Computing risk averse equilibrium in incomplete market

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CERMICS - EPOC
Today, wholesale electricity markets take the form of an auction that matches supply and demand.

But, the demand cannot be predicted with absolute certainty. Day-ahead markets must be augmented with balancing ones.

To reduce CO$_2$ emissions and increase the penetration of renewables, there are increasing amounts of electricity from intermittent sources such as wind and solar.

Equilibrium on the market are then set in a stochastic setting.
Figure 1: Social planner
Figure 1: Social planner

Figure 2: Equilibrium
To do optimization, we aggregate uncertainty using a risk measure which turns a random variable into a real number

- the expectation $E_P$: risk neutral
- a risk measure $F$: risk averse
  - Worst Case
  - Best Case
  - Quantile
  - Median
  - Any convex combination

**Figure 3:** Aggregating uncertainty with a risk measure to obtain real value
### Definition

A complete market is a market in which the number of different Arrow–Debreu securities equals the number of states of nature.

- We will define an Arrow-Debreu security later.
- We will retain for the moment that

<table>
<thead>
<tr>
<th>Complete market</th>
<th>Incomplete market</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stage 1</strong></td>
<td><strong>Stage 2</strong></td>
</tr>
<tr>
<td>buy and sell contracts</td>
<td>buy and sell products</td>
</tr>
<tr>
<td>do nothing</td>
<td>buy and sell products</td>
</tr>
</tbody>
</table>

Stage 1 and Stage 2 represent the two stages in which market participants can act. Stage 1 involves buying and selling contracts, while Stage 2 involves buying and selling products. In an incomplete market, there may be actions that are not represented in the table, such as doing nothing, which might be permissible or necessary in certain situations.
In Philpott, Ferris, and Wets (2013), the authors present a framework for multistage stochastic equilibria. They show an equivalence between global risk neutral optimization problem and equilibrium in risk-neutral market. This allows us to decompose per agent.

They extend the implication of the result to the risk averse case with complete markets.
### Relations between Optimization and Equilibrium problems

<table>
<thead>
<tr>
<th>Optimization with Social Planner</th>
<th>Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Neutral $E_P$</td>
<td>RnSp $\iff$ RnEq</td>
</tr>
<tr>
<td>Risk Averse $F$</td>
<td>RaSp $\Rightarrow$ RaEq-AD</td>
</tr>
</tbody>
</table>

- Two questions
  - What about the reverse statement?
  - What about equilibrium in risk averse incomplete markets?
Multiple equilibrium in a incomplete market

- We show a reverse statement in the risk averse case with complete markets.
- We present a toy problem with agreeable properties (strong concavity of utility) that displays multiple equilibrium.
- Classical computing methods fail to find all equilibria.
Ingredients of the toy problem

Social planner problem (Optimization problem)

Equilibriums problem

Links between optimization problems and equilibrium problems

Multiple risk averse equilibrium
Ingredients of the toy problem

Social planner problem (Optimization problem)

Equilibriums problem

Links between optimization problems and equilibrium problems

Multiple risk averse equilibrium
Ingredients of the problem

- Two time-step market
- One good traded
- Two agents: producer and consumer
- Finite number of scenario \( \omega \in \Omega \)
- Consumption on second stage only

Figure 4: Illustration of the toy problem
Producer’s welfare and Consumer’s welfare

- Step 1: production of \( x \) at a marginal cost \( cx \)
- Step 2: random production \( x_r \) at uncertain marginal cost \( c_r x_r \)

\[
W_p(\omega) = -\frac{1}{2}cx^2 - \frac{1}{2}c_r(\omega)x_r(\omega)^2
\]

- Step 1: no consumption \( \emptyset \)
- Step 2: random consumption \( y \) at marginal utility \( V - ry \)

\[
W_c(\omega) = V(\omega)y(\omega) - \frac{1}{2}r(\omega)y(\omega)^2
\]
Ingredients of the toy problem

Social planner problem (Optimization problem)

Equilibriums problem

Links between optimization problems and equilibrium problems

Multiple risk averse equilibrium
The welfare of the social planner can be defined by

\[ W_{sp}(\omega) = W_p(\omega) + W_c(\omega) \]

- Social planner’s welfare
- Producer’s welfare
- Consumer’s welfare
Risk neutral social planner problem

