Algorithms for train scheduling on a single line

Laurent DAUDET[1]

PhD advisor: Frédéric MEUNIER[1]

[1]CERMICS, Centre d'Enseignement et de Recherche en Mathématiques et Calcul Scientifique, ENPC

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The context

→ Scientific chair between

- École Nationale des Ponts et Chaussées
- Eurotunnel

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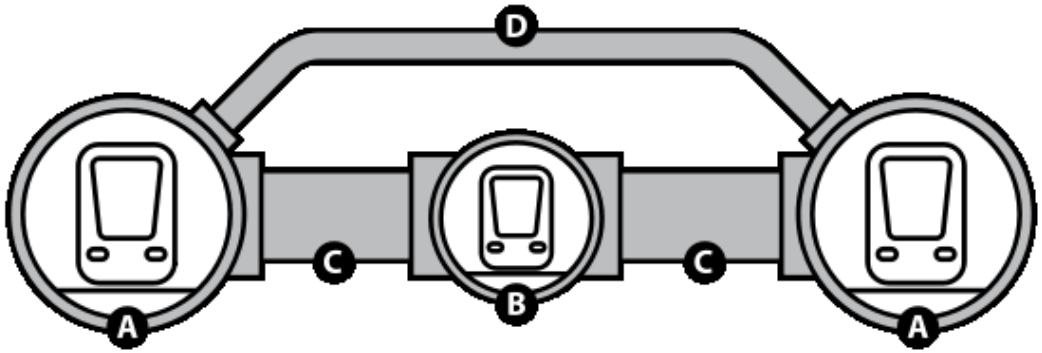
Questions?

- How to increase the global capacity with current means?
- How to improve the quality of service?

The tunnel under the Channel



- Line