Partial hedging: numerical methods.

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What is partial hedging ?

Partial hedging aims to determine:

- prices to sell products with respect to a risk constraint (e.g. Value at Risk, or the probability of a successful hedge: quantile hedging),
- associated strategies to satisfy this constraint.

Why is it useful?

- Super-replication price can be high for insurance (complex, long-term and with high notional) products: quantile hedging allows a price reduction.
- Insurance companies need to control their balance sheet with Value at Risk constraints.

The Markovian model

Consider a risky asset with price given, for an initial condition $(t, x) \in [0, T] \times (0, \infty)^d$, by:

$$X_s^{t,x} = x + \int_t^s \mathrm{diag}(X_u^{t,x})\mu(X_u^{t,x})\mathrm{d}u + \int_t^s \mathrm{diag}(X_u^{t,x})\sigma(X_u^{t,x})\mathrm{d}W_u\,,$$
$$= x + \int_t^s \mu_X(X_u^{t,x})\mathrm{d}u + \int_t^s \sigma_X(X_u^{t,x})\mathrm{d}W_u\,, s \in [t, T]\,.$$

where W is a Brownian motion.
Given an initial wealth $y \geq 0$ and a process ν modeling the amount of wealth invested in the asset, the wealth process is:

$$Y_s^{t,x,y,\nu} = y + \int_t^s f(u, X_u^{t,x}, Y_u^{t,x,y,\nu}, \nu_u)\mathrm{d}u + \int_t^s \nu_u \sigma(X_u^{t,x})\mathrm{d}W_u, s \in [t, T].$$

Assume that the coefficients are Lipschitz continuous: we then have existence and uniqueness for every initial condition and control such that the wealth stays non-negative (such a control is called admissible).

The stochastic control problem

The stochastic control problem we are interested in takes the form

$$v(t,x,p) = \inf\left\{y \geq 0 : \exists \nu, \mathbb{E}\left[\ell(Y_T^{t,x,y,\nu} - g(X_T^{t,x}))\right] \geq p\right\},$$

where ℓ satisfies hypothesis so that v is finite and with polynomial growth, and $\overline{\text{conv}}\,\ell(\mathbb{R})$ is compact.
For example, if $\ell(x) = \mathbb{1}_{\mathbb{R}_+}(x)$, we find the quantile hedging problem:

$$v(t,x,p) = \inf\left\{y \geq 0 : \exists \nu : \mathbb{P}\left(Y_T^{t,x,y,\nu} \geq g(X_T^{t,x})\right) \geq p\right\}.$$

In the sequel we will only consider quantile hedging, so $\overline{\text{conv}}\,\ell(\mathbb{R}) = [0,1]$.
A few basic properties about v:

- $v(t,x,\cdot)$ is increasing,
- $v(t,x,p) = 0$ if $p \leq p_{min}(t,x) = \mathbb{P}\left(g(X_T^{t,x}) = 0\right)$.
- $v(t,x,1) = \overline{v}(t,x)$ is the super-replication price of the derivative.

An example

The first work about quantile hedging was done by Föllmer and Leukert [5], in the Black&Scholes model.
Suppose the underlying is a 1-dimensional geometric Brownian motion:

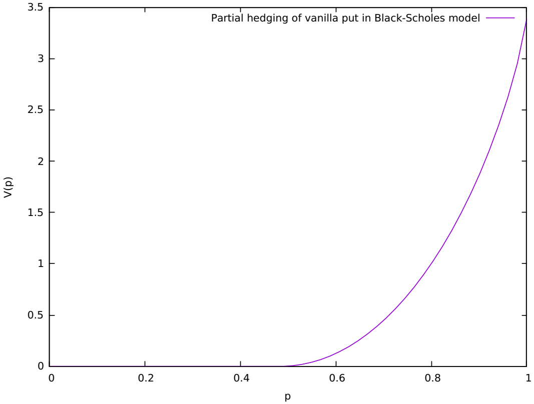
$$X_s^{t,x} = x + \int_t^s \mu X_u \mathrm{d}u + \int_t^s \sigma X_u \mathrm{d}W_u, s \in [t, T].$$

Suppose a hedging strategy is only possible by buying and selling the underlying in a linear market (with zero interest rate for simplicity). Given such a strategy ν and an initial wealth $y \geq 0$, the associated wealth process $Y^{y,\nu}$ is given by (recall: ν is the wealth invested in the asset):

$$Y_s^{t,y,\nu} = y + \int_t^s \mu \nu_u \mathrm{d}u + \int_t^s \sigma \nu_u \mathrm{d}W_u, s \in [t, T].$$

They provide, thanks to the Neyman-Pearson lemma from statistics, closed-form expressions for the quantile hedging problem for vanilla options.

