Dirichlet Forms for Poisson Measures and Lévy Processes: The Lent Particle Method

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Abstract

We present a new approach to absolute continuity of laws of Poisson functionals. The theoretical framework is that of local Dirichlet forms as a tool to study probability spaces. The method gives rise to a new explicit calculus that we show on some simple examples : it consists in adding a particle and taking it back after computing the gradient.

1 Introduction

In order to situate the method it is worth to emphasize some features of the Dirichlet forms approach with comparison to the Malliavin calculus which is generally better known among probabilists.

First the arguments hold under only Lipschitz hypotheses : for example the method applies to a stochastic differential equation with Lipschitz coefficients (cf. our second lecture in this volume). Second a general criterion exists, (EID) the Energy Image Density property, (proved on the Wiener space for the Ornstein-Uhlenbeck form, still a conjecture in general cf. Bouleau-Hirsch [6] but established in the case of random Poisson measures with natural hypotheses) which ensures the existence of a density for a $\mathbb{R}^d$-valued random variable. Third, Dirichlet forms are easy to construct in the infinite dimensional frameworks encountered in probability theory and this yields a theory of errors propagation through the stochastic calculus, especially for finance and physics cf. Bouleau [2], but also for numerical analysis of PDE's and SPDE's cf. Scotti [14].

Our aim is to extend, thanks to Dirichlet forms, the Malliavin calculus applied to the case of Poisson measures and SDE's with jumps. Let us recall that in the case of jumps, there are several ways for applying the ideas of Malliavin calculus. The works are based either on the chaos decomposition (Nualart-Vives [11]) and provide tools in analogy with the Malliavin calculus on Wiener space, but non-local (Picard [12], Ishikawa-Kunita [9]) or dealing with local operators acting on the size of the jumps using the expression of the generator on a sufficiently rich class and closing the structure, for instance by Friedrichs' argument (cf. especially Bichteler-Gravereaux-Jacod [1] Coquio[7] Ma-Röckner[10]).

We follow a way close to this last one. We will first expose the method from a practical point of view, in order to show how it acts on concrete cases. Then in a separate part we shall give the main elements of the proof of the main theorem on the lent particle formula. Eventually we will display several examples where the method improves known results. Applications to stochastic differential equations driven by Lévy processes will be gathered in our second article in this volume. Complete details of the proofs and hypotheses for getting (EID) are published in [3] and [4].

2 The lent particle method

Consider a random Poisson measure as a distribution of points, and let us see a Lévy process as defined by a Poisson measure, that is let us think on the *configuration space.* We suppose the particles live in a space (called the bottom space) equipped with a local Dirichlet form with carré du champ and gradient. This makes it possible to construct a local Dirichlet form with carré du champ on the configuration space (called the upper space). To calculate for some functional the Malliavin matrix – which in the framework of Dirichlet forms becomes the carré du champ matrix – the method consists first of adding a particle to the system. The functional then possesses a new argument which is due to this new particle. We can compute the bottom-gradient of the functional with respect to this argument and as well its bottom carré du champ. Then taking back the particle we have added does not erase the new argument of the obtained functional. We can integrate the new argument with respect to the Poisson measure and this gives the upper carré du champ matrix – that is the Malliavin matrix. This is the exact summary of the method.

2.1 Let us give more details and notation.

Let $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ be a local symmetric Dirichlet structure which admits a carré du champ operator. This means that $(X, \mathcal{X}, \nu)$ is a measured space, ν is σ-finite and the bilinear form $e[f, g] = \frac{1}{2}\int \gamma[f, g]\, d\nu$, is a local Dirichlet form with domain $\mathbf{d} \subset L^2(\nu)$ and carré du champ γ (cf Fukushima-Oshima-Takeda [8] and Bouleau-Hirsch [6]). $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ is called the bottom space.

