Continuous-time Principal-Agent Problem in Partially Observed System and Path-dependent FBSDEs

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Outline

Moral hazard

- Adam Smith (1723-1790): moral hazard is a major risk in economics:

In a situation where an agent may benefit from an action whose cost is supported by others,

one should not count on agents' morality.

Principal-Agent Problem

• The Principal delegate the management of the output process $(X_t)t \in [0, T]$ to the Agent. He alone can only oversee X and decide the salary of the Agent.

• By receiving the salary (and signing the contract), the Agent devotes his effort and manage the output process. He chooses his optimal control by solving his optimization problem:

$$V_A(\xi) = \max_{\alpha} \mathbb{E}^{\alpha}[U_A(\xi - \int_0^T c_t(\alpha_t)\mathrm{d}t)]. \tag{1}$$

• The Principal chooses the optimal contract by solving the non-zero sum Stackelberg game:

$$V_P = \max_{\xi} \mathbb{E}^{\alpha^*(\xi)}[U_P(X_T - \xi)]. \tag{2}$$

Continuous time Principal-Agent Literature

Holström & Milgrom 1987 ⋮ Cvitanić & Zhang 2012: book

Sannikov 2008 ⋮ Cvitanic, Possamaï & Touzi 2015

Elie, Mastrolia, Possamaï 2016 (Many Agents)

Mastrolia, Ren 2017 (Many Principals)

Outline

Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. Let (X, W) be a standard 2-dimensional Brownian motion, μ a Gaussian variable. We assume that X, W, μ are mutually independent. Denote $\mathcal{F}_t := \sigma\{X_s, W_s, \mu, s \leq t\}$. Consider $(\mathbb{P}^{\alpha,\beta}, B)$ as a weak solution of the following controlled system

$$\begin{cases} \mathrm{d}\mu_t = (f(t)\mu_t + \alpha_t)\mathrm{d}t + \sigma(t)\mathrm{d}W_t & \mu_0 = \mu, \quad (3) \\ \mathrm{d}X_t = (h(t)\mu_t + \beta_t)\mathrm{d}t + \mathrm{d}B_t & X_0 = 0. \quad (4) \end{cases}$$

Here, $(\mu_t, \mathcal{F}_t)$, $0 \leq t \leq T$, is the unobservable process while X is the observation of the system.

Assumptions

(i) The prior distribution of μ is normal of mean and variance m_0 and σ_0 respectively, which is known to both the Agent and the Principal.

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(i) The prior distribution of μ is normal of mean and variance m_0 and σ_0 respectively, which is known to both the Agent and the Principal.

(ii) The unobservable process μ_t however is unknown to neither the Principal nor the Agent.

(iii) The Principal doesn't observe the Agent's effort, namely α and β.

Prior Analysis

Proposition

Let $\hat{\mu}_t = \mathbb{E}^{\alpha,\beta}[\mu_t|\mathcal{F}_t^X]$. *We have the following control filter:*

$$\begin{cases} \mathrm{d}\hat{\mu}_t = (f(t)\hat{\mu}_t + \alpha_t)\mathrm{d}t + h(t)V(t)\mathrm{d}I_t & \hat{\mu}_0 = m_0, \quad (5) \\ \mathrm{d}X_t = (h(t)\hat{\mu}_t + \beta_t)\mathrm{d}t + \mathrm{d}I_t & X_0 = 0, \quad (6) \\ \frac{\mathrm{d}V(t)}{\mathrm{d}t} = 2f(t)V(t) - h(t)^2V(t)^2 + \sigma(t) & V_0 = \sigma_0. \quad (7) \end{cases}$$

Here I *is the Innovation process define by*

$$I_t := B_t + \int_0^t h(s)(\mu_s - \hat{\mu}_s)\mathrm{d}s, \quad (8)$$

which is a $\mathbb{F}^X$*-adapted* $\mathbb{P}^{\alpha,\beta}$*-Brownian motion.*

Agent's Optimization Problem

We assume that the agent will only receive his pay ξ at the end of the contract. Denote c_t the agent's cost function at time t. The agent's optimization problem can be then written as follow

$$V_A = \sup_{\alpha,\beta} \mathbb{E}^{\alpha,\beta}[e^{-\int_0^T \rho_t^A \mathrm{d}t} \xi - \int_0^T c_t(\alpha_t, \beta_t)\mathrm{d}t]. \tag{9}$$

Here we only consider implementable contracts such that the Agent has at least one optimal control.

