Let us also remark that if F belongs to $\mathcal{D}_0$,

$$A[F] = N(\varepsilon^-(a[\varepsilon^+ F])). \tag{21}$$

3.2.5. Link with the Fock space

The aim of this subsection is to make the link with other existing works and to present another approach based on the Fock space. It is independent of the rest of this article.
Let $g \in \mathcal{D}(a) \cap L^1(\nu)$ such that $-\frac{1}{2} \leqslant g \leqslant 0$ and $a[g] \in L^1(\nu)$. Clearly, $f = -\log(1+g)$ is non-negative and belongs to $\mathbf{d}$. We have for all $v \in \mathbf{d} \cap L^1(\nu)$

$$\begin{aligned}
\mathcal{E}[e^{-N(f)}, e^{-N(v)}] &= \frac{1}{2}\mathbb{E}\left[e^{-N(f)}e^{-N(v)}\Gamma[N(f), N(v)]\right] \\
&= \frac{1}{2}\mathbb{E}\left[e^{-N(f)}e^{-N(v)}N(\gamma[f,v])\right] \\
&= \frac{1}{2}e^{\int_X (1-e^{-f-v})d\nu} \int_X \gamma[f,v]e^{-f-v}d\nu.
\end{aligned}$$

As a consequence of the functional calculus

$$\int_X \gamma[f,v]e^{-f-v}d\nu = \int_X \gamma[g,e^{-v}]d\nu = -2\int_X a[g]e^{-v}d\nu,$$

this yields

$$\mathcal{E}[e^{-N(f)},e^{-N(v)}] = -\mathbb{E}[e^{-N(f)}e^{-N(v)}N(\frac{a[g]}{1+g})]. \tag{22}$$

Thus by Lemma 8, we obtain

Proposition 15. *Let $g \in \mathcal{D}(a)\cap L^1(\nu)$ such that $-\frac{1}{2} \leqslant g \leqslant 0$ and $a[g] \in L^1(\nu)$ then*

$$e^{N(\log(1+g))} \in \mathcal{D}(A) \text{ and } A[e^{N(\log(1+g))}] = e^{N(\log(1+g))}N(\frac{a[g]}{1+g}). \tag{23}$$

Let us recall that (p_t) is the semigroup associated to the bottom structure. If g satisfies the hypotheses of the previous proposition, p_tg also satisfies them. The map $\Psi : t \mapsto e^{N(\log(1+p_tg))}$ is differentiable and $\frac{d\Psi}{dt} = A\Psi$ with $\Psi(0) = e^{N(\log(1+g))}$ hence $\Psi(t) = P_t[e^{N(\log(1+g))}]$ where (P_t) is the strongly continuous semigroup generated by A. So, we have proved

Proposition 16. *Let g be a measurable function with $-\frac{1}{2} \leqslant g \leqslant 0$, then*

$$\forall t \geqslant 0,\ P_t[e^{N(\log(1+g))}] = e^{N(\log(1+p_tg))}.$$

For any $m \in \mathbb{N}^*$, we denote by $L^2_{sym}(X^m,\mathcal{X}^{\otimes m},\nu^{\times m})$ the set of symmetric functions in $L^2(X^m,\mathcal{X}^{\otimes m},\nu^{\times m})$ and we recall that ν is diffuse.
For all $F \in L^2_{sym}(X^m,\mathcal{X}^{\otimes m},\nu^{\times m})$, we put

$$I_m(F) = \int_{X^m} F(x_1,\cdots,x_m)\mathbf{1}_{\{\forall i\neq j, x_i\neq x_j\}}\,\tilde{N}(dx_1)\cdots\tilde{N}(dx_m).$$

One can easily verify that for all $F,G \in L^2_{sym}(X^m,\mathcal{X}^{\otimes m},\nu^{\times m})$ and all $n,m \in \mathbb{N}^*$, $\mathbb{E}[I_m(F)I_n(G)] = 0$ if $n \neq m$ and

$$\mathbb{E}[I_n(F)I_n(G)] = n!\langle F,G\rangle_{L^2_{sym}(X^n,\mathcal{X}^{\otimes n},\nu^{\times n})},$$

where $\langle\cdot,\cdot\rangle_{L^2_{sym}(X^n,\mathcal{X}^{\otimes n},\nu^{\times n})}$ denotes the scalar product in $L^2_{sym}(X^n,\mathcal{X}^{\otimes n},\nu^{\times n})$. For all $n \in \mathbb{N}^*$, we consider C_n, the Poisson chaos of order n, i.e. the sub-vector space of $L^2(\Omega,\mathcal{A},\mathbb{P})$ generated by the variables $I_n(F)$, $F \in L^2_{sym}(X^n,\mathcal{X}^{\otimes n},\nu^{\times n})$. The fact that

$$L^2(\Omega,\mathcal{A},\mathbb{P}) = \mathbb{R}\oplus_{n=1}^{+\infty} C_n.$$

has been proved by K. Ito (see [18]) in 1956. This proof is based on the fact that the set $\{N(E_1)\cdots N(E_k),\ (E_i) \text{ disjoint sets in } \mathcal{X}\}$ is total in $L^2(\Omega,\mathcal{A},\mathbb{P})$.

Another approach, quite natural, consists in studying carefully, for $g \in L^1\cap L^\infty(\nu)$, what has to be subtracted from the integral with respect to the product measure

$$\int_{X^n} g(x_1)\cdots g(x_n)\,\tilde{N}(dx_1)\cdots\tilde{N}(dx_n)$$

to obtain the Poisson stochastic integral

$$I_n(g^{\otimes n}) = \int_{X^n} g(x_1)\cdots g(x_n)\mathbf{1}_{\{\forall i\neq j, x_i\neq x_j\}}\,\tilde{N}(dx_1)\cdots\tilde{N}(dx_n).$$