Consider a Poisson random measure on this state space with intensity measure ν. A Dirichlet structure may be constructed canonically on the probability space of this Poisson measure that we denote $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$. We call this space the upper space.

$\mathbb{D}$ is a set of functions in the domain of Γ, in other words a set of random variables which are functionals of the random distribution of points. The main result is the following formula:

For all $F \in \mathbb{D}$

$$\Gamma[F] = \int_X \varepsilon^-(\gamma[\varepsilon^+ F])\, dN$$

in which ε^+ and ε^- are the creation and annihilation operators.

Let us explain the meaning and the use of this formula on an example.

2.2 First example.

Let Y_t be a centered Lévy process with Lévy measure σ integrating x^2. We assume that σ is such that a local Dirichlet structure may be constructed on $\mathbb{R}\backslash\{0\}$ with carré du champ $\gamma[f] = x^2 f'^2(x)$.

The notion of gradient in the sense of Dirichlet forms is explained in [6] Chapter V. It is a linear operator with values in an auxiliary Hilbert space giving the carré du champ by taking the square of the Hilbert norm. It is convenient to choose for the Hilbert space a space L^2 of a probability space.

Here we define a gradient $\flat$ associated with γ by choosing ξ such that $\int_0^1 \xi(r)dr = 0$ and $\int_0^1 \xi^2(r)dr = 1$ and putting

$$f^\flat = xf'(x)\xi(r).$$

Practically $\flat$ acts as a derivation with the chain rule $(\varphi(f))^\flat = \varphi'(f).f^\flat$ (for $\varphi \in \mathcal{C}^1 \cap Lip$ or even only Lipschitz).

N is the Poisson random measure associated with Y with intensity $dt \times \sigma$ such that $\int_0^t h(s)\, dY_s = \int \mathbf{1}_{[0,t]}(s)h(s)x\tilde{N}(dsdx)$ for $h \in L^2_{loc}(\mathbb{R}_+)$.

We study the regularity of

$$V = \int_0^t \varphi(Y_{s-})dY_s$$

where φ is Lipschitz and $\mathcal{C}^1$.

1^o. First step. We add a particle (α, x) i.e. a jump to Y at time α with size x what gives

$$\varepsilon^+ V = V + \varphi(Y_{\alpha-})x + \int_{]\alpha}^t (\varphi(Y_{s-} + x) - \varphi(Y_{s-}))dY_s$$

2^o. $V^\flat = 0$ since V does not depend on x, and

$$(\varepsilon^+ V)^\flat = \left(\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)xdY_s\right)\xi(r)$$

because $x^\flat = x\xi(r)$.

3^o. We compute

$$\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^{\flat 2} dr = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)xdY_s)^2$$

4^o. We take back the particle what gives $\varepsilon^- \gamma[\varepsilon^+ V] = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-})xdY_s)^2$ and compute $\Gamma[V] = \int \varepsilon^- \gamma[\varepsilon^+ V]dN$ (lent particle formula)

$$\Gamma[V] = \int \left(\varphi(Y_{\alpha-}) + \int_{]\alpha}^t \varphi'(Y_{s-})dY_s\right)^2 x^2\, N(d\alpha dx)$$

$$= \sum_{\alpha \leqslant t} \Delta Y_\alpha^2 (\int_{]\alpha}^t \varphi'(Y_{s-})dY_s + \varphi(Y_{\alpha-}))^2.$$

where $\Delta Y_\alpha = Y_\alpha - Y_{\alpha-}$.

For real functional (EID) is always true : V possesses a density as soon as $\Gamma[V] > 0$. Then the above expression may be used to discuss the strict positivity of $\Gamma[V]$ depending on the finite or infinite mass of σ cf. [4] Example 5.2.

2.3 Energy Image Density property (EID).

Before taking an example with a multivariate functional, let us explicit the (EID) property.