An illustration



The parameters used are: $\mu = 0.05, \sigma = 0.25$, and we are plotting the graph of $p \mapsto v(0, 30, p)$ for a put of maturity $T = 1$ and strike price $K = 30$.

General case : reduction to a stochastic target problem

If α is a control, let $P^{t,p,\alpha}$ be the process defined by:

$$P_s^{t,p,\alpha} = p + \int_t^s \alpha_u \mathrm{d}W_u, s \in [t, T].$$

The control α is admissible if $P_T^{t,p,\alpha} \in [0,1]$ a.s..
Then, by the martingale reprensentation theorem, Bouchard, Elie and Touzi [3] prove the following:

Lemma (The associated stochastic target problem)

For every $(t,x,p) \in [0,T] \times (0,\infty) \times [0,1]$, we have:

$$v(t,x,p) = \inf \left\{ y \geq 0 : \exists (\nu,\alpha), \mathbb{1}_{\{Y_T^{t,x,y,\nu} - g(X_T^{t,x})\}} \geq P_T^{t,p,\alpha} a.s. \right\}.$$

This lemma allows them to obtain a PDE representation for v, but with a discontinuous operator.

The PDE

Following an idea from Bokanowski et al. [2], Bouveret and Chassagneux [4] obtain a new PDE representation for v, together with a comparison theorem which implies uniqueness:

Theorem

v is the unique positive viscosity solution, on $[0,T) \times (0,\infty)^d \times (0,1)$, of:

$$\sup_{a \in \mathbb{R}^d} \frac{1}{1+|a|^2}\Big(-\partial_t \varphi - \mu_X(x)^\top D_x \varphi + f\left(t, x, \varphi, \nabla^a \varphi\right) \qquad (1)$$

$$-\frac{1}{2}\operatorname{Tr}\left[\sigma_X(x)\sigma_X(x)^\top D^2_{xx}\varphi\right] - \frac{|a|^2}{2}\partial^2_{pp}\varphi - a^\top \sigma_X(x)^\top D^2_{xp}\varphi\Big) = 0,$$

where $\nabla^a \varphi = D_x \varphi^\top \operatorname{diag}(x) + \partial_p \varphi \, a^\top \sigma^{-1}(x)$, satisfying to the following:

$$\begin{aligned} v(t,x,0) &= 0 \text{ on } [0,T] \times (0,\infty)^d, \\ v(t,x,1) &= \overline{v}(t,x) \text{ on } [0,T] \times (0,\infty)^d, \\ v(T,x,p) &= g(x) 1_{p \neq 0} \text{ on } (0,\infty)^d \times [0,1], \\ v(T^-,x,p) &= pg(x) \text{ on } (0,\infty)^d \times [0,1]. \end{aligned}$$

Numerical approximation of v: remarks and strategy

Our goal is to provide a numerical method to numerically approximate the solution of this PDE.
First, in view of the discontinuity at time $t = T$, it is convenient for us to take the $v(T, x, p) = pg(x)$ on $(0, \infty)^d \times (0, 1)$ as our terminal condition.
There are several numerical difficulties we have to deal with.

- Unboundedness of the control space,
- The "nonlinearity" $\frac{1}{1+|a|^2}$ in front of the ∂_t term, which makes it impossible to use usual schemes.
- The semilinear term f in the PDE.

In a first step, we truncate the control space in order to solve the two first issues.
Then, it allows us to discretise the control space.
Last, we introduce a *piecewise constant policy iteration scheme* to solve the PDE numerically.

First step: control space truncation

For each $n \geq 1$, let $K_n := [-n, n]^d \subset \mathbb{R}^d$, and we define v_n as the unique viscosity solution of:

$$-\partial_t \varphi - \mu_X(x)^\top D_x \varphi - \frac{1}{2} \mathrm{Tr}\left[\sigma_X(x)\sigma_X(x)^\top D^2_{xx}\varphi\right]$$
$$+ \sup_{a \in K_n} \left(f\left(t, x, \varphi, \nabla^a \varphi\right) - \frac{|a|^2}{2} \partial^2_{pp}\varphi - a^\top \sigma_X(x)^\top D^2_{xp}\varphi \right) = 0,$$

satisfying to the same boundary conditions as v.
It is straightforward to see that v_n is the solution of (1) where the supremum is to be taken over K_n. Then, we prove the following:

Theorem

The sequence $(v_n)_{n \geq 1}$ converges to v uniformly on compact sets, as n goes to infinity.