This can be done in an elegant way by the use of lattices of partitions and the Möbius inversion formula (see Rota-Wallstrom [30]). This leads to the following formula (observe the tilde on the first N only) :

$$I_n(g^{\otimes n}) = \sum_{k=1}^{n} B_{n,k}(\tilde{N}(g), -1!N(g^2), 2!N(g^3), \ldots, (-1)^{n-k}(n-k)!N(g^{n-k+1})),$$

where the $B_{n,k}$ are the exponential Bell polynomials given by

$$B_{n,k} = \sum \frac{n!}{c_1!c_2!\cdots(1!)^{c_1}(2!)^{c_2}\cdots} x_1^{c_1}x_2^{c_2}\cdots$$

the sum being taken over all the non-negative integers $c_1, c_2, \cdots$ such that

$$\begin{aligned} c_1 + 2c_2 + 3c_3 + \cdots &= n \\ c_1 + c_2 + \cdots = k. \end{aligned}$$

$I_n(g^{\otimes n})$ is a homogeneous function of order n with respect to g. If we express the Taylor expansion of $e^{N(\log(1+tg))}$ and compute the n-th derivate with respect to t thanks to the formula of the composed functions (see Comtet [11]) we obtain

$$e^{N(\log(1+tg))-t\nu(g)} = 1 + \sum_{n=1}^{+\infty}\frac{t^n}{n!}\sum_{k=1}^{n} B_{n,k}(\tilde{N}(g), -1!N(g^2), \ldots, (-1)^{n-k}(n-k)!N(g^{n-k+1}))$$

this yields

$$e^{N(\log(1+g))-\nu(g)} = 1 + \sum_{n=1}^{+\infty}\frac{1}{n!}I_n(g^{\otimes n}). \tag{24}$$

The density of the chaos is now a consequence of Lemma 8.

Conversely, one can prove formula (24) thanks to the density of the chaos, see for instance Surgailis [32]. By transportation of structure, the density of the chaos has a short proof using stochastic calculus for the Poisson process on $\mathbb{R}_+$, cf Dellacherie, Maisonneuve and Meyer [13] p207, see also Applebaum [3] Theorems 4.1 and 4.3.

3.3 Extension of the representation of the gradient and the lent particle method

3.3.1. Extension of the representation of the gradient

The goal of this subsection is to extend formula of Definition 14 to any $F \in \mathbb{D}$.
To this aim, we introduce an auxiliary vector space $\underline{\mathbb{D}}$ which is the completion of the algebraic tensor product $\mathcal{D}_0 \otimes \mathbf{d}$ with respect to the norm $\|\ \|_{\underline{\mathbb{D}}}$ which is defined as follows.

Considering η, a fixed strictly positive function on X such that $N(\eta)$ belongs to $L^2(\mathbb{P})$, we set for all $H \in \mathcal{D}_0 \otimes \mathbf{d}$:

$$
\begin{aligned}
\|H\|_{\underline{\mathbb{D}}} &= \left(\mathbb{E}\int_X \varepsilon^-(\gamma[H])(w,x)N(dx)\right)^{\frac{1}{2}} + \mathbb{E}\int (\varepsilon^-|H|)(w,x)\eta(x)N(dx) \\
&= \left(\mathbb{E}\int_X \gamma[H])(w,x)\nu(dx)\right)^{\frac{1}{2}} + \mathbb{E}\int |H|(w,x)\eta(x)\nu(dx)
\end{aligned}
$$

One has to note that if $F \in \mathcal{D}_0$ then $\varepsilon^+ F - F \in \mathcal{D}_0 \otimes \mathbf{d}$ and if $F = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$, we have

$$
\gamma[\varepsilon^+ F - F] = \sum_{p,q} \lambda_p \overline{\lambda_q} e^{i\tilde{N}(f_p - f_q)} e^{i(f_p - f_q)} \gamma[f_p, f_q],
$$

so that

$$
\int_X \varepsilon^- \gamma[\varepsilon^+ F - F]\, dN = \int \sum_{p,q} \lambda_p \overline{\lambda_q} e^{i\tilde{N}(f_p - f_q)} \gamma[f_p, f_q]\, dN,
$$

by the construction of Proposition 10, this last term is nothing but $\Gamma[F]$. Thus, if $F \in \mathcal{D}_0$ then $\varepsilon^+ F - F \in \underline{\mathbb{D}}$ and

$$
\begin{aligned}
\|\varepsilon^+ F - F\|_{\underline{\mathbb{D}}} &= (\mathbb{E}\Gamma[F])^{\frac{1}{2}} + \mathbb{E}[\int |\varepsilon^+ F - F|\eta\, dN] \\
&\leqslant (2\mathcal{E}[F])^{\frac{1}{2}} + 2\|F\|_{L^2(\mathbb{P})}\|N(\eta)\|_{L^2(\mathbb{P})}
\end{aligned}
$$

As a consequence, $\varepsilon^+ - I$ admits a unique extension on $\mathbb{D}$. It is a continuous linear map from $\mathbb{D}$ into $\underline{\mathbb{D}}$. Since by (13) $\gamma[\varepsilon^+ F - F] = \gamma[\varepsilon^+ F]$ and $(\varepsilon^+ F - F)^\flat = (\varepsilon^+ F)^\flat$, this leads to the following theorem :

Theorem 17. *The formula*

$$
\forall F \in \mathbb{D}, \qquad F^\sharp = \int_{X\times R} \varepsilon^-((\varepsilon^+ F)^\flat)\, dN \odot \rho, \tag{25}
$$

is justified by the following decomposition:

$$
F \in \mathbb{D} \quad \overset{\varepsilon^+ - I}{\longmapsto} \quad \varepsilon^+ F - F \in \underline{\mathbb{D}} \quad \overset{\varepsilon^-((.)^\flat)}{\longmapsto} \quad \varepsilon^-((\varepsilon^+ F)^\flat) \in L^2_0(\mathbb{P}_N \times \rho) \quad \overset{d(N\odot\rho)}{\longmapsto} \quad F^\sharp \in L^2(\mathbb{P}\times\hat{\mathbb{P}})
$$

where each operator is continuous on the range of the preceding one and where $L^2_0(\mathbb{P}_N \times \rho)$ is the closed set of elements G in $L^2(\mathbb{P}_N \times \rho)$ such that $\int_R G d\rho = 0$ $\mathbb{P}_N$-a.e.
Moreover, we have for all $F \in \mathbb{D}$

$$
\Gamma[F] = \hat{\mathbb{E}}(F^\sharp)^2 = \int_X \varepsilon^-(\gamma[\varepsilon^+ F])\, dN.
$$