A Dirichlet form on $L^2(\Lambda)$ (Λ σ-finie) with carré du champ γ satisfies (EID) if, for any d and all U with values in $\mathbb{R}^d$ whose components are in the domain of the form, the image by U of the measure with density with respect to Λ the determinant of the carré du champ matrix is absolutely continuous with respect to the Lebesgue measure i.e.

$$U_*[(\det\underline{\underline{\gamma}}[U,U^t])\cdot\Lambda] \ll \lambda^d$$

This property is true for the Ornstein-Uhlenbeck form on the Wiener space, and in several other cases cf. Bouleau-Hirsch [6]. It was conjectured in 1986 that it were always true. It is still a conjecture.

It is therefore necessary to prove this property in the context or Poisson random measures. With natural hypotheses, cf. [4] Parts 2 and 4, as soon as EID is true for the bottom space, EID is true for the upper space. Our proof uses a result of Shiqi Song [15].

2.4 Multivariate example.

Consider the same hypotheses as before on Y (which imply $1+\Delta Y_s \neq 0$ a.s.). We want to study the existence of density for the pair $(Y_t, \mathcal{E}xp(Y)_t)$ where $\mathcal{E}xp(Y)$ is the Doléans exponential of Y.

$$\mathcal{E}xp(Y)_t = e^{Y_t}\prod_{s\leqslant t}(1+\Delta Y_s)e^{-\Delta Y_s}.$$

1^0/ We add a particle (α, y) i.e. a jump to Y at time $\alpha \leqslant t$ with size y :

$$\varepsilon^+_{(\alpha,y)}(\mathcal{E}xp(Y)_t) = e^{Y_t+y}\prod_{s\leqslant t}(1+\Delta Y_s)e^{-\Delta Y_s}(1+y)e^{-y} = \mathcal{E}xp(Y)_t(1+y).$$

2^0/ We compute $\gamma[\varepsilon^+\mathcal{E}xp(Y)_t](y) = (\mathcal{E}xp(Y)_t)^2y^2$.
3^0/ We take back the particle :

$$\varepsilon^-\gamma[\varepsilon^+\mathcal{E}xp(Y)_t] = \left(\mathcal{E}xp(Y)_t(1+y)^{-1}\right)^2 y^2$$

we integrate in N and that gives the upper carré du champ operator (lent particle formula):

$$\begin{aligned}\Gamma[\mathcal{E}xp(Y)_t] &= \textstyle\int_{[0,t]\times\mathbb{R}}\left(\mathcal{E}xp(Y)_t(1+y)^{-1}\right)^2 y^2 N(d\alpha dy)\\ &= \textstyle\sum_{\alpha\leqslant t}\left(\mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1}\right)^2\Delta Y_\alpha^2.\end{aligned}$$

By a similar computation the matrix $\underline{\underline{\Gamma}}$ of the pair $(Y_t, \mathcal{E}xp(Y_t))$ is given by

$$\underline{\underline{\Gamma}} = \sum_{\alpha \leqslant t} \begin{pmatrix} 1 & \mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1} \\ \mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1} & \left(\mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1}\right)^2 \end{pmatrix} \Delta Y_\alpha^2.$$

Hence under hypotheses implying (EID) the density of the pair $(Y_t, \mathcal{E}xp(Y_t))$ is yielded by the condition

$$\dim \mathcal{L}\left(\begin{pmatrix} 1 \\ \mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1} \end{pmatrix} \quad \alpha \in JT\right) = 2$$

where JT denotes the jump times of Y between 0 and t.

Making this in details we obtain

Let Y be a real Lévy process with infinite Lévy measure with density dominating a positive continuous function $\neq 0$ near 0, then the pair $(Y_t, \mathcal{E}xp(Y)_t)$ possesses a density on $\mathbb{R}^2$.