Given a probability distribution $\mathbb{P}$ on $\Omega$, we can define a risk neutral social planner problem

$$\text{RnSp}(\mathbb{P}): \max_{x, x_r, y} \underbrace{\mathbb{E}_\mathbb{P}[W_{sp}]}_{\text{expected welfare}}$$

s.t. $x + x_r(\omega) = y(\omega), \ \forall \omega \in \Omega$

supply $\rightarrow$ demand
Given a risk measure $F$, we can define a risk averse social planner problem:

$$\text{RaSp}(F): \max_{x,x_r,y} F[W_{sp}]$$

risk adjusted welfare

s.t. $x + x_r(\omega) = y(\omega)$, $\forall \omega \in \Omega$

supply  demand
We study **coherent risk measures** defined by
(see Artzner, Delbaen, Eber, and Heath (1999))

\[
F[Z] = \min_{Q \in \mathcal{Q}} E_Q[Z]
\]

where \( \mathcal{Q} \) is a **convex set** of probability distributions over \( \Omega \)
Risk averse social planner problem with polyhedral risk measure

- If $\mathcal{Q}$ is a polyhedron defined by $K$ extreme points $(\mathcal{Q}_k)_{k \in [1; K]}$, then the risk measure $F$ is said to be polyhedral and is defined by

$$F[\mathcal{Z}] = \min_{\mathcal{Q}_1, \ldots, \mathcal{Q}_K} \mathbb{E}_{\mathcal{Q}_k}[\mathcal{Z}]$$

- The problem $\text{RaSp}(F)$ where $F$ is polyhedral can be written in a more convenient form for optimization

$$\max_{\theta, x, x_r, y} \theta$$

$$\text{s.t. } \theta \leq \mathbb{E}_{\mathcal{Q}_k}[\mathcal{W}_{sp}], \; k \in [1; K]$$

$$x + x_r(\omega) = y(\omega), \; \forall \omega \in \Omega$$
We have presented Optimization problems

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<td>RnEq</td>
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<td>RaEq(-AD)</td>
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Ingredients of the toy problem

Social planner problem (Optimization problem)

Equilibriums problem
  General equilibrium
  Trading risk with Arrow-Debreu securities

Links between optimization problems and equilibrium problems

Multiple risk averse equilibrium
Equilibriums problem

General equilibrium

Trading risk with Arrow-Debreu securities
Agent are price takers

Definition
An agent is *price taker* if she acts as if she has no influence on the price.

In the remain of the presentation, we consider that agents are price takers
Definition risk neutral equilibrium

Definition ((See Arrow and Debreu (1954) or Uzawa (1960)))

Given a probability $\mathbb{P}$ on $\Omega$, a risk neutral equilibrium $\text{RnEq}(\mathbb{P})$ is a set of prices $\{\pi(\omega), \omega \in \Omega\}$ such that there exists a solution to the system

\[
\text{RnEq}(\mathbb{P}): \begin{align*}
\max_{x, x_r} & \quad \mathbb{E}_\mathbb{P} \left[ W_p + \pi(x + x_r) \right] \\
\max_y & \quad \mathbb{E}_\mathbb{P} \left[ W_c - \pi y \right] \\
0 & \leq x + x_r(\omega) - y(\omega) \perp \pi(\omega) \geq 0, \quad \forall \omega \in \Omega
\end{align*}
\]

expected profit

expected utility

market clears
Remark on complementarity constraints

- Complementarity constraints are defined by
  \[ \begin{align*}
  0 \leq x + x_r(\omega) - y(\omega) \perp \pi(\omega) & \geq 0, \quad \forall \omega \in \Omega 
  \end{align*} \]

- If \( \pi > 0 \) then supply = demand

- If \( \pi = 0 \) then supply \( \geq \) demand
Definition of risk averse equilibrium

Definition
Given two risk measures $F_p$ and $F_c$, a risk averse equilibrium $RaEq(F_p, F_c)$ is a set of prices $\{\pi(\omega) : \omega \in \Omega\}$ such that there exists a solution to the system

$$RaEq(F_p, F_c): \max_{x,x_r} F_p \left[ W_p + \pi(x + x_r) \right] \quad \text{risk adjusted profit}$$

$$\max_y F_c \left[ W_c - \pi y \right] \quad \text{risk adjusted consumption}$$

$$0 \leq x + x_r(\omega) - y(\omega) \perp \pi(\omega) \geq 0, \quad \forall \omega \in \Omega \quad \text{market clears}$$

- If $F_p = F_c$ then we write $RaEq(F)$
Consumer is insensitive to the choice of risk measure

Assuming that the risk measure $\mathbb{F}_c$ of the consumer is monotonic, she can optimize scenario per scenario as she has no first stage decision.