Proof. Let $H \in \underline{\mathbb{D}}$, there exists a sequence (H_n) of elements in $\mathcal{D}_0 \otimes \mathbf{d}$ which converges to H in $\underline{\mathbb{D}}$ and we have for all $n \in \mathbb{N}$

$$
\int \varepsilon^-(H_n^\flat)^2\, d\mathbb{P}_N d\rho = \mathbb{E}\int \varepsilon^- \gamma[H_n]\, dN \leqslant \|H_n\|^2_{\underline{\mathbb{D}}},
$$

therefore $(H_n^\flat)$ is a Cauchy sequence in $L_0^2(\mathbb{P}_N \times \rho)$ hence converges to an element in $L_0^2(\mathbb{P}_N \times \rho)$ that we denote by $\varepsilon^-(H^\flat)$.
Moreover, if $K \in L_0^2(\mathbb{P}_N \times \rho)$, we have

$$\mathbb{E}\hat{\mathbb{E}}\left(\int_{X\times R} K(w,x,r)\, N \odot \rho(dxdr)\right)^2 = \mathbb{E}\int_{X\times R} K^2\, dN d\rho = \|K\|^2_{L^2(\mathbb{P}_N\times\rho)}.$$

This provides the assertion of the statement. □

The functional calculus for $\sharp$ and Γ involves mutually singular measures and may be followed step by step :

Let us first recall that by Lemma 13 the map $(w,x) \mapsto (\varepsilon_x^+(w), x)$ applied to classes of functions $\mathbb{P}_N$-a.e. yields classes of functions $\mathbb{P}\times\nu$-a.e. and also the map $(w,x) \mapsto (\varepsilon_x^-(w), x)$ applied to classes of functions $\mathbb{P}\times\nu$-a.e. yields classes of functions $\mathbb{P}_N$-a.e.

But product functionals of the form $F(w,x) = G(w)g(x)$ where G is a class $\mathbb{P}$-a.e. and g a class ν-a.e. belong necessarily to a single class $\mathbb{P}_N$-a.e. Hence, if we applied ε^+ to such a functional, this yields a unique class $\mathbb{P}\times\nu$-a.e. In particular with $F = e^{i\tilde{N}f}g$:

$$\varepsilon^+(e^{i\tilde{N}f}g) = e^{i\tilde{N}f}e^{if}g \quad \mathbb{P}\times\nu\text{-a.e.}$$

from this class the operator ε^- yields a class $\mathbb{P}_N$-a.e.

$$\varepsilon^-(e^{i\tilde{N}f}e^{if}g) = e^{i\tilde{N}f}g \quad \mathbb{P}_N\text{-a.e.}$$

and this result is the same as F $\mathbb{P}_N$-a.e.

This applies to the case where F depends only on w and is defined $\mathbb{P}$-a.e. then

$$\varepsilon^-(\varepsilon^+F)) = F \quad \mathbb{P}_N\text{-a.e.}$$

Thus the functional calculus decomposes as follows :

Proposition 18. *Let us consider the subset of $\mathbb{D}$ of functionals of the form $H = \Phi(F_1,\ldots,F_n)$ with $\Phi \in \mathcal{C}^1 \cap Lip(\mathbb{R}^n)$ and $F_i \in \mathbb{D}$, putting $F = (F_1,\ldots,F_n)$ we have the following :*

$$\begin{array}{llll}
a) & (\varepsilon^+H)^\flat & = \sum_i \Phi_i'(\varepsilon^+F)(\varepsilon^+F_i)^\flat & \mathbb{P}\times\nu\times\rho\text{-a.e.} \\
 & \gamma[\varepsilon^+H] & = \sum_{ij} \Phi_i'(\varepsilon^+F)\Phi_j'(\varepsilon^+F)\gamma[\varepsilon^+F_i,\varepsilon^+F_j] & \mathbb{P}\times\nu\text{-a.e.} \\
b) & \varepsilon^-(\varepsilon^+H)^\flat & = \sum_i \Phi_i'(F)\varepsilon^-(\varepsilon^+F_i)^\flat & \mathbb{P}_N\times\rho\text{-a.e.} \\
 & \varepsilon^-\gamma[\varepsilon^+H] & = \sum_{ij} \Phi_i'(F)\Phi_j'(F)\varepsilon^-\gamma[\varepsilon^+F_i,\varepsilon^+F_j] & \mathbb{P}_N\text{-a.e.} \\
c) & H^\sharp = \int \varepsilon^-((\varepsilon^+H)^\flat)\, dN\odot\rho & = \sum_i \Phi_i'(F)\int \varepsilon^-(\varepsilon^+F_i)^\flat\, dN\odot\rho & \mathbb{P}\times\hat{\mathbb{P}}\text{-a.e.} \\
 & \Gamma[H] = \int \varepsilon^-\gamma[\varepsilon^+H]dN & = \sum_{ij} \Phi_i'(F)\Phi_j'(F)\int \varepsilon^-\gamma[\varepsilon^+F_i,\varepsilon^+F_j]dN & \mathbb{P}\text{-a.e.}
\end{array}$$

Remark 2. *The projection of the measure $\mathbb{P}_N$ on Ω is a (possibly non σ-finite) measure equivalent to $\mathbb{P}$ only if $\nu(X) = +\infty$, i.e. if $\mathbb{P}\{N(1) > 0\} = 1$.*

If $\nu(X) = \|\nu\| < +\infty$, then $\mathbb{P}\{N(1) = 0\} = e^{-\|\nu\|} > 0$, and the sufficient condition for existence of density $\Gamma[F] > 0$ $\mathbb{P}$-a.s. is never fulfilled because $\Gamma[F] = \int \varepsilon^-(\gamma[\varepsilon^+F])\, dN$ vanishes on $\{N(1) = 0\}$. Conditioning arguments with respect to the set $\{N(1) > 0\}$ have to be used.

3.3.2. The lent particle method: first application

The preceding theorem provides a new method to study the regularity of Poisson functionals, that we present on an example.