3 Demonstration of the lent particle formula.

3.1. The construction.

Let us recall that $(E, \mathcal{X}, m, \mathbf{d}, \gamma)$ is a local Dirichlet structure with carré du champ called the bottom space, m is σ-finite and the bilinear form $e[f,g] = \frac{1}{2}\int \gamma[f,g]\,dm$ is a local Dirichlet form with domain $\mathbf{d} \subset L^2(m)$ and with carré du champ γ. For all $x \in X$, $\{x\}$ is supposed to belong to $\mathcal{X}$, m is diffuse. The associated generator is denoted a, its domain is $\mathcal{D}(a) \subset \mathbf{d}$.

We consider a random Poisson measure N, on $(E, \mathcal{X}, m)$ with intensity m. It is defined on $(\Omega, \mathcal{A}, \mathbb{P})$ where Ω is the configuration space of countable sums of Dirac masses on E, $\mathcal{A}$ is the σ-field generated by N and $\mathbb{P}$ is the law of N.

$(\Omega, \mathcal{A}, \mathbb{P})$ is called the upper space. The question is to construct a Dirichlet structure on the upper space, induced "canonically" by the Dirichlet structure of the bottom space.

This question is natural by the following interpretation. The bottom structure may be thought as the elements for the description of a single particle moving according to a symmetric Markov process associated with the bottom Dirichlet form. Then considering an infinite family of independent such particles with initial law given by $(\Omega, \mathcal{A}, \mathbb{P})$ shows that a Dirichlet structure can be canonically considered on the upper space (cf. the introduction of [4] for different ways of tackling this question).

Because of some formulas on functions of the form $e^{iN(f)}$ related to the Laplace functional, we consider the space of test functions

$$\mathcal{D}_0 = \mathcal{L}\{e^{i\tilde{N}(f)} \text{ with } f \in \mathcal{D}(a) \cap L^1(m) \text{ et } \gamma[f] \in L^2\}.$$

and for $U = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$ in $\mathcal{D}_0$, we put

$$A_0[U] = \sum_p \lambda_p e^{i\tilde{N}(f_p)}(i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p])).$$

In order to show that A_0 is uniquely defined and is the generator of a Dirichlet form satisfying the needed properties, starting from a gradient of the bottom structure we construct a gradient for the upper structure defined first on the test functions. Then we show that this gradient do not depend on the form of the test function and this allows to extend the operators thanks to Friedrichs' property yielding the closedness of the upper structure.

3.2. The bottom gradient.

We suppose the space $\mathbf{d}$ separable, then there exists a gradient for the bottom space : There is a separable Hilbert space and a linear map D from $\mathbf{d}$ into $L^2(X,m;H)$ such that $\forall u \in \mathbf{d}$, $\|D[u]\|_H^2 = \gamma[u]$, then necessarily

- If $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz then $\forall u \in \mathbf{d}$,
$D[F \circ u] = (F' \circ u)Du$,
- If F is $\mathcal{C}^1$ and Lipschitz from $\mathbb{R}^d$ into $\mathbb{R}$ then
$D[F \circ u] = \sum_{i=1}^d (F_i' \circ u)D[u_i] \;\; \forall u = (u_1, \cdots, u_d) \in \mathbf{d}^d$.

We take for H a space $L^2(R,\mathcal{R},\rho)$ where $(R,\mathcal{R},\rho)$ is a probability space s.t. $L^2(R,\mathcal{R},\rho)$ be infinite dimensional. The gradient D is denoted $\flat$:

$$\forall u \in \mathbf{d}, \;\; Du = u^\flat \in L^2(X \times R, \mathcal{X} \otimes \mathcal{R}, m \otimes \rho).$$

Without loss of generality, we assume moreover that operator $\flat$ takes its values in the orthogonal space of 1 in $L^2(R,\mathcal{R},\rho)$. So that we have

$$\forall u \in \mathbf{d}, \;\; \int u^\flat d\rho = 0 \;\; \nu\text{-}a.e.$$

3.3. Candidate gradient for the upper space.

We introduce the operators ε^+ and ε^- :

$$\begin{aligned} \forall x, w \in \Omega, \quad \varepsilon_x^+(w) \; &= w\mathbf{1}_{\{x \in supp\, w\}} + (w + \varepsilon_x)\mathbf{1}_{\{x \notin supp\, w\}}. \\ \varepsilon_x^-(w) \; &= (w - \varepsilon_x)\mathbf{1}_{\{x \in supp\, w\}} + w\mathbf{1}_{\{x \notin supp\, w\}}. \end{aligned}$$