Necessary Condition

Given a contract ξ, assume that an optimal control (α^*, β^*) exists for the Agent's problem, then we have the following result.

Theorem (Necessary Condition)

Under $\mathbb{P}^{\alpha^*,\beta^*}$, the agent's value function satisfies the following FBSDE:

$$\begin{cases} \mathrm{d}Y_t = c_t(\alpha_t^*, \beta_t^*)\mathrm{d}t + Z_t \mathrm{d}I_t^* & Y_T = \Gamma_T^A \xi, \\ \mathrm{d}P_t = (Z_t - f(t)P_t + V(t)h^2(t)P_t)\mathrm{d}t + Q_t \mathrm{d}I_t^* & P_T = 0, \\ \mathrm{d}X_t = (h(t)\hat{\mu}_t + \beta_t^*)\mathrm{d}t + \mathrm{d}I_t^* & X_0 = 0, \\ \mathrm{d}\hat{\mu}_t = (f(t)\hat{\mu}_t + \alpha_t^*)\mathrm{d}t + h(t)V(t)\mathrm{d}I_t^* & \hat{\mu}_0 = m_0, \\ \partial_\alpha c_t(\alpha^*, \beta_t^*) + P_t = 0, & \\ \partial_\beta c_t(\alpha^*, \beta_t^*) - Z_t - V(t)h(t)P_t = 0. & \end{cases}$$

Necessary Condition

Given a contract of the following form (if such a solution exists)

$$\begin{cases} \mathrm{d}Y_t = c_t(\alpha_t^*, \beta_t^*)\mathrm{d}t + Z_t\mathrm{d}(\mathrm{d}X_t - (h(t)\hat{\mu}_t + \beta_t^*)\mathrm{d}t), \\ \mathrm{d}P_t = (Z_t - (f(t) - V(t)h^2(t))P_t)\mathrm{d}t + Q_t(\mathrm{d}X_t - (h(t)\hat{\mu}_t + \beta_t^*)\mathrm{d}t), \\ \mathrm{d}\hat{\mu}_t = (f(t)\hat{\mu}_t + \alpha_t^*)\mathrm{d}t + h(t)V(t)(\mathrm{d}X_t - (h(t)\hat{\mu}_t + \beta_t^*)\mathrm{d}t), \end{cases}$$

with terminal condition $Y_T = \Gamma_T^A \xi(X)$ and $P_T = 0$, where

$$\begin{cases} \partial_\alpha c_t(\alpha^*, \beta_t^*) + P_t = 0, \\ \partial_\beta c_t(\alpha^*, \beta_t^*) - Z_t - V(t)h(t)P_t = 0. \end{cases}$$

We can show that if the optimal control exists, necessarily the optimal control is (α^*, β^*).

Sufficient Condition

Define

$$\mathcal{H}_t(\alpha,\beta,\mu) := -c_t(\alpha,\beta) - Q_t\mu^2 - Q_t\mu\beta - P_t\alpha + (Z_t + V(t)h(t)P_t - Q\mu_t^*)\beta.$$

We have

$$V_A(\alpha,\beta) - V_A(\alpha^*,\beta^*) = \mathbb{E}^{\alpha,\beta}\Big[\int_0^T (\mathcal{H}_t(\alpha_t,\beta_t,\hat{\mu}_t) - \mathcal{H}_t(\alpha_t^*,\beta_t^*,\hat{\mu}_t^*) - \partial_\mu\mathcal{H}_t(\alpha_t^*,\beta_t^*,\hat{\mu}_t^*)\delta\mu)\mathrm{d}t\Big].$$

The right hand side of the above equality is negative if for all $t \in [0,T]$, $Hess(\mathcal{H})$ has only negative Eigenvalues.