Let us consider, for instance, a real process Y_t with independent increments and Lévy measure σ integrating x^2, Y_t being supposed centered without Gaussian part. We assume that σ has a density satisfying Hamza's condition (Fukushima-Oshima-Takeda [16] p105) so that a local Dirichlet structure may be constructed on $\mathbb{R}\backslash\{0\}$ with carré du champ $\gamma[f] = x^2 f'^2(x)$. We suppose also hypothesis (BC) (cf §3.2.1). If N is the random Poisson measure with intensity $dt \times \sigma$ we have $\int_0^t h(s)\, dY_s = \int \mathbf{1}_{[0,t]}(s)h(s)x\tilde{N}(dsdx)$ and the choice done for γ gives $\Gamma[\int_0^t h(s)dY_s] = \int_0^t h^2(s)d[Y,Y]_s$ for $h \in L^2_{loc}(dt)$. In order to study the regularity of the random variable $V = \int_0^t \varphi(Y_{s-})dY_s$ where φ is Lipschitz and $\mathcal{C}^1$, we have two ways:

a) We may represent the gradient $\sharp$ as $Y_t^\sharp = B_{[Y,Y]_t}$ where B is a standard auxiliary independent Brownian motion. Then by the chain rule

$$V^\sharp = \int_0^t \varphi'(Y_{s-})(Y_{s-})^\sharp dY_s + \int_0^t \varphi(Y_{s-})dB_{[Y]_s}$$

now using $(Y_{s-})^\sharp = (Y_s^\sharp)_-$, a classical but rather tedious stochastic calculus yields

$$\Gamma[V] = \hat{\mathbb{E}}[V^{\sharp 2}] = \sum_{\alpha \leqslant t} \Delta Y_\alpha^2 (\int_{]\alpha}^t \varphi'(Y_{s-})dY_s + \varphi(Y_{\alpha-}))^2. \tag{26}$$

where $\Delta Y_\alpha = Y_\alpha - Y_{\alpha-}$. Since V has real values the *energy image density property* holds for V, and V has a density as soon as $\Gamma[V]$ is strictly positive a.s. what may be discussed using the relation (26).

b) An other more direct way consists in applying the theorem. For this we define $\flat$ by choosing ξ such that $\int_0^1 \xi(r)dr = 0$ and $\int_0^1 \xi^2(r)dr = 1$ and putting $f^\flat = xf'(x)\xi(r)$.

1^o. First step. We add a particle (α, x) i.e. a jump to Y at time α with size x what gives

$\varepsilon^+ V - V = \varphi(Y_{\alpha-})x + \int_{]\alpha}^t (\varphi(Y_{s-} + x) - \varphi(Y_{s-}))dY_s$

2^o. $V^\flat = 0$ since V does not depend on x, and

$(\varepsilon^+ V)^\flat = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)xdY_s)\xi(r)$ because $x^\flat = x\xi(r)$.

3^o. We compute $\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^{\flat 2} dr = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)xdY_s)^2$

4^o. We take back the particle we gave, in order to compute $\int \varepsilon^- \gamma[\varepsilon^+ V]dN$. That gives

$$\int \varepsilon^- \gamma[\varepsilon^+ V]dN = \int \left(\varphi(Y_{\alpha-}) + \int_{]\alpha}^t \varphi'(Y_{s-})dY_s \right)^2 x^2\, N(d\alpha dx)$$

and (26).

We remark that both operators $F \mapsto \varepsilon^+ F$, $F \mapsto (\varepsilon^+ F)^\flat$ are non-local, but instead $F \mapsto \int \varepsilon^- (\varepsilon^+ F)^\flat\, d(N \odot \rho)$ and $F \mapsto \int \varepsilon^- \gamma[\varepsilon^+ F]\, dN$ are local : taking back the lent particle gives the locality. We will deepen this example in dimension p in Part 5.

4 (EID) property on the upper space from (EID) property on the bottom space and the domain $\mathbb{D}_{loc}$

From now on, we make additional hypotheses on the bottom structure $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ which are stronger but satisfied in most of the examples.
Hypothesis (H1): X admits a partition of the form: $X = B \bigcup (\bigcup_{k=1}^{+\infty} A_k)$ where for all k, $A_k \in \mathcal{X}$ with $\nu(A_k) < +\infty$ and $\nu(B) = 0$, in such a way that for any $k \in \mathbb{N}^*$ may be defined a local Dirichlet structure with carré du champ:

$$\mathcal{S}_k = (A_k, \mathcal{X}_{|A_k}, \nu_{|A_k}, \mathbf{d}_k, \gamma_k),$$

with

$$\forall f \in \mathbf{d},\ f_{|A_k} \in \mathbf{d}_k \text{ and } \gamma[f]_{|A_k} = \gamma_k[f_{|A_k}].$$

Hypothesis (H2): Any finite product of structures $\mathcal{S}_k$ satisfies (EID).

Remark 3. *In many examples where X is a topological space, (H1) is satisfied by choosing for (A_k) $k \in \mathbb{N}^*$ a regular open set.*
Let us remark that (H2) is satisfied for the structures studied in Part 2.

The main result of this section is the following:

Proposition 19. *If the bottom structure $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ satisfies (H1) and (H2) then the upper structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies (EID).*

Proof. For all $k \in \mathbb{N}^*$, since $\nu(A_k) < +\infty$, we consider an upper structure $S_k = (\Omega_k, \mathcal{A}_k, \mathbb{P}_k, \mathbb{D}_k, \Gamma_k)$ associated to $\mathcal{S}_k$ as a direct application of the construction by product (see §3.3.2 above or Bouleau [7] Chap. VI.3).
Let $k \in \mathbb{N}^*$, we denote by N_k the corresponding random Poisson measure on A_k with intensity $\nu_{|A_k}$ and we consider N^* the random Poisson measure on X with intensity ν, defined on the product probability space

$$(\Omega^*, \mathcal{A}^*, \mathbb{P}^*) = \prod_{k=1}^{+\infty} (\Omega_k, \mathcal{A}_k, \mathbb{P}_k),$$

by

$$N^* = \sum_{k=1}^{+\infty} N_k.$$

In a natural way, we consider the product Dirichlet structure

$$S^* = (\Omega^*, \mathcal{A}^*, \mathbb{P}^*, \mathbb{D}^*, \Gamma^*) = \prod_{k=1}^{+\infty} S_k.$$