So that for all $w \in \Omega$,

$$\varepsilon_x^+(w) = w \;\text{ et }\; \varepsilon_x^-(w) = w - \varepsilon_x \qquad \text{for } N_w\text{-almost every } x$$

$$\varepsilon_x^+(w) = w + \varepsilon_x \;\text{ et }\; \varepsilon^-(w) = w \qquad \text{for } m\text{-almost every } x$$

Definition. *For $F \in \mathcal{D}_0$, we define the pre-gradient*

$$F^\sharp = \int \varepsilon^-((\varepsilon^+ F)^\flat)\, dN \odot \rho.$$

where $N \odot \rho$ is the point process N "marked" by ρ

i.e. if N is the family of points X_i, $N \odot \rho$ is the family of pairs (X_i, r_i) where the r_i are new independent random variables mutually independent and identically distributed with law ρ. So $N \odot \rho$ is a Poisson random measure on $E \times R$.

3.4 Main result.

The above candidate may be shown to extend in a true gradient for the upper structure. The argument is based on the extension of the pregenerator A_0 thanks to Friedrichs' property (cf. for instance [6] p4) : A_0 is shown to be well defined on $\mathcal{D}_0$ which is dense, A_0

is negative and symmetric and therefore possesses a selfadjoint extension. This is stated as follows:
Theorem. *The formula*

$$\forall F \in \mathbb{D},\ F^\sharp = \int_{E\times R} \varepsilon^-((\varepsilon^+ F)^\flat)\, dN \odot \rho,$$

extends from $\mathcal{D}_0$ *to* $\mathbb{D}$*, it is justified by the following decomposition :*

$$F \in \mathbb{D} \overset{\varepsilon^+-I}{\mapsto} \varepsilon^+ F - F \in \underline{\mathbb{D}} \overset{\varepsilon^-((.)^\flat)}{\mapsto} \varepsilon^-((\varepsilon^+ F)^\flat) \in L^2_0(\mathbb{P}_N \times \rho) \overset{d(N\odot\rho)}{\mapsto} F^\sharp \in L^2(\mathbb{P}\times\hat{\mathbb{P}})$$

where each operator is continuous on the range of the preceding one and where $L^2_0(\mathbb{P}_N \times \rho)$ *is the closed set of elements* G *in* $L^2(\mathbb{P}_N \times \rho)$ *such that* $\int_R G d\rho = 0$ $\mathbb{P}_N$*-a.s.*

Furthermore for all $F \in \mathbb{D}$

$$\Gamma[F] = \hat{\mathbb{E}}(F^\sharp)^2 = \int_E \varepsilon^-\gamma[\varepsilon^+ F]\, dN.$$

Let us explicit the steps of a typical calculation applying this theorem.

Let be $H = \Phi(F_1, \dots, F_n)$ avec $\Phi \in \mathcal{C}^1 \cap Lip(\mathbb{R}^n)$ and $F = (F_1, \dots, F_n)$ with $F_i \in \mathbb{D}$, we have :

a) $\gamma[\varepsilon^+ H] = \sum_{ij} \Phi'_i(\varepsilon^+ F)\Phi'_j(\varepsilon^+ F)\gamma[\varepsilon^+ F_i, \varepsilon^+ F_j]$ $\quad \mathbb{P}\times\nu$-a.e.

b) $\varepsilon^-\gamma[\varepsilon^+ H] = \sum_{ij} \Phi'_i(F)\Phi'_j(F)\varepsilon^-\gamma[\varepsilon^+ F_i, \varepsilon^+ F_j]$ $\quad \mathbb{P}_N$-a.e.

c) $\Gamma[H] = \int \varepsilon^-\gamma[\varepsilon^+ H] dN = \sum_{ij} \Phi'_i(F)\Phi'_j(F) \int \varepsilon^-\gamma[\varepsilon^+ F_i, \varepsilon^+ F_j] dN$ $\quad \mathbb{P}$-a.e.