Principal's Optimization Problem

The Principal has to maximize his utility by anticipating Agent's action and choosing an optimal contract.

$$
\begin{aligned}
V_P &:= \sup_{\xi} \mathbb{E}^{\alpha^*(\xi),\beta^*(\xi)}[e^{-\int_0^T \rho_t^P \mathrm{d}t}(X_T - \xi)] \\
&:= e^{-\int_0^T \rho_t^P \mathrm{d}t} \sup_{Y_0,Z,Q,P_T=0} \mathbb{E}^*[\int_0^T (h(t)\hat{\mu}_t + \beta_t^*)\mathrm{d}t - e^{\int_0^T \rho_t^A \mathrm{d}t} Y_T^Z] \\
&:= e^{-\int_0^T \rho_t^P \mathrm{d}t} \Big(m_0 \int_0^T h(t)\mathrm{d}t + \sup_{Y_0,Z,Q,P_T=0} \mathbb{E}^*[\int_0^T (\beta_t^* \\
&\qquad - e^{\int_0^T \rho_t^A \mathrm{d}t} c_t(\alpha_t^*, \beta_t^*)\mathrm{d}t] \Big).
\end{aligned}
$$

Principal's Optimization Problem

Theorem

Assume that for all $\alpha \in \mathcal{U}_\alpha$, *the function* $\beta \to \beta - e^{\int_0^T \rho_t^A \mathrm{d}t} c_t(\alpha, \beta)$ *has a maximiser, denoted* $\bar{\beta}_t(\alpha)$ *and the equation*

$$\partial_\alpha c_t(\alpha, \bar{\beta}_t(\alpha)) + P_t = 0 \tag{10}$$

has at least one solution, where P_t *is given by the ODE*

$$\mathrm{d}P_t = (1 - f(t)P_t)\mathrm{d}t. \tag{11}$$

Then there exists an optimal contract for the Principal's problem.

An Example

Consider the following output process X given the Agent's effort process α:

$$X_t = \int_0^t (\mu + \alpha_s)\mathrm{d}s + B_t. \tag{12}$$

The time-invariant productivity is denoted by μ whereas $\alpha_t \in \mathcal{U}$ is the effort provided by the Agent. The Agent's action thus shifts the average output but does not directly affect its volatility. However, μ is unknown at time 0 and the common priors are normal with mean m_0 and precision h_0.

An Example

We will be consider the following Principal-Agent problem:

$$V_A = \sup_{\alpha} \mathbb{E}^{\alpha}[e^{-\int_0^T \rho_t^A \mathrm{d}t} \xi - \int_0^T c_t(\alpha_t) \mathrm{d}t]. \tag{13}$$

$$V_P = \sup_{\xi, V_A \geq R_0} \mathbb{E}^{\alpha^*(\xi)}[e^{-\int_0^T \rho_t^P \mathrm{d}t}(X_T - \xi)] \tag{14}$$

An Example

Theorem

Assume that for all $0 \leq t \leq T$, *the function* $\alpha \to \alpha - (\Gamma^A_T)^{-1} c_t(\alpha)$ *has a global maximum and denote* $\bar{\alpha}^*_t$ *a maximizer. If in addition* $\alpha \to c'_t(\alpha)$ *is onto and* $\mathcal{U} = \mathbb{R}$, *we have*

$$V_P = \Gamma^P_T \int_0^T (\bar{\alpha}^*_t - (\Gamma^A_T)^{-1} c_t(\bar{\alpha}^*_t)) \mathrm{d}t + \Gamma^P_T (\Gamma^A_T)^{-1} R_0 + (\Gamma^A_T)^{-1} m_0 T.$$