In the third Part, we have built using the Friedrichs argument, the Dirichlet structure

$$S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma),$$

let us now make the link between those structures.
First of all, thanks to Theorem 2.2.1 and Proposition 2.2.2. of Chap. V in Bouleau-Hirsch [10], we know that a function φ in $L^2(\mathbb{P}^*)$ belongs to $\mathbb{D}^*$ if and only if

1. For all $k \in \mathbb{N}^*$ and $\prod_{n \neq k} \mathbb{P}_n$-almost all $\xi_1, \cdots, \xi_{k_1}, \xi_{k+1}, \cdots$, the map
$$\xi \mapsto \varphi(\xi_1, \cdots, \xi_{k_1}, \xi, \xi_{k+1}, \cdots)$$
belongs to $\mathbb{D}_k$.

2. $\sum_k \Gamma_k[\varphi] \in L^1(\mathbb{P}^*)$ and we have $\Gamma^*[\varphi] = \sum_k \gamma_k[\varphi]$.

Consider $f \in \mathbf{d} \cap L^1(\gamma)$ then clearly
$$N(f) = \sum_k N_k(f_{|A_k})$$
belongs to $\mathbb{D}^*$ and in the same way
$$e^{i\tilde{N}(f)} = \prod_k e^{i\tilde{N}_k(f_{|A_k})} \in \mathbb{D}^*.$$
Moreover, by hypothesis (H1):
$$\begin{aligned} \Gamma^*[e^{i\tilde{N}(f)}] &= \sum_k |\prod_{l \neq k} e^{i\tilde{N}_l(f_{|A_l})}|^2 \Gamma_k[e^{i\tilde{N}_k(f_{|A_k})}] = \sum_k N_k(\gamma[f]_{|A_k}) \\ &= N(\gamma[f]) = \Gamma[e^{i\tilde{N}(f)}]. \end{aligned}$$
Thus as $\mathcal{D}_0$ is dense in $\mathbb{D}$, we conclude that $\mathbb{D} \subset \mathbb{D}^*$ and $\Gamma = \Gamma^*$ on $\mathbb{D}$.
As for all k, S_k is a product structure, thanks to hypothesis (H2) and Proposition 2.2.3 in Bouleau-Hirsch [10] Chapter V, we conclude that S^* satisfies (EID) hence S too. □

Main case. Let N be a random Poisson measure on $\mathbb{R}^d$ with intensity measure ν satisfying one of the following conditions :

i) $\nu = k\,dx$ and a function ξ (the carré du champ coefficient matrix) may be chosen such that hypotheses (HG) hold (cf §2.1)

ii) ν is the image by a Lipschitz injective map of a measure satisfying (HG) on $\mathbb{R}^q$, $q \leqslant d$,

iii) ν is a product of measures like ii),

then the associated Dirichlet structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ constructed (cf §3.2.4) with ν and the carré du champ obtained by the ξ of i) or induced by operations ii) or iii) satisfies (EID).

We end this section by a few remarks on the localization of this structure which permits to extend the functional calculus related to Γ or $\sharp$ to bigger spaces than $\mathbb{D}$, which is often convenient from a practical point of view.
Following Bouleau-Hisrch (see [10] p. 44-45) we recall that $\mathbb{D}_{loc}$ denotes the set of functions $F : \Omega \to \mathbb{R}$ such that there exists a sequence $(E_n)_{n \in \mathbb{N}^*}$ in $\mathcal{A}$ such that
$$\Omega = \bigcup_n E_n \text{ and } \forall n \in \mathbb{N}^*,\ \exists F_n \in \mathbb{D}\ F_n = F \text{ on } E_n.$$
Moreover if $F \in \mathbb{D}_{loc}$, $\Gamma[F]$ is well-defined and satisfies (EID) in the sense that
$$F_*(\Gamma[F] \cdot P) \ll \lambda^1.$$

More generally, if $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies (EID),

$$\forall F \in (\mathbb{D}_{loc})^n,\ F_*(det\Gamma[F] \cdot \mathbb{P}) \ll \lambda^n.$$

We can consider another space bigger than $\mathbb{D}_{loc}$ by considering a partition of Ω consisting in a sequence of sets with negligible boundary. More precisely, we denote by $\mathbb{D}_{LOC}$ the set of functions $F : \Omega \to \mathbb{R}$ such that there exists a sequence of disjoint sets $(A_n)_{n\in\mathbb{N}^*}$ in $\mathcal{A}$ such that $\mathbb{P}(\Omega \setminus \bigcup_n A_n) = 0$ and

$$\forall n \in \mathbb{N}^*,\ \exists F_n \in \mathbb{D}\ F_n = F \text{ on } A_n.$$

One can easily verify that it contains the localized domain of any structure S^* as considered in the proof of Proposition 19, that Γ is well-defined on $\mathbb{D}_{LOC}$, that the functional calculus related to Γ or $\sharp$ remains valid and that it satisfies (EID) i.e. if $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies (EID),

$$\forall F \in (\mathbb{D}_{LOC})^n,\ F_*(det\Gamma[F] \cdot \mathbb{P}) \ll \lambda^n.$$

5 Examples

5.1 Upper bound of a process on [0,t]

Let Y be a real process with stationary independent increments satisfying the hypotheses of example 3.3.2.

We consider a real càdlàg process K independent of Y and put $H_s = Y_s + K_s$.