A key ingredient in the proof Dini's theorem, as the sequence of operators $(\mathcal{H}_n)$ such that $\mathcal{H}_n(v_n) = 0$ is increasing and simply converging.

Second step: control space discretisation

For each $m \geq 1$, let $K_{n,m}$ be a finite subset of K_n satisfying:

$$\max_{a \in K_n} \min_{b \in K_{n,m}} |a - b| \leq m^{-1}.$$

Let $v_{n,m}$ be the unique viscosity solution of the following PDE:

$$-\partial_t \varphi - \mu_X(x)^\top D_x \varphi - \frac{1}{2} \operatorname{Tr}\left[\sigma_X(x)\sigma_X(x)^\top D^2_{xx}\varphi\right] + \sup_{a \in K_{n,m}} \left(f\left(t, x, \varphi, \nabla^a \varphi\right) - \frac{|a|^2}{2} \partial^2_{pp}\varphi - a^\top \sigma_X(x)^\top D^2_{xp}\varphi \right) = 0,$$

satisfying the same boundary conditions as v.

Then we have:

Theorem

As $m \to \infty$, $v_{m,n} \to v_n$, uniformly on compact sets.

Piecewise constant policy timestepping scheme

We now introduce the piecewise constant policy timestepping scheme. We fix a grid $\pi = \{t_0 = 0 < \cdots < t_j < \cdots < t_\kappa = T\}$ ($\kappa \geq 1$) for the time-discretisation. The backward algorithm for the approximation $\widehat{v}$ of $v_{n,m}$ is given by $\widehat{v}(T, x, p) = g(x)p$ on $(0, \infty)^d \times [0, 1]$, and $\widehat{v}(t, x, p) = \min_{a \in K} v^a(t, x, p)$ on $[t_j, t_{j+1}) \times (0, \infty) \times [0, 1]$ for $j < \kappa$. Here, for each control $a \in K$ and $j < \kappa$, v^a is the solution on $[t_j, t_{j+1})$ to:

$$\begin{aligned} -\partial_t \varphi - \mu_X(x)^\top D_x \varphi - \frac{1}{2} \operatorname{Tr}\left[\sigma_X(x)\sigma_X(x)^\top D^2_{xx}\varphi\right] + f\left(t, x, \varphi, \nabla^a \varphi\right) \\ - \frac{|a|^2}{2} \partial^2_{pp}\varphi - a^\top \sigma_X(x)^\top D^2_{xp}\varphi \; = 0, \end{aligned}$$

with the boundary conditions:

$$\begin{aligned} \varphi(t_{j+1}, x, p) = \widehat{v}(t_{j+1}, x, p) \text{ on } (0, \infty)^d \times [0, 1], \\ \varphi(t, x, p) = \overline{v}(t, x) 1_{p=1} \text{ on } [t_j, t_{j+1}) \times (0, \infty)^d \times \{0, 1\}. \end{aligned}$$

Quantile hedging in the Black & Scholes model

In the 1-dimensional Black & Scholes setting where $\mu(x) \equiv \mu \in \mathbb{R}$ and $\sigma(x) \equiv \sigma > 0$, we can first perform a change of variable $w(t, y, p) = v_{n,m}(t, e^y, p)$ on $[0, T] \times \mathbb{R} \times [0, 1]$.
Then, for $a \in K$, the PDE to solve on each interval $[t_j, t_{j+1}), j < \kappa$ rewrites:

$$-D^a\varphi + \left(\frac{1}{2}\sigma^2 - \mu\right)\widetilde{\nabla}^a\varphi - \frac{1}{2}\Delta^a\varphi + f(t, e^y, \varphi, \widetilde{\nabla}^a\varphi) = 0,$$

with:

$$\widetilde{\nabla}^a\varphi := \partial_y\varphi + \frac{a}{\sigma}\partial_p\varphi,$$
$$\Delta^a\varphi := \sigma^2\partial^2_{yy}\varphi + 2a\sigma\partial^2_{yp}\varphi + a^2\partial^2_{pp}\varphi,$$
$$D^a\varphi := \partial_t\varphi + \frac{a}{\sigma}\left(\frac{1}{2}\sigma^2 - \mu\right)\partial_p\varphi.$$

We discuss next the discretisation operators, the discretisation grids and the numerical scheme.