An Example

Theorem

The optimal contract is given by

$$\xi = R + \int_0^T (c'(\bar{\alpha}_t^*) - Z_t^* b_t(X_t, A_t^*, \bar{\alpha}_t^*))\mathrm{d}t + \int_0^T Z_t^* \mathrm{d}X_t,$$

where

$$Z_t^* = c'(\bar{\alpha}_t^*) + \int_t^T e^{\int_t^s \frac{\mathrm{d}r}{h_r}} \frac{c'(\bar{\alpha}_s^*)}{h_s}\mathrm{d}s,$$

and

$$A_t^* := \int_0^t \bar{\alpha}_s^* \mathrm{d}s, \; h_t = h_0 + t.$$

Outline

Markovian FBSDEs

Let $T > 0$ be a fixed time horizon We consider the following system of strongly coupled forward-backward stochastic differential equation (abbreviated FBSDE):

$$\begin{cases} dX_t = b_t(X_t, Y_t, Z_t)\mathrm{d}t + \sigma_t(X_t, Y_t, Z_t)\mathrm{d}W_t & X_0 = x \\ \mathrm{d}Y_t = -f_t(X_t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t & Y_T = g(X_T). \end{cases}$$

Markovian FBSDEs Literature

Antonelli 1993 ⋮ Pardoux & Tang 1999 (Contracting Mapping)

Ma, Protter & Yong 1994 (Four Step Scheme)

Hu & Peng 1995 ⋮ Peng & Wu 1999 (Method of Continuation)

Ma, Wu, Zhang (Detao) & Zhang (Jianfeng) 2015 (Unified Approach)

Path-dependent FBSDEs

Let $T > 0$ be a fixed time horizon We consider the following system of strongly coupled forward-backward stochastic differential equation:

$$\begin{cases} dX_t = b_t(X, Y_t, Z_t)\mathrm{d}t + \sigma_t(X, Y_t, Z_t)\mathrm{d}W_t & X_0 = x \\ \mathrm{d}Y_t = -f_t(X, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t & Y_T = g(X). \end{cases}$$

Assumptions

Assumption

- *The coefficients*

$$b : [0,T] \times \Omega \times \mathcal{C}([0,T], \mathbb{R}^d) \times \mathbb{R}^n \times \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}^d$$

$$f : [0,T] \times \Omega \times \mathcal{C}([0,T], \mathbb{R}^d) \times \mathbb{R}^n \times \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}^n$$

$$\sigma : [0,T] \times \Omega \times \mathcal{C}([0,T], \mathbb{R}^d) \times \mathbb{R}^n \times \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_{d,n}(\mathbb{R})$$

are $\mathbb{F}$-progressively measurable, for fixed

$$(X, y, z) \in \mathcal{C}([0,T], \mathbb{R}^d) \times \mathbb{R}^n \times \mathcal{M}_n(\mathbb{R});$$

the function $g : \mathcal{C}([0,T], \mathbb{R}^d) \times \Omega \to \mathbb{R}^n$ is $\mathcal{F}_T$-measurable.

Assumptions

Assumption

- *The following integrability condition holds:*

$$\mathbb{E}\Big[\Big(\int_0^T \big[\|b\| + \|f\|\big](t,\bar{0},0,0)\mathrm{d}t\Big)^2 + \int_0^T \|\sigma\|^2(t,\bar{0},0,0)\mathrm{d}t + \|g(\bar{0})\|\Big] < \infty,$$

where $\bar{0}$ *is the function constantly equals to* 0 *and* $\|\cdot\|$ *is the Euclidean distance.*

Assumptions

Assumption

- *The coefficients* b, σ, f *satisfy the following Lipschitz condition in the spacial variable* $(X, y, z) \in \mathcal{C}([0, T], \mathbb{R}^d) \times \mathbb{R}^n \times \mathcal{M}_n(\mathbb{R})$*:* $\exists K_0 > 0$ *such that*

$$\|\xi(t, X, y, z) - \xi(t, X', y', z')\| \leq K_0 \left[\|X - X'\|_{2,t} + \|y - y'\| + \|z - z'\|\right]$$

uniformly in $\omega \in \Omega$*, where* ξ *can be* b, σ, f *and*

$$\|X - X'\|_{2,t}^2 := \int_0^t \|X(s) - X'(s)\|^2 \mathrm{d}s + \|X(t) - X'(t)\|^2.$$

Assumptions

Assumption

- *The function g satisfies the following Lipschitz condition:* $\exists K_1 > 0, \forall X, X' \in \mathcal{C}([0,T], \mathbb{R}^d)$:

$$\|g(X) - g(X')\|^2 \leq K_1^2 \int_0^T \|X - X'\|^2(t)\mathrm{d}t + K_1^2 \|X(T) - X'(T)\|^2$$

uniformly in $\omega \in \Omega$.