Proposition 20. *If $\sigma(\mathbb{R}\backslash\{0\}) = +\infty$ and if $\mathbb{P}[\sup_{s \leqslant t} H_s = H_0] = 0$, the random variable $\sup_{s \leqslant t} H_s$ possesses a density.*

Proof. a) We may suppose that K satisfies $\sup_{s \leqslant t} |K_s| \in L^2$. Indeed, if random variables X_n have densities and $\mathbb{P}[X_n \neq X] \to 0$, then X has a density. Hence the assertion is obtained by considering $(K_s \wedge k) \vee (-k)$.

b) Let us put $M = \sup_{s \leqslant t} H_s$. Applying the lent particle method gives

$$\begin{aligned}(\varepsilon^+ M)(\alpha, x) &= \sup_{s \leqslant t}((Y_s + K_s)1_{\{s<\alpha\}} + (Y_s + x + K_s)1_{\{s \geqslant \alpha\}}) \\ &= \max(\sup_{s<\alpha}(Y_s + K_s), \sup_{s \geqslant \alpha}(Y_s + x + K_s)) \\ \gamma[\varepsilon^+ M](\alpha, x) &= \mathbf{1}_{\{\sup_{s \geqslant \alpha}(Y_s+x+K_s) \geqslant \sup_{s<\alpha}(Y_s+K_s)\}}\gamma[j](x)\end{aligned}$$

where j is the identity map $j(x) = x$.

We take back the lent particle before integrating with respect to N and obtain, since $\gamma[j](x) = x^2$,

$$\Gamma[M] = \int \varepsilon^- \gamma[\varepsilon^+ M]\, N(d\alpha dx) = \sum_{\alpha \leqslant t} \Delta Y_\alpha^2 \mathbf{1}_{\{\sup_{s \geqslant \alpha}(Y_s+K_s) \geqslant \sup_{s<\alpha}(Y_s+K_s)\}}.$$

As $\sigma(\mathbb{R}\backslash\{0\}) = +\infty$, Y has infinitely many jumps on every time interval, so that

$$\Gamma[M] = 0 \quad \Rightarrow \quad \forall \alpha \leqslant t\ \sup_{s \geqslant \alpha}(Y_s + K_s) < \sup_{s<\alpha}(Y_s + K_s)$$

and choosing α decreasing to zero, we obtain

$$\Gamma[M]=0 \quad \Rightarrow \quad \sup_{t \geqslant s \geqslant 0} H_s = H_0$$

and the proposition. □

It follows that any real Lévy process X starting at zero and immediately entering $\mathbb{R}_+^*$, whose Lévy measure dominates a measure σ satisfying Hamza's condition and infinite, is such that $\sup_{s \leqslant t} X_s$ has a density.

5.2 Regularizing properties of Lévy processes

Let Y be again a real process with stationary independent increments satisfying the hypotheses of example 3.3.2. By Hamza's condition, hypothesis (H1) is fulfilled and hypothesis (H2) ensues from Theorem 2, so that the upper structure verifies (EID).

Let S be an $\mathbb{R}^p$-valued semi-martingale independent of Y. We will say that S is pathwise p-dimensional on $[0,t]$ if almost every sample path of S on $[0,t]$ spans a p-dimensional vector space.

We consider the $\mathbb{R}^p$-valued process Z whose components are given by

$$Z_t^1 = S_t^1 + Y_t^1 \quad \text{and} \quad Z_t^i = S_t^i \quad \forall i \geqslant 2$$

and the stochastic integral

$$R = \int_0^t \psi(Z_{s-})\, dZ_s$$

where ψ is a Lipschitz and $\mathcal{C}^1$ mapping from $\mathbb{R}^p$ into $\mathbb{R}^{p\times p}$.

Proposition 21. *If $\sigma(\mathbb{R}\backslash\{0\})=+\infty$, if the Jacobian determinant of the column vector $\psi_{.1}$ does not vanish and if R is pathwise p-dimensional on $[0,t]$, then the law of R is absolutely continuous with respect to λ^p.*

Proof. We apply the lent particle method. Putting $\overline{x} = (x, 0, \ldots, 0)$ and $R^i = \sum_j \int_0^t \psi_{ij}(Z_{s-})\, dZ_s^j$, we have

$$\varepsilon^+ R^i - R^i = \psi_{i1}(Z_{\alpha-})x + \int_{]\alpha}^t (\psi_{i1}(Z_{s-}+\overline{x}) - \psi_{i1}(Z_{s-}))\, dY_s$$

as in example 3.3.2,

$$(\varepsilon^+ R^i)^\flat = (\psi_{i1}(Z_{\alpha-})x + \int_{[\alpha}^t \partial_1 \psi_{i1}(Z_{s-}+\overline{x})x\, dY_s)\xi(r)$$

and
$\gamma[\varepsilon^+ R^i, \varepsilon^+ R^j] =$

$$\left(\psi_{i1}(Z_{\alpha-}) + \int_{[\alpha}^t \partial_1\psi_{i1}(Z_{s-}+\overline{x})\, dY_s\right)\left(\psi_{j1}(Z_{\alpha-}) + \int_{[\alpha}^t \partial_1\psi_{j1}(Z_{s-}+\overline{x})\, dY_s\right) x^2.$$

We take back the lent particle before integrating in N :

$$\Gamma[R^i, R^j] = \int \varepsilon^-(\gamma[\varepsilon^+ R^i, \varepsilon^+ R^j])\, dN = \sum_{\alpha \leqslant t} \Delta Y_\alpha^2 U_\alpha U_\alpha^t$$

where U_α is the column vector $\psi_{.1}(Z_{\alpha-}) + \int_{[\alpha}^t \partial_1 \psi_{.1}(Z_{s-})\, dY_s$.

Let JT be the set of jump times of Y on $[0,t]$, we conclude that

$$\det \Gamma[R, R^t] = 0 \quad \Leftrightarrow \quad \dim \mathcal{L}(U_\alpha\,;\, \alpha \in JT) < p.$$

Let $A = \{\omega\,:\, \dim \mathcal{L}(U_\alpha\,;\, \alpha \in JT) < p\}$. Reasoning on A, there exist $\lambda_1, \ldots, \lambda_p$ such that

$$\sum_{k=1}^p \lambda_k \left(\psi_{k1}(Z_{\alpha-}) + \int_{[\alpha}^t \partial_1 \psi_{k1}(Z_{s-})\, dY_s \right) = 0 \quad \forall \alpha \in JT, \tag{27}$$

now, since $\sigma(\mathbb{R}_+ \backslash \{0\}) = +\infty$, JT is a dense countable subset of $[0,t]$, so that taking left limits in (27), using (27) anew and the fact that ψ is $\mathcal{C}^1$, we obtain

$$\sum_{k=1}^p \lambda_k \psi_{k1}(Z_{\alpha-}) = 0 \quad \forall \alpha \in JT \quad \text{hence} \quad \forall \alpha \in]0,t]$$

thus, on A, we have $\dim \mathcal{L}(\psi_{.1}(Z_{s-}); s \in]0,t]) < p$.