$$\max_y \mathbb{F}_c \left[ W_c - \pi y \right]$$

risk adjusted consumption

$\iff$

$$\forall \omega \in \Omega, \max_{y(\omega)} W_c(\omega) - \pi(\omega)y(\omega)$$

scenario independant
Risk averse equilibrium with polyhedral risk measure

If the risk measure $\mathbb{F}$ is polyhedral, then RaEq($\mathbb{F}$) reads

\[
\text{RaEq: } \max_{\theta, x, x_r} \theta \\
\text{s.t. } \theta \leq \mathbb{E}_{Q_k} [W_p + \pi(x + x_r)] , \quad \forall k \in [1; K]
\]

\[
\max_{y(\omega)} \mathcal{W}_c(\omega) - \pi y(\omega) , \quad \forall \omega \in \Omega
\]

\[
0 \leq x + x_r(\omega) - y(\omega) \perp \pi(\omega) \geq 0 , \quad \forall \omega \in \Omega
\]
Equilibriums problem

General equilibrium

Trading risk with Arrow-Debreu securities
Definition of an Arrow-Debreu security

Definition

An *Arrow-Debreu security* for node $\omega \in \Omega$ is a contract that charges a price $\mu(\omega)$ in the first stage, to receive a payment of 1 in scenario $\omega$.

```
quantity q1
price p1
scenario \omega_1

scenario \omega_1 occurs
payment of q1

If scenario \omega_2 occurs
payment of 0

quantity q2
price p2
scenario \omega_2

If scenario \omega_1 occurs
payment of 0

scenario \omega_2 occurs
payment of q2
```
A *risk trading equilibrium* is sets of prices \( \{ \pi(\omega) , \omega \in \Omega \} \) and \( \{ \mu(\omega) , \omega \in \Omega \} \) such that there exists a solution to the system:

\[
\text{RaEq-AD: } \max_{x,x_r} - \sum_{\omega \in \Omega} \mu(\omega) a(\omega) + \mathbb{I} \left[ W_p + \pi(x + x_r) + a \right] \\
\text{max } \phi,y - \sum_{\omega \in \Omega} \mu(\omega) b(\omega) + \mathbb{I} \left[ W_c - \pi y + b \right] \\
0 \leq x + x_r(\omega) - y(\omega) \perp \pi(\omega) \geq 0 , \forall \omega \in \Omega \\
0 \leq -a(\omega) - b(\omega) \perp \mu(\omega) \geq 0 , \forall \omega \in \Omega \\
"supply \geq demand"
\]
RaEq with trading and polyhedral risk measure

A *risk trading equilibrium* is sets of prices \{\pi(\omega), \omega \in \Omega\} and \{\mu(\omega), \omega \in \Omega\} such that there exists a solution to the system:

**RaEq-AD:**

\[
\max_{\theta, x, x_r} \quad \theta - \sum_{\omega \in \Omega} \mu(\omega)a(\omega)
\]

value of contracts purchased

s.t.
\[
\theta \leq \mathbb{E}_{Q_k}\left[W_p + \pi(x + x_r) + a\right], \quad \forall k \in [1; K]
\]

\[
\max_{\phi, y} \quad \phi - \sum_{\omega \in \Omega} \mu(\omega)b(\omega)
\]

value of contracts purchased

s.t.
\[
\phi \leq \mathbb{E}_{Q_k}\left[W_c - \pi y + b\right], \quad \forall k \in [1; K]
\]

\[
0 \leq x + x_r(\omega) - y(\omega) \perp \pi(\omega) \geq 0, \quad \forall \omega \in \Omega
\]

\[
0 \leq -a(\omega) - b(\omega) \perp \mu(\omega) \geq 0, \quad \forall \omega \in \Omega
\]

"supply \geq demand"
We have presented Equilibrium problems

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<td>RaSp</td>
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Ingredients of the toy problem

Social planner problem (Optimization problem)

Equilibriums problem

Links between optimization problems and equilibrium problems
  In the risk neutral case
  In the risk averse case

Multiple risk averse equilibrium
Links between optimization problems and equilibrium problems

In the risk neutral case

In the risk averse case
Proposition

Let $\mathbb{P}$ be a probability measure over $\Omega$. The elements $(x^*, x_r^*, y_r^*)$ are optimal solutions to $\text{RnSp}(\mathbb{P})$ if and only if there exist non trivial equilibrium prices $\pi$ for $\text{RnEq}(\mathbb{P})$ with associated optimal controls $(x^*, x_r^*, y^*)$.