Differential operators approximation

The operators $\widetilde{\nabla}^a$ and $\widetilde{\Delta}^a$ are defined to be approximated by an implicit finite difference operators in a suitable direction, and D^a is approximated by an explicit finite difference operator:

$$\begin{aligned}
\widehat{\Delta}^a(\delta)\varphi(t,y,p) &:= \frac{\sigma^2}{\delta^2}\Big(\varphi(t,y+\delta,p+\frac{a}{\sigma}\delta)+\varphi(t,y-\delta,p-\frac{a}{\sigma}\delta) \\
&\quad -2\varphi(t,y,p)\Big), \\
\widehat{\nabla}^a(\delta)\varphi(t,y,p) &:= \frac{1}{2\delta}\left(\varphi(t,y+\delta,p+\frac{a}{\sigma}\delta)-\varphi(t,y-\delta,p-\frac{a}{\sigma}\delta)\right), \\
\widehat{D}^a(h)\varphi(t,y,p) &:= \frac{1}{h}\left(\varphi(t+h,y,p+\frac{a}{\sigma}\left(\frac{1}{2}\sigma^2-\mu\right)h)-\varphi(t,y,p)\right).
\end{aligned}$$

The discretisation grids

The very definition of the discretisation operators $\widehat{\nabla}^a$ and $\widehat{\Delta}^a$ suggests to use a (y, p)-grid of the form $\Gamma_y \times \Gamma_p^a$ with:

$$\Gamma_y := \delta\mathbb{Z}, \Gamma_p := \left(\frac{|a|}{\sigma}\delta\mathbb{Z}\right) \cap [0,1].$$

However, since we want $\{0,1\} \in \Gamma_p$, we need to slightly modify the control considered: let $N_a := \min\{n \geq 1 : n\frac{|a|}{\sigma}\delta \geq 1\}$, and set $\mathfrak{a}(a,\delta) := \frac{\sigma}{N_a\delta}$.
We then set, for $a \in K$:

$$\Gamma_a := \left(\frac{\mathfrak{a}(a,\delta)}{\sigma}\delta\mathbb{Z}\right) \cap [0,1].$$

Thus, for any $\delta > 0$, our numerical scheme will consider the control set $\{\mathfrak{a}(a,\delta) : a \in K\}$ rather than K. However, to prove convergence, we have the following:

Lemma

For all $a \in K$, we have $0 \leq |a| - \mathfrak{a}(a,\delta) \leq \frac{n^2}{\sigma}\delta$.

Constant policy solver

For each $a \in K$, we solve on $[t_j, t_{j+1}) \times \Gamma_y \times \Gamma_a$ the PDE with control $\mathfrak{a}(a, \delta)$ using a 1-step scheme.
The last operation we need to deal with is the minimisation to get $\widehat{w}(t_j, y, p)$ on $\bigcup_{a \in K} (\Gamma_y \times \Gamma_a)$.
It needs an interpolation, but only in the p-variable. However, we recall that the value function is increasing in the p-variable. This allows to use a monopole interpolation as described in [6].

Remark

We also use this interpolation to evaluate $\widehat{w}\left(t_{j+1}, y, p + \frac{a}{\sigma}\left(\frac{1}{2}\sigma^2 - \mu\right)\right)$, *needed in the explicit finite differential operator* $\widehat{D}^a$.

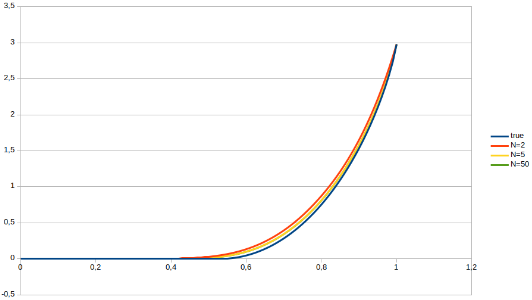
Convergence of the scheme

We now have all the ingredients to produce the numerical scheme to implement in practice the piecewise constant policy timestepping scheme. Using techniques introduced in the so-called article by Barles and Souganidis [1], we show:

Theorem

As $\delta \to 0$ and $h \to 0$, the solution of the numerical scheme converges to w.

Numerics



The parameters used are: $\mu = 0.05, \sigma = 0.25$, and we are plotting the graph of $p \mapsto v(0, 30, p)$ for a put of maturity $T = 1$ and strike price $K = 30$.

Thank you for your attention!

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