Then EID property yields the conclusion. □

The lent particle method and (EID) property may be applied to density results for solutions of stochastic differential equations driven by Lévy processes or random measures under Lipschitz hypotheses. Let us mention also that the gradient $\sharp$ defined in §3.2 has the property to be easily iterated, this allows to obtain conclusions on C^∞-regularity in the case of smooth coefficients. These applications will be investigated in forthcoming articles.

5.3 A regular case violating Hörmander conditions

In spite of the difficulty of the proofs, applying the method is quite easy. This will be pushed forward in another article, we are just showing here an extremely simple case, example of situations rarely taken in account in the literature.

a) Let us consider the following sde driven by a two dimensional Brownian motion

$$\left\{ \begin{array}{ll} X_t^1 & = z_1 + \int_0^t dB_s^1 \\ X_t^2 & = z_2 + \int_0^t 2X_s^1 dB_s^1 + \int_0^t dB_s^2 \\ X_t^3 & = z_3 + \int_0^t X_s^1 dB_s^1 + 2\int_0^t dB_s^2. \end{array} \right. \tag{28}$$

This diffusion is degenerate and the Hörmander conditions are not fulfilled. The generator is $A = \frac{1}{2}(U_1^2 + U_2^2) + V$ and its adjoint $A^* = \frac{1}{2}(U_1^2 + U_2^2) - V$ with $U_1 = \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$, $U_2 = \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}$ and $V = -\frac{\partial}{\partial z_2} - \frac{1}{2}\frac{\partial}{\partial z_3}$. The Lie brackets of these vectors vanish and the Lie algebra is of dimension 2 : the diffusion remains on the quadric of equation $\frac{3}{4}x_1^2 - x_2 + \frac{1}{2}x_3 - \frac{3}{4}t = C$.

b) Let us now consider the same equation driven by a Lévy process :

$$\begin{cases} Z_t^1 &= z_1 + \int_0^t dY_s^1 \\ Z_t^2 &= z_2 + \int_0^t 2Z_{s-}^1\, dY_s^1 + \int_0^t dY_s^2 \\ Z_t^3 &= z_3 + \int_0^t Z_{s-}^1\, dY_s^1 + 2\int_0^t dY_s^2 \end{cases}$$

under hypotheses on the Lévy measure such that the bottom space may be equipped with the carré du champ operator $\gamma[f] = y_1^2 f_1'^2 + y_2^2 f_2'^2$ satisfying (BC) and our hypotheses yielding EID. Let us apply the lent particle method.

$$\text{For } \alpha \leqslant t \qquad \varepsilon^+_{(\alpha,y_1,y_2)} Z_t = Z_t + \begin{pmatrix} y_1 \\ 2Y^1_{\alpha-}y_1 + 2\int_{]\alpha}^t y_1 dY_s^1 + y_2 \\ Y^1_{\alpha-}y_1 + \int_{]\alpha}^t y_1 dY_s^1 + 2y_2 \end{pmatrix} = Z_t + \begin{pmatrix} y_1 \\ 2y_1 Y_t^1 + y_2 \\ y_1 Y_t^1 + 2y_2 \end{pmatrix}.$$

where we have used $Y^1_{\alpha-} = Y^1_\alpha$ because ε^+ send into $\mathbb{P} \times \nu$ classes. That gives

$$\gamma[\varepsilon^+ Z_t] = \begin{pmatrix} y_1^2 & y_1^2 2Y_t^1 & y_1^2 Y_t^1 \\ id & y_1^2 4(Y_t^1)^2 + y_2^2 & y_1^2 2(Y_t^1)^2 + 2y_2^2 \\ id & id & y_1^2 (Y_t^1)^2 + 4y_2^2 \end{pmatrix}$$

and

$$\varepsilon^-\gamma[\varepsilon^+ Z_t] = \begin{pmatrix} y_1^2 & y_1^2 2(Y_t^1 - \Delta Y_\alpha^1) & y_1^2 (Y_t^1 - \Delta Y_\alpha^1) \\ id & y_1^2 4(Y_t^1 - \Delta Y_\alpha^1)^2 + y_2^2 & y_1^2 2(Y_t^1 - \Delta Y_\alpha^1)^2 + 2y_2^2 \\ id & id & y_1^2 (Y_t^1 - \Delta Y_\alpha^1)^2 + 4y_2^2 \end{pmatrix}$$

hence

$$\Gamma[Z_t] = \sum_{\alpha \leqslant t} (\Delta Y_\alpha^1)^2 \begin{pmatrix} 1 & 2(Y_t^1 - \Delta Y_\alpha^1) & (Y_t^1 - \Delta Y_\alpha^1) \\ id & 4(Y_t^1 - \Delta Y_\alpha^1)^2 & 2(Y_t^1 - \Delta Y_\alpha^1)^2 \\ id & id & (Y_t^1 - \Delta Y_\alpha^1)^2 \end{pmatrix} + (\Delta Y_\alpha^2)^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

If the Lévy measures of Y^1 and Y^2 are infinite, it follows that Z_t has a density as soon as

$$\dim \mathcal{L} \left\{ \begin{pmatrix} 1 \\ 2(Y_t^1 - \Delta Y_\alpha^1) \\ (Y_t^1 - \Delta Y_\alpha^1) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \alpha \in JT \right\} = 3.$$

But Y^1 possesses necessarily jumps of different sizes, hence Z_t has a density on $\mathbb{R}^3$.

It follows that the integro-differential operator

$$\tilde{A}f(z) = \int \left[f(z) - f\begin{pmatrix} z_1 + y_1 \\ z_2 + 2z_1 y_1 + y_2 \\ z_3 + z_1 y_1 + 2y_2 \end{pmatrix} - (f_1'(z)\ f_2'(z)\ f_3'(z)) \begin{pmatrix} y_1 \\ 2z_1 y_1 + y_2 \\ z_1 y_1 + 2y_2 \end{pmatrix} \right] \sigma(dy_1 dy_2)$$

is hypoelliptic at order zero, in the sense that its semigroup P_t has a density. No minoration is supposed of the growth of the Lévy measure near 0 as assumed by many authors.

