Partial hedging: numerical methods.

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Motivation, framework, example

- Motivations
- Framework
- The Black&Scholes case

PDE characterisation, comparison theorem

- PDE characterisation

Numerical method

- Control space truncation
- Control space discretisation
- Piecewise constant policy timestepping scheme

The numerical scheme - Black & Scholes setting
What is partial hedging?

Partial hedging aims to determine:
- prices to sell products with respect to a risk constraint (e.g. Value at Risk, or the probability of a successful hedge: quantile hedging),
- associated strategies to satisfy this constraint.

Why is it useful?
- Super-replication price can be high for insurance (complex, long-term and with high notional) products: quantile hedging allows a price reduction.
- Insurance companies need to control their balance sheet with Value at Risk constraints.
The Markovian model

Consider a risky asset with price given, for an initial condition 
\((t, x) \in [0, T] \times (0, \infty)^d\), by:

\[
X_{s}^{t, x} = x + \int_{t}^{s} \text{diag}(X_{u}^{t, x}) \mu(X_{u}^{t, x}) \, du + \int_{t}^{s} \text{diag}(X_{u}^{t, x}) \sigma(X_{u}^{t, x}) \, dW_{u}, 
\]

\[
= x + \int_{t}^{s} \mu(X_{u}^{t, x}) \, du + \int_{t}^{s} \sigma(X_{u}^{t, x}) \, dW_{u}, \, s \in [t, T].
\]

where \(W\) is a Brownian motion.

Given an initial wealth \(y \geq 0\) and a process \(\nu\) modeling the amount of wealth invested in the asset, the wealth process is:

\[
Y_{s}^{t, x, y, \nu} = y + \int_{t}^{s} f(u, X_{u}^{t, x}, Y_{u}^{t, x, y, \nu}, \nu_{u}) \, du + \int_{t}^{s} \nu_{u} \sigma(X_{u}^{t, x}) \, dW_{u}, \, s \in [t, T].
\]

Assume that the coefficients are Lipschitz continuous: we then have existence and uniqueness for every initial condition and control such that the wealth stays non-negative (such a control is called admissible).
The stochastic control problem we are interested in takes the form

\[ v(t, x, p) = \inf \{ y \geq 0 : \exists \nu, \mathbb{E} [\ell(Y^t_{\nu}, x, y, \nu - g(X^t_{\nu}))] \geq p \}, \]

where \( \ell \) satisfies hypothesis so that \( v \) is finite and with polynomial growth, and \( \text{conv} \ell(\mathbb{R}) \) is compact.

For example, if \( \ell(x) = 1_{\mathbb{R}_+}(x) \), we find the quantile hedging problem:

\[ v(t, x, p) = \inf \{ y \geq 0 : \exists \nu : \mathbb{P}(Y^t_{\nu}, x, y, \nu \geq g(X^t_{\nu})) \geq p \}. \]

In the sequel we will only consider quantile hedging, so \( \text{conv} \ell(\mathbb{R}) = [0, 1] \).

A few basic properties about \( v \):

- \( v(t, x, \cdot) \) is increasing,
- \( v(t, x, p) = 0 \) if \( p \leq p_{\text{min}}(t, x) = \mathbb{P}(g(X^t_{\nu}) = 0) \).
- \( v(t, x, 1) = \bar{v}(t, x) \) is the super-replication price of the derivative.
The first work about quantile hedging was done by Föllmer and Leukert [5], in the Black&Scholes model.

Suppose the underlying is a 1-dimensional geometric Brownian motion:

$$X_s^{x, \mu} = x + \int_t^s \mu X_u \, du + \int_t^s \sigma X_u \, dW_u, \quad s \in [t, T].$$

Suppose a hedging strategy is only possible by buying and selling the underlying in a linear market (with zero interest rate for simplicity). Given such a strategy $\nu$ and an initial wealth $y \geq 0$, the associated wealth process $Y_{y, \nu}^{y, \nu}$ is given by (recall: $\nu$ is the wealth invested in the asset):

$$Y_s^{y, \nu} = y + \int_t^s \mu \nu u \, du + \int_t^s \sigma \nu u \, dW_u, \quad s \in [t, T].$$

They provide, thanks to the Neyman-Pearson lemma from statistics, closed-form expressions for the quantile hedging problem for vanilla options.
Partial hedging of vanilla put in Black-Scholes model

The parameters used are: \( \mu = 0.05, \sigma = 0.25 \), and we are plotting the graph of \( p \mapsto v(0,30,p) \) for a put of maturity \( T = 1 \) and strike price \( K = 30 \).
If $\alpha$ is a control, let $P^{t,p,\alpha}$ be the process defined by:

$$P^{t,p,\alpha}_s = p + \int_t^s \alpha_u dW_u, \, s \in [t, T].$$

The control $\alpha$ is admissible if $P^{t,p,\alpha}_T \in [0, 1]$ a.s.

