Numerical Approximations of McKean Anticipative Backward Stochastic Differential Equations Arising in Initial Margin Requirements

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Outline

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Classical theory for option pricing and hedging

- Let us assume that a risky asset S follows the dynamics

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$

and a riskless asset S^0, where $\frac{dS^0_t}{S^0_t} = rdt$

- Given an European option with payoff $g(S_T)$, the dynamics of a self-financing hedging portfolio in S, S^0 is, using the notation $\theta_t = \frac{\mu_t - r}{\sigma_t}$ and $Z_t = \sigma_t \pi_t$

$$\begin{aligned} dV_t =& r\left(V_t - \pi_t\right) dt + \frac{\pi_t}{S_t} dS_t = rV_t dt + \theta_t Z_t dt + Z_t dW_t \\ V_T =& g(S_T) \end{aligned}$$

- Linear BSDE, the price of the portfolio is given by

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} g(S_T) \middle| \mathcal{F}_t \right] = v(t, S_t)$$

where $W + \int_0^{\cdot} \theta_s ds$ is a Brownian motion in the probability measure $\mathbb{Q}$.

A first BSDE nonlinear in the sense of McKean

- New regulation since 2008: change in pricing and hedging rules (from linear to nonlinear BSDEs)
- New constraint: post collateral to Central Counterparty (CCP) to secure position
- CVaR variation margin: the value of the deposit depends on the CVaR of the portfolio computed over 10 days

- The equation satisfied by the hedging portfolio becomes (MKA)

$$\begin{aligned} dV_t &= rV_t dt + \theta_t Z_t dt - R\lambda \mathbf{CVaR}_{\alpha,\mathcal{F}_t}\left(V_t - V_{(t+\Delta)\wedge T}\right) dt + Z_t dW_t \\ V_T &= g(S_T) \end{aligned}$$

- Anticipative BSDE, with dependence in conditional law (MKABSDE)
- Correction w.r.t. the price without collateralization ?
- Assumption: here, no default, safe products

Objectives

- Is (MKA) a well-posed problem?
- How do we compute the correction induced by the nonlinear CVaR term on the price of an European option?

Outline

Framework

- We are looking for a process (Y, Z), such that (MKABSDE)

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s, \Lambda_s\left(Y_{s:T}\right)\right) ds - \int_t^T Z_s dW_s, \ t \in [0, T].$$

- For $t \in [0, T]$ and $X \in \mathbb{H}^2_{0,T}(\mathbb{R})$, $\Lambda_t(X)$ is a $\mathcal{F}_t$-measurable random variable with values in $\mathbb{R}$.
- Λ_t is a functional of the future path of $(Y_s)_{s\in[t,T]} =: Y_{s:T}$.
- For $s, y, z, x \in [0, T] \times \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}$, $f(s, y, z, x)$ is a $\mathcal{F}_s$-adapted random variable with values in $\mathbb{R}$.
- ξ is a square integrable $\mathcal{F}_T$-measurable random variable.

Existence and Uniqueness

Theorem

Under integrability Assumptions on ξ, f and Lipschitz properties on f, Λ, the BSDE (MKABSDE) has a unique solution in $\mathbb{S}^2_{0,T}(\mathbb{R}) \times \mathbb{H}^2_{0,T}(\mathbb{R}^q)$.

Proof.

Standard a priori estimates in $\mathbb{S}^2_{\beta,T} \times \mathbb{H}^2_{\beta,T} = \mathbb{S}^2_{0,T} \times \mathbb{H}^2_{0,T}$. For $\beta \geq 0$ large enough, the map $\phi : (U, V) \in \mathbb{S}^2_{\beta,T} \times \mathbb{H}^2_{\beta,T} \to (Y, Z) := \phi(U, V) \in \mathbb{S}^2_{\beta,T} \times \mathbb{H}^2_{\beta,T}$,where

$$Y_t = \xi + \int_t^T f\left(s, U_s, V_s, \Lambda_s\left(U_{s:T}\right)\right) ds - \int_t^T Z_s dW_s, \;\; t \in [0, T],$$

is a contraction under Assumptions $(L), (I), (R)$, hence existence and uniqueness. □

Corollary

BSDE (MKA) has a unique solution (Y^{MK}, Z^{MK}) in $\mathbb{S}^2_{0,T}(\mathbb{R}) \times \mathbb{H}^2_{0,T}(\mathbb{R}^q)$.