Corollary

If producer’s criterion and consumer’s criterion are strictly concave, then $\text{RnSp}(\mathbb{P})$ admit a unique solution and $\text{RnEq}(\mathbb{P})$ admit a unique equilibrium.
Links between optimization problems and equilibrium problems

In the risk neutral case

In the risk averse case
RaEq-AD is equivalent to RaSp

Theorem

Let \((x^\#_h, x^\#_r, y^\#_r)\) be optimal solutions to RaSp, with associated worst case probability measure \(\mu\). Then there exists prices \(\pi\) such that \((\pi, \mu)\) forms a risk trading equilibrium for RaEq-AD with optimal solutions \((x^\#_h, x^\#_r, y^\#_r)\)

- We adapt a result of Ralph and Smeers (2015)

Theorem

Let \((\pi, \mu)\) be equilibrium prices such that \((x^\#_h, x^\#_r, y^\#_r, a, b, \theta, \phi)\) solves RaEq-AD. Then \((x^\#_h, x^\#_r, y^\#_r)\) solves RaSP, with worst case measure \(\mu\).
Corollary

If both the producer’s and consumer’s criterion are strictly concave and some technical assumptions, then RaSp admits a unique solution and RaEq-AD admits a unique equilibrium.
### Summing up equivalences

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<td>RnSp</td>
</tr>
<tr>
<td>Risk Averse $\mathbb{F}$</td>
<td>RaSp</td>
</tr>
</tbody>
</table>

- This leads to result about **uniqueness** of equilibrium and methods of **decomposition**
- What can we say about **RaEq**? ** incomplete market **
Ingredients of the toy problem

Social planner problem (Optimization problem)

Equilibriums problem

Links between optimization problems and equilibrium problems

Multiple risk averse equilibrium
  - Numerical results
  - Analytical results
Multiple risk averse equilibrium

Numerical results

Analytical results
Recall on the problem

Recall:
- Two time-step market
- One good traded
- Two agents
- Consumption on second stage only

We focus on:
- Two scenarios $\omega_1$ and $\omega_2$
- Two prices: $\pi_1$ and $\pi_2$
- Five controls: $x$, $x_1$, $x_2$, $y_1$ and $y_2$
- Two probabilities $(p, 1 - p)$ and $(\bar{p}, 1 - \bar{p})$
- $p = \frac{1}{4}$, $\bar{p} = \frac{3}{4}$
- prices $0 < \pi_1 < \pi_2$

Figure 5: Illustration of the toy problem
Computing an equilibrium with GAMS

- GAMS with the solver PATH in the EMP framework (See Britz et al. (2013), Brook et al. (1988), Ferris and Munson (2000) and Ferris et al. (2009))

- different starting points defined by a grid $100 \times 100$ over the square $[1.220; 1.255] \times [2.05; 2.18]$

- We find one equilibrium defined by

$$\pi = (\pi_1, \pi_2) = (1.23578; 2.10953)$$
A second algorithm: the idea of tâtonnement method

\[ \pi_{k+1} < \pi_k \]

\[ \pi_{k+1} > \pi_k \]
Then we compute the equilibrium using a tâtonnement algorithm

**Data:** MAX-ITER, \((\pi_1^0, \pi_2^0), \tau\)

**Result:** A couple \((\pi_1^*, \pi_2^*)\) approximating equilibrium price \(\pi^*_1\)

1. for \(k\) from 0 to MAX-ITER do
2.   **Compute an optimal decision** for each player given a price:
3.     \(x, x_1, x_2 = \arg \max F[W_p + \pi(x + x_r)];\)
4.     \(y(\omega) = \arg \max F[W_c - \pi y];\)
5.   **Update the price**:
6.     \(\pi_1 = \pi_1 - \tau \max \{0; y_1 - (x + x_1)\};\)
7.     \(\pi_2 = \pi_2 - \tau \max \{0; y_2 - (x + x_2)\};\)
8. end
9. return \((\pi_1, \pi_2)\)

**Algorithm 1:** Walras’ tâtonnement
Running Walras’s tâtonnement algorithm starting from (1.25; 2.06), respectively from (1.22; 2.18), with 100 iterations and a step size of 0.1, we find two new equilibria

\[ \pi = (1.2256; 2.0698) \text{ and } \pi = (1.2478; 2.1564) \]

An alternative tâtonnement method called FastMarket (seeFacchinei and Kanzow (2007)) find the same equilibria
Summing up about computing equilibrium