This result implies that for any Lévy process Y satisfying the above hypotheses, even a subordinated one in the sense of Bochner, the process Z is never subordinated of the Markov process X solution of equation (28).

References

[1] ALBEVERIO S., KONDRATIEV Y. and RÖCKNER M. "Differential geometry of Poisson space" *C. R. Acad. Sci. Paris*, t. 323, sI, 1129-1134, (1996); and "Analysis and geometry on configuration spaces" *J. Funct. Analysis* 154, 444-500, (1998).

[2] ALBEVERIO S. and RÖCKNER M. "Classical Dirichlet forms on topological vector spaces — closability and a Cameron-Martin formula" *J. Funct. Analysis*, 88, 395-436, (1990).

[3] APPLEBAUM D. "Universal Malliavin calculus in Fock space and Lévy-Itô spaces" preprint (2008).

[4] BICHTELER K., GRAVEREAUX J.-B., JACOD J. *Malliavin Calculus for Processes with Jumps* (1987)

[5] BOULEAU N. "Décomposition de l'énergie par niveau de potentiel" *Lect. Notes in M. 1096*, Springer(1984).

[6] BOULEAU N. "Construction of Dirichlet structures" *in : Potential Theory-ICPT94*, Král, Lukeš, Netuka, Veselý, eds, De Gruyter (1996).

[7] BOULEAU N. *Error Calculus for Finance and Physics, the Language of Dirichlet Forms*, De Gruyter (2003).

[8] BOULEAU N. "Error calculus and regularity of Poisson functionals: the lent particle method" C. R. Acad. Sc. Paris, Mathématiques, Vol 346, n13-14, (2008), p779-782.

[9] BOULEAU N. and HIRSCH F."Formes de Dirichlet générales et densité des variables aléatoires réelles sur l'espace de Wiener" *J. Funct. Analysis* 69, 2, 229-259, (1986).

[10] BOULEAU N. and HIRSCH F. *Dirichlet Forms and Analysis on Wiener Space* De Gruyter (1991).

[11] COMTET L. *Advanced Combinatorics*, Springer (1974).

[12] COQUIO A. "Formes de Dirichlet sur l'espace canonique de Poisson et application aux équations différentielles stochastiques" *Ann. Inst. Henri Poincaré* vol 19, n1, 1-36, (1993)

[13] DELLACHERIE C., MAISONNEUVE B. and MEYER P.-A. *Probabilités et Potentiel* Chap XVII à XXIV, Hermann 1992.

[14] DENIS L. "A criterion of density for solutions of Poisson-driven SDEs" *Probab. Theory Relat. Fields* 118, 406-426, (2000).

[15] FEDERER H. *Geometric Measure Theory*, Springer (1969).

[16] FUKUSHIMA M., OSHIMA Y. and TAKEDA M. *Dirichlet Forms and Symmetric Markov Processes* De Gruyter (1994).

[17] Ishikawa Y. and Kunita H. "Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps" *Stoch. Processes and their App.* 116, 1743-1769, (2006).

[18] Ito K. "Spectral type of the shift transformation of differential processes with stationary increments" *Trans. Amer. Math. Soc.* 81, 253-263, (1956)

[19] Kondratiev Y., Lytvynov E. and Röckner M. "The semigroup of the Glauber dynamics of a continuous system of free particles" arXiv: math/0407359v1 (2004).

[20] Martin-Löf A. "Limit theorems for the motion of a Poisson system of independent Markovian particles with high density" *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 34, 205-223, (1976).

[21] Meyer P.-A. "Eléments de probabilités quantiques" *Sém. Prob. XX*, Lect. Notes in M. 1204, Springer (1986).

[22] Ma Z.-M. and Röckner M. *Introduction to the Theory of (non-symmetric) Dirichlet Forms*, Springer 1992).

[23] Ma and Röckner M. "Construction of diffusion on configuration spaces" *Osaka J. Math.* 37, 273-314, (2000).

[24] Neveu J. *Processus Ponctuels*, Lect. Notes in M. 598, Springer (1977).

[25] Nualart D. and Vives J. "Anticipative calculus for the Poisson process based on the Fock space", *Sém. Prob. XXIV*, Lect. Notes in M. 1426, Springer (1990).

[26] Picard J."On the existence of smooth densities for jump processes" *Probab. Theorie Relat. Fields* 105, 481-511, (1996)

[27] Picard J. "Brownian excursions, stochastic integrals and representation of Wiener functionals" *Elec. Jour. Probability* 11, 199-248, (2006).

[28] Privault N. "A pointwise equivalence of gradients on configuration spaces", *C. Rendus Acad. Sc. Paris*, 327, 7, 677-682, (1998).

[29] Röckner M. and Wielens N. "Dirichlet forms — Closability and change of speed measure", *Infinite Dimensional Analysis and Stochastic Processes*, Research Notes in M., Albeverio ed. Pitman 124, 119-144, (1985).

[30] Rota G.-C. and Wallstrom T. "Stochastic integrals: a combinatorial approach" *Ann. of Probability* 25, 3, 1257-1283, (1997).

[31] Song Sh. "Admissible vectors and their associated Dirichlet forms" *Potential Analysis* 1, 4, 319-336, (1992).

[32] Surgailis D. "On multiple Poisson stochastic integrals and associated Markov processes" *Probability and Mathematical Statistics* 3, 2, 217-239, (1984)

[33] Wu L. "Construction de l'opérateur de Malliavin sur l'espace de Poisson" *Sém. Probabilité XXI* Lect. Notes in M. 1247, Springer (1987).

Ecole des Ponts,
ParisTech, Paris-Est
6 Avenue Blaise Pascal
77455 Marne-La-Vallée Cedex 2 FRANCE
bouleau@enpc.fr

Equipe Analyse et Probabilités,
Université d'Evry-Val-d'Essonne,
Boulevard François Mitterrand
91025 EVRY Cedex FRANCE
ldenis@univ-evry.fr