As we see above, an peculiarity of the method comes from the fact that it involves, in the computation, successively mutually singular measures, as measures $\mathbb{P}_N = \mathbb{P}(d\omega)N(\omega, dx)$ and $\mathbb{P}\times\nu$. This imposes some care in the applications.

4 Applications

4.1. Sup of a stochastic process on $[0, t]$.

The fact that the operation of taking the maximum is typically a Lipschitz operation makes it easy to apply the method.

Let Y be a centered Lévy process as in §2.2. Let K be a càdlàg process independent of Y. We put

$$H_s = Y_s + K_s.$$

Proposition. *If* $\sigma(\mathbb{R}\backslash\{0\}) = +\infty$ *and if* $\mathbb{P}[\sup_{s\leqslant t} H_s = H_0] = 0$*, the random variable* $\sup_{s\leqslant t} H_s$ *has a density.*

As a consequence, *any Lévy process starting from zero and immediately entering* $\mathbb{R}^*_+$*, whose Lévy measure dominates a measure* σ *satisfying Hamza condition and infinite, is such that* $\sup_{s\leqslant t} X_s$ *has a density.*

Let us recall that the Hamza condition (cf. Fukushima and al.[8] Chapter 3) gives a necessary and sufficient condition of existence of a Dirichlet structure on $L^2(\sigma)$. Such a necessary and sufficient condition is only known in dimension one.

4.2. Regularity withour Hörmander.

Consider the following SDE driven by a two dimensional Brownian motion

$$\left\{\begin{array}{ll} X_t^1 & = z_1 + \int_0^t dB_s^1 \\ X_t^2 & = z_2 + \int_0^t 2X_s^1 dB_s^1 + \int_0^t dB_s^2 \\ X_t^3 & = z_3 + \int_0^t X_s^1 dB_s^1 + 2\int_0^t dB_s^2. \end{array}\right. \tag{1}$$

This diffusion is degenerate and the Hörmander conditions are not fulfilled. The generator is $A = \frac{1}{2}(U_1^2 + U_2^2) + V$ and its adjoint $A^* = \frac{1}{2}(U_1^2 + U_2^2) - V$ with $U_1 = \frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}$, $U_2 = \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}$ and $V = -\frac{\partial}{\partial z_2} - \frac{1}{2}\frac{\partial}{\partial z_3}$. The Lie brackets of these vectors vanish and the Lie algebra is of dimension 2 : the diffusion remains on the quadric of equation $\frac{3}{4}x_1^2 - x_2 + \frac{1}{2}x_3 - \frac{3}{4}t = C$.
Consider now the same equation driven by a Lévy process :

$$\left\{\begin{array}{ll} Z_t^1 & = z_1 + \int_0^t dY_s^1 \\ Z_t^2 & = z_2 + \int_0^t 2Z_{s_-}^1 dY_s^1 + \int_0^t dY_s^2 \\ Z_t^3 & = z_3 + \int_0^t Z_{s_-}^1 dY_s^1 + 2\int_0^t dY_s^2 \end{array}\right.$$

under hypotheses on the Lévy measure such that the bottom space may be equipped with the carré du champ operator $\gamma[f] = y_1^2 f_1'^2 + y_2^2 f_2'^2$ satisfying the hypotheses yielding EID. Let us apply in full details the lent particle method.