Outline

A first approximation

- We do not know how to simulate the original CVaR BSDE
- Typically, $\Delta = 1$ week and thus $\Delta \ll 1$.
- First approximation (MKA to nonlinear classical BSDE): in the CVaR term, at the lowest order,

$$V_t - V_{t+\Delta} \approx \int_t^{t+\Delta} Z_s dW_s \approx \int_t^{t+\Delta} Z_t dW_s = Z_t(W_{t+\Delta} - W_t)$$

  For $C_\alpha := \mathbf{CVarR}_\alpha(\mathcal{N}(0,1))$, the CVAR term simplifies to

$$\mathbf{CVaR}_{\alpha,\mathcal{F}_t}(V_t - V_{t+\Delta}) \approx C_\alpha |Z_t| \sqrt{(t+\Delta) \wedge T - t}$$

- We obtain the nonlinear BSDE

$$V_t^{NL} = g\left(S_T\right) + \int_t^T \left(-rV_s^{NL} - \theta_s Z_s^{NL} + R\lambda C_\alpha |Z_s^{NL}| \sqrt{(s+\Delta) \wedge T - s}\right) ds - \int_t^T Z_s^{NL} dW_s$$

A second approximation

- Let us remark that in order 0 in the parameter $\sqrt{\Delta}$, we have the solution of classical linear pricing framework:

$$V_t^{BS} = g\left(S_T\right) + \int_t^T \left(-rV_s^{BS} - \theta_s Z_s^{BS}\right) ds - \int_t^T Z_s^{BS} dW_s$$

- Second Approximation (nonlinear to linear): we make a formal derivation of the nonlinear BSDE w.r.t the parameter $\sqrt{\Delta}$ to obtain a linear BSDE (Gobet-Pagliarani 2015)

$$\begin{aligned} V_t^L =& g\left(S_T\right) + \int_t^T \left(-rV_s^L - \theta_s Z_s^L + R\lambda C_\alpha |Z_s^{BS}| \sqrt{(s+\Delta)\wedge T - s}\right) ds \\ & - \int_t^T Z_s^L dW_s \end{aligned}$$

Numerical estimates

Proposition

If μ, σ are Markovian and satisfy Lipschitz properties, there exists $K_1, K_2, K_3 > 0$, independent from Δ, such that

$$\begin{aligned}
||V^L - V^{BS}||^2_{\mathbb{S}^2_{0,T}(\mathbb{R})} + ||Z^L - Z^{BS}||^2_{\mathbb{H}^2_{0,T}(\mathbb{R}^q)} &\leq K_1 \Delta \\
||V^{NL} - V^L||^2_{\mathbb{S}^2_{0,T}(\mathbb{R})} + ||Z^{NL} - Z^L||^2_{\mathbb{H}^2_{0,T}(\mathbb{R}^q)} &\leq K_2 \Delta^2 \\
||V^{MK} - V^{NL}||^2_{\mathbb{S}^2_{0,T}(\mathbb{R})} + ||Z^{MK} - Z^{NL}||^2_{\mathbb{H}^2_{0,T}(\mathbb{R}^q)} &\leq K_3 \Delta^2
\end{aligned}$$

Conclusion and future research

- We obtained existence and uniqueness for general MKABSDEs
- We do not know how to simulate the CVaR BSDE, but in this case, as $\Delta \ll 1$, we approximated the CVaR BSDE with standard BSDEs
- We obtained estimates to control the error between the real solution and its approximation
- How to compute efficiently the approximations? How to simulate general MKABSDEs?

Thank you!

Thank you for your attention!