<table>
<thead>
<tr>
<th></th>
<th>Equilibrium prices</th>
<th>Risk adjusted welfares</th>
</tr>
</thead>
<tbody>
<tr>
<td>red (Tâtonnement)</td>
<td>(1.2478; 2.1564)</td>
<td>(2.113; 0.845)</td>
</tr>
<tr>
<td>blue (GAMS)</td>
<td>(1.2358; 2.1095)</td>
<td>(2.134; 0.821)</td>
</tr>
<tr>
<td>green (Tâtonnement)</td>
<td>(1.2256; 2.0698)</td>
<td>(2.152; 0.798)</td>
</tr>
</tbody>
</table>

- No equilibrium dominates another

Figure 6: Representation of equilibrium in terms of welfare
Multiple risk averse equilibrium

Numerical results

Analytical results
Optimal control of agents with respect to a price $\pi$

There are three regimes

![Figure 7: Illustration of the three regimes](image)

<table>
<thead>
<tr>
<th>Condition</th>
<th>$X^#_c$</th>
<th>$X^#_i$</th>
<th>$Y_i^#$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E}_p\left[\pi\right]/c \leq X_c \leq \mathbb{E}_p\left[\pi\right]/c$</td>
<td>$\mathbb{E}_p\left[\pi\right]/c$</td>
<td>$\pi_i/c_i$</td>
<td>$\frac{V_i-\pi_i}{r_i}$</td>
</tr>
<tr>
<td>$\mathbb{E}_p\left[\pi\right]/c \leq X_c \leq \mathbb{E}_p\left[\pi\right]/c$</td>
<td>$\pi_i/c_i$</td>
<td>$\frac{V_i-\pi_i}{r_i}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Optimal control for producer and consumer problems

where $X_c(\pi) = \frac{1}{2(\pi_1 - \pi_2)} \left( \frac{\pi_2^2}{2c_2} - \frac{\pi_1^2}{2c_1} \right)$
Excess production function

- We have optimal control as a function of price in three regions.
- We look for prices \((\pi_1, \pi_2)\) such that supply = demands.
- The complementarity constraints are satisfied if

\[
0 = z_i(\pi) = x^\#(\pi) + x_i^\#(\pi) - y_i^\#(\pi), \quad i \in \{1, 2\}
\]

market clears for equilibrium prices

- This excess functions have three regime.
- In the green and red part the equation is linear, in the blue part the equation is quadratic.
Regimes of excess production function in scenario 1 \((z_1(\pi) = 0)\)

\[
\begin{align*}
1.220 & \quad 1.225 & \quad 1.230 & \quad 1.235 & \quad 1.240 & \quad 1.245 & \quad 1.250 & \quad 1.255 \\
2.06 & \quad 2.08 & \quad 2.10 & \quad 2.12 & \quad 2.14 & \quad 2.16 & \quad 2.18 & \\
\end{align*}
\]

**Figure 8:** Null excess function per scenario manifold for \(V_1 = 4, V_2 = \frac{48}{5}, c = \frac{23}{2}, c_1 = 1, c_2 = \frac{7}{2}, r_1 = 2, r_2 = 10.\)
Figure 9: Null excess function per scenario manifold for \( V_1 = 4, V_2 = \frac{48}{5}, c = \frac{23}{2}, c_1 = 1, c_2 = \frac{7}{2}, r_1 = 2, r_2 = 10. \)
Figure 10: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$. 
Figure 11: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$. 
Figure 12: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$. 
Remark
The **PATH solver** find the **blue equilibrium**, while the tatônements methods find equilibrium green and red. Interestingly it can be shown that the blue equilibrium is **unstable** in the sense that the dynamical system driven by $\pi' = z(\pi)$ is unstable around the blue equilibrium.

Remark
There exists a set of **non-zero measure of parameters** $V_1, V_2, c, c_1, c_2, r_1$, and $r_2$ (albeit small), that have **three distinct equilibrium** with the same properties.

Remark
We can show that the **blue equilibrium is a convex combination of red and green equilibrium**.
Stability of equilibriums (red equilibrium)

Figure 13: Representation of vector field $\pi' = z(\pi)$ around green equilibrium
Stability of equilibriums (blue equilibrium)

Figure 14: Representation of vector field $\pi' = z(\pi)$ around green equilibrium
Stability of equilibriums (green equilibrium)

Figure 15: Representation of vector field $\pi' = z(\pi)$ around green equilibrium
Figure 16: Representation of vector field $\pi' = z(\pi)$ around green equilibrium
Conclusion

In this talk we have

- shown an equivalence between risk averse social planner problem and risk trading equilibrium (respectively risk neutral equivalence)
- given theorems of uniqueness of equilibrium
- shown non uniqueness of equilibrium in incomplete market

On going work

- Extend the counter example with multiple agents and scenarios
- Do we have uniqueness with bounds on the number of Arrow-Debreu securities exchanged?