$$\text{For } \alpha \leqslant t \qquad \varepsilon^+_{(\alpha,y_1,y_2)} Z_t = Z_t + \begin{pmatrix} y_1 \\ 2Y_{\alpha-}^1 y_1 + 2\int_{]\alpha}^t y_1 dY_s^1 + y_2 \\ Y_{\alpha-}^1 y_1 + \int_{]\alpha}^t y_1 dY_s^1 + 2y_2 \end{pmatrix} = Z_t + \begin{pmatrix} y_1 \\ 2y_1 Y_t^1 + y_2 \\ y_1 Y_t^1 + 2y_2 \end{pmatrix}.$$

where we have used $Y_{\alpha-}^1 = Y_\alpha^1$ because ε^+ send into $\mathbb{P}\times\nu$ classes. That gives

$$\gamma[\varepsilon^+ Z_t] = \begin{pmatrix} y_1^2 & y_1^2 2Y_t^1 & y_1^2 Y_t^1 \\ id & y_1^2 4(Y_t^1)^2 + y_2^2 & y_1^2 2(Y_t^1)^2 + 2y_2^2 \\ id & id & y_1^2 (Y_t^1)^2 + 4y_2^2 \end{pmatrix}$$

and

$$\varepsilon^-\gamma[\varepsilon^+ Z_t] = \begin{pmatrix} y_1^2 & y_1^2 2(Y_t^1 - \Delta Y_\alpha^1) & y_1^2(Y_t^1 - \Delta Y_\alpha^1) \\ id & y_1^2 4(Y_t^1 - \Delta Y_\alpha^1)^2 + y_2^2 & y_1^2 2(Y_t^1 - \Delta Y_\alpha^1)^2 + 2y_2^2 \\ id & id & y_1^2 (Y_t^1 - \Delta Y_\alpha^1)^2 + 4y_2^2 \end{pmatrix}$$

hence

$$\Gamma[Z_t] = \sum_{\alpha\leqslant t} (\Delta Y_\alpha^1)^2 \begin{pmatrix} 1 & 2(Y_t^1 - \Delta Y_\alpha^1) & (Y_t^1 - \Delta Y_\alpha^1) \\ id & 4(Y_t^1 - \Delta Y_\alpha^1)^2 & 2(Y_t^1 - \Delta Y_\alpha^1)^2 \\ id & id & (Y_t^1 - \Delta Y_\alpha^1)^2 \end{pmatrix} + (\Delta Y_\alpha^2)^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

With this formula we can reason, trying to find conditions for the determinant of $\Gamma[Z]$ to be > 0. For instance if the Lévy measures of Y^1 and Y^2 are infinite, it follows that Z_t has a density as soon as

$$\dim \mathcal{L}\left\{\begin{pmatrix} 1 \\ 2(Y_t^1 - \Delta Y_\alpha^1) \\ (Y_t^1 - \Delta Y_\alpha^1) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \alpha \in JT\right\} = 3.$$

But Y^1 possesses necessarily jumps of different sizes, hence Z_t has a density on $\mathbb{R}^3$.

It follows that the integro-differential operator

$$\tilde{A}f(z) = \int \left[f(z) - f\begin{pmatrix} z_1 + y_1 \\ z_2 + 2z_1y_1 + y_2 \\ z_3 + z_1y_1 + 2y_2 \end{pmatrix} - (f_1'(z)\ f_2'(z)\ f_3'(z)) \begin{pmatrix} y_1 \\ 2z_1y_1 + y_2 \\ z_1y_1 + 2y_2 \end{pmatrix}\right] \sigma(dy_1dy_2)$$

is hypoelliptic at order zero, in the sense that its semigroup P_t has a density. No minoration is supposed of the growth of the Lévy measure near 0 as assumed by many authors.

This result implies that for any Lévy process Y satisfying the above hypotheses, even a subordinated one in the sense of Bochner, the process Z is never subordinated of the Markov process X solution of equation (1) (otherwise it would live on the same manifold as the initial diffusion.)

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