Well-posedness on Small Time Interval

Theorem

Under the above Assumptions, if $\sigma_z < \frac{1}{K_1}$, *then* $\exists \delta > 0$ *such that* $\forall T < \delta$, *the FBSDE*

$$\begin{cases} dX_t = b_t(X, Y_t, Z_t)\mathrm{d}t + \sigma_t(X, Y_t, Z_t)\mathrm{d}W_t & X_0 = x \\ \mathrm{d}Y_t = -f_t(X, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t & Y_T = g(X) \end{cases}$$

has an unique solution (X, Y, Z).

Some Examples of Non-solvable FBSDEs

Case: $\sigma_z K_1 < 1$ **isn't satisfied:**

$$\begin{cases} \mathrm{d}X_t = (\sigma_0 + Z_t)\mathrm{d}W_t & X_0 = x_0 \quad (15) \\ \mathrm{d}Y_t = Z_t \mathrm{d}W_t & Y_T = X_T. \quad (16) \end{cases}$$

From (15) we get $X_T - X_t = \int_t^T (\sigma_0 + Z_s)\mathrm{d}s$. Combing with (16), we get

$$Y_t = X_t + \sigma_0(W_T - W_t). \quad (17)$$

In particular, $Y_0 = x_0 + \sigma_0 W_T$, where lays the contradiction.

Some Examples of Non-solvable FBSDEs

Case: No solution on $\mathbb{R}$:

$$\begin{cases} \mathrm{d}X_t = Y_t \mathrm{d}t & X_0 = x_0 \\ \mathrm{d}Y_t = Z_t \mathrm{d}W_t & Y_T = X_T. \end{cases}$$

We can actually solve explicitly the above FBSDE on $[t, T]$:

$$\begin{cases} X_t = x_0 \dfrac{1-(T-s)}{1-(T-t)} \\ Y_t = \dfrac{x_0}{1-(T-t)}. \\ Z_t = 0. \end{cases}$$

We can see that there is explosion in both forward and backward process when $T-t$ approach 1.

Decoupling Field

Definition

An $\mathbb{F}$-progressively measurable random field $u : [0, T] \times \Omega \times \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}^n$ with $u(T, X) = g(X)$ is said to be a *decoupling field* of FBSDE if there exists a constant $\delta > 0$ such that, for any $0 \leq t_1 < t_2 \leq T$ with $t_2 - t_1 \leq \delta$ and any $x \in L^2(\mathcal{F}_{t_1})$, the FBSDE with initial value $x \in \mathcal{C}([0, T], \mathbb{R}^d)$ and terminal condition $u(t_2, \cdot)$ has an unique solution that satisfies

$$Y_t = u(t, X_{\wedge t}) = u(t, X), \ t \in [0, T], \ \mathbb{P} - a.s.$$

Well-posedness on Larger Time Interval

Theorem

Under the above Assumptions, if σ depends only on X, then there exists $T_{max} > 0$ depending on the Lipschitz constants of the coefficients of the following FBSDE

$$\begin{cases} dX_t = b_t(X, Y_t, Z_t)\mathrm{d}t + \sigma_t(X)\mathrm{d}W_t & X_0 = x \\ \mathrm{d}Y_t = -f_t(X, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t & Y_T = g(X) \end{cases}$$

such that the FBSDE has an unique solution (X, Y, Z) on $[0, T_{max}]$. Furthermore, there exists an unique decoupling field u such that $\forall X, X'$,

$$\|u(t, X) - u(t, X')\|^2 \leq K(t)||X - X'||_{2,t}, \tag{18}$$

where $t \to K(t)$ is a deterministic function.

Estimation of Z

Theorem

Assume that all the coefficients of the FBSDE are deterministic and all the assumptions in the above theorem are satisfied, denote (X, Y, Z) the solution of the FBSDE, then

$$|Z_t| \leq K(t) \cdot ||\sigma||_\infty, \tag{19}$$

where K is the deterministic function given by the previous theorem.

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Thanks for your attention!