Then, by the martingale representation theorem, Bouchard, Elie and Touzi [3] prove the following:

**Lemma (The associated stochastic target problem)**

For every $(t, x, p) \in [0, T] \times (0, \infty) \times [0, 1]$, we have:

$$v(t, x, p) = \inf \left\{ y \geq 0 : \exists (\nu, \alpha), 1 \{Y^{t,x,y,\nu}_T - g(X^{t,x}_T)\} \geq P^{t,p,\alpha}_T \text{ a.s.} \right\}.$$

This lemma allows them to obtain a PDE representation for $v$, but with a discontinuous operator.
The PDE

Following an idea from Bokanowski et al. [2], Bouveret and Chassagneux [4] obtain a new PDE representation for $v$, together with a comparison theorem which implies uniqueness:

**Theorem**

$v$ is the unique positive viscosity solution, on $[0, T) \times (0, \infty)^d \times (0, 1)$, of:

\[
\begin{align*}
\sup_{a \in \mathbb{R}^d} \frac{1}{1 + |a|^2} & \left( -\partial_t \varphi - \mu X(x)^\top D_x \varphi + f(t, x, \varphi, \nabla^a \varphi) \\
& - \frac{1}{2} \text{Tr} \left[ \sigma X(x) \sigma X(x)^\top D^2_{xx} \varphi \right] - \frac{|a|^2}{2} \partial^2_{pp} \varphi - a^\top \sigma X(x)^\top D^2_{xp} \varphi \right) = 0,
\end{align*}
\]

where $\nabla^a \varphi = D_x \varphi^\top \text{diag}(x) + \partial_p \varphi \ a^\top \sigma^{-1}(x)$, satisfying to the following:

- $v(t, x, 0) = 0$ on $[0, T] \times (0, \infty)^d$,
- $v(t, x, 1) = \bar{v}(t, x)$ on $[0, T] \times (0, \infty)^d$,
- $v(T, x, p) = g(x)1_{p \neq 0}$ on $(0, \infty)^d \times [0, 1]$,
- $v(T^-, x, p) = pg(x)$ on $(0, \infty)^d \times [0, 1]$. 
Our goal is to provide a numerical method to numerically approximate the solution of this PDE.
First, in view of the discontinuity at time $t = T$, it is convenient for us to take the $v(T, x, p) = pg(x)$ on $(0, \infty)^d \times (0, 1)$ as our terminal condition. There are several numerical difficulties we have to deal with.

- Unboundedness of the control space,
- The “nonlinearity” $\frac{1}{1+|a|^2}$ in front of the $\partial_t$ term, which makes it impossible to use usual schemes.
- The semilinear term $f$ in the PDE.

In a first step, we truncate the control space in order to solve the two first issues.
Then, it allows us to discretise the control space.
Last, we introduce a piecewise constant policy iteration scheme to solve the PDE numerically.
First step: control space truncation

For each $n \geq 1$, let $K_n := [-n, n]^d \subset \mathbb{R}^d$, and we define $\nu_n$ as the unique viscosity solution of:

$$
- \partial_t \varphi - \mu X(x)^\top D_x \varphi - \frac{1}{2} \text{Tr} \left[ \sigma X(x) \sigma X(x)^\top D^2_{xx} \varphi \right] \\
+ \sup_{a \in K_n} \left( f(t, x, \varphi, \nabla^a \varphi) - \frac{|a|^2}{2} \partial^2_{pp} \varphi - a^\top \sigma X(x)^\top D^2_{xp} \varphi \right) = 0,
$$

satisfying to the same boundary conditions as $\nu$.

It is straightforward to see that $\nu_n$ is the solution of (1) where the supremum is to be taken over $K_n$. Then, we prove the following:

**Theorem**

The sequence $(\nu_n)_{n \geq 1}$ converges to $\nu$ uniformly on compact sets, as $n$ goes to infinity.

A key ingredient in the proof Dini's theorem, as the sequence of operators $(\mathcal{H}_n)$ such that $\mathcal{H}_n(\nu_n) = 0$ is increasing and simply converging.
Second step: control space discretisation

For each $m \geq 1$, let $K_{n,m}$ be a finite subset of $K_n$ satisfying:

$$\max_{a \in K_n} \min_{b \in K_{n,m}} |a - b| \leq m^{-1}.$$

Let $v_{n,m}$ be the unique viscosity solution of the following PDE:

$$-\partial_t \phi - \mu x(x)^\top D_x \phi - \frac{1}{2} \text{Tr} \left[ \sigma x(x) \sigma x(x)^\top D_{xx}^2 \phi \right]$$

$$+ \sup_{a \in K_{n,m}} \left( f(t, x, \phi, \nabla^a \phi) - \frac{|a|^2}{2} \partial_{pp}^2 \phi - a^\top \sigma x(x)^\top D_{xp}^2 \phi \right) = 0,$$

satisfying the same boundary conditions as $v$.

Then we have:

**Theorem**

As $m \to \infty$, $v_{m,n} \to v_n$, uniformly on compact sets.
We now introduce the piecewise constant policy timestepping scheme. We fix a grid \( \pi = \{ t_0 = 0 < \cdots < t_j < \cdots < t_\kappa = T \} \) (\( \kappa \geq 1 \)) for the time-discretisation. The backward algorithm for the approximation \( \hat{v} \) of \( v_{n,m} \) is given by

\[
\hat{v}(T, x, p) = g(x)p \text{ on } (0, \infty)^d \times [0, 1], \quad \hat{v}(t, x, p) = \min_{a \in K} v^a(t, x, p) \text{ on } [t_j, t_{j+1}) \times (0, \infty) \times [0, 1] \text{ for } j < \kappa.
\]

Here, for each control \( a \in K \) and \( j < \kappa \), \( v^a \) is the solution on \( [t_j, t_{j+1}) \) to:

\[
-\partial_t \varphi - \mu X(x)^\top D_x \varphi - \frac{1}{2} \text{Tr} \left[ \sigma X(x) \sigma X(x)^\top D_{xx}^2 \varphi \right] + f(t, x, \varphi, \nabla^a \varphi) - \frac{|a|^2}{2} \partial_{pp}^2 \varphi - a^\top \sigma X(x)^\top D_{xp}^2 \varphi = 0,
\]

with the boundary conditions:

\[
\varphi(t_{j+1}, x, p) = \hat{v}(t_{j+1}, x, p) \text{ on } (0, \infty)^d \times [0, 1],
\]

\[
\varphi(t, x, p) = \overline{v}(t, x)1_{p=1} \text{ on } [t_j, t_{j+1}) \times (0, \infty)^d \times \{0, 1\}.
\]
Quantile hedging in the Black & Scholes model

In the 1-dimensional Black & Scholes setting where \( \mu(x) \equiv \mu \in \mathbb{R} \) and \( \sigma(x) \equiv \sigma > 0 \), we can first perform a change of variable \( w(t, y, p) = v_{n,m}(t, e^y, p) \) on \([0, T] \times \mathbb{R} \times [0, 1]\).

Then, for \( a \in K \), the PDE to solve on each interval \([t_j, t_{j+1}), j < \kappa\) rewrites:

\[
-D^a \varphi + \left( \frac{1}{2} \sigma^2 - \mu \right) \tilde{\nabla}^a \varphi - \frac{1}{2} \Delta^a \varphi + f(t, e^y, \varphi, \tilde{\nabla}^a \varphi) = 0,
\]

with:

\[
\tilde{\nabla}^a \varphi := \frac{a}{\sigma} \partial_p \varphi,
\]

\[
\Delta^a \varphi := \sigma^2 \partial_{yy} \varphi + 2a \sigma \partial_{yp} \varphi + a^2 \partial_{pp} \varphi,
\]

\[
D^a \varphi := \partial_t \varphi + \frac{a}{\sigma} \left( \frac{1}{2} \sigma^2 - \mu \right) \partial_p \varphi.
\]

We discuss next the discretisation operators, the discretisation grids and the numerical scheme.
The operators $\tilde{\nabla}^a$ and $\tilde{\Delta}^a$ are defined to be approximated by an implicit finite difference operators in a suitable direction, and $D^a$ is approximated by an explicit finite difference operator:

$$
\hat{\Delta}^a(\delta)\varphi(t, y, p) := \frac{\sigma^2}{\delta^2} \left( \varphi(t, y + \delta, p + \frac{a}{\sigma} \delta) + \varphi(t, y - \delta, p - \frac{a}{\sigma} \delta) \right) - 2\varphi(t, y, p),
$$

$$
\hat{\nabla}^a(\delta)\varphi(t, y, p) := \frac{1}{2\delta} \left( \varphi(t, y + \delta, p + \frac{a}{\sigma} \delta) - \varphi(t, y - \delta, p - \frac{a}{\sigma} \delta) \right),
$$

$$
\hat{D}^a(h)\varphi(t, y, p) := \frac{1}{h} \left( \varphi(t + h, y, p + \frac{a}{\sigma} \left( \frac{1}{2} \sigma^2 - \mu \right) h) - \varphi(t, y, p) \right).
$$
The discretisation grids

The very definition of the discretisation operators $\hat{\nabla}^a$ and $\hat{\Delta}^a$ suggests to use a $(y, p)$-grid of the form $\Gamma_y \times \Gamma_p^a$ with:

$$\Gamma_y := \delta \mathbb{Z}, \quad \Gamma_p := \left( \frac{|a|}{\sigma} \delta \mathbb{Z} \right) \cap [0, 1].$$

However, since we want $\{0, 1\} \in \Gamma_p$, we need to slightly modify the control considered: let $N_a := \min\{n \geq 1 : n \frac{|a|}{\sigma} \delta \geq 1\}$, and set $a(a, \delta) := \frac{\sigma}{N_a \delta}$. We then set, for $a \in K$:

$$\Gamma_a := \left( \frac{a(a, \delta)}{\sigma} \delta \mathbb{Z} \right) \cap [0, 1].$$

Thus, for any $\delta > 0$, our numerical scheme will consider the control set $\{a(a, \delta) : a \in K\}$ rather than $K$. However, to prove convergence, we have the following:

**Lemma**

For all $a \in K$, we have $0 \leq |a| - a(a, \delta) \leq \frac{n^2}{\sigma} \delta$. 
For each $a \in K$, we solve on $[t_j, t_{j+1}) \times \Gamma_y \times \Gamma_a$ the PDE with control $a(a, \delta)$ using a 1-step scheme. The last operation we need to deal with is the minimisation to get $\hat{w}(t_j, y, p)$ on $\bigcup_{a \in K} (\Gamma_y \times \Gamma_a)$. It needs an interpolation, but only in the $p$-variable. However, we recall that the value function is increasing in the $p$-variable. This allows to use a monopole interpolation as described in [6].

**Remark**

*We also use this interpolation to evaluate* $\hat{w}(t_{j+1}, y, p + \frac{a}{\sigma} \left(\frac{1}{2}\sigma^2 - \mu\right))$, *needed in the explicit finite differential operator $\hat{D}^a$.***
We now have all the ingredients to produce the numerical scheme to implement in practice the piecewise constant policy timestepping scheme. Using techniques introduced in the so-called article by Barles and Souganidis [1], we show:

**Theorem**

As $\delta \to 0$ and $h \to 0$, the solution of the numerical scheme converges to $w$. 
The parameters used are: $\mu = 0.05, \sigma = 0.25$, and we are plotting the graph of $p \mapsto v(0, 30, p)$ for a put of maturity $T = 1$ and strike price $K = 30$. 
Thank you for your attention!
Guy Barles and Panagiotis E Souganidis.
Convergence of approximation schemes for fully nonlinear second order equations.

Olivier Bokanowski, Benjamin Bruder, Stefania Maroso, and Hasnaa Zidani.
Numerical approximation for a superreplication problem under gamma constraints.

Bruno Bouchard, Romuald Elie, and Nizar Touzi.
Stochastic target problems with controlled loss.

Géraldine Bouveret and Jean-François Chassagneux.
A comparison principle for pdes arising in approximate hedging problems: application to bermudan options.

Hans Föllmer and Peter Leukert.
Quantile hedging.

Frederick N Fritsch and Ralph E Carlson.
Monotone piecewise cubic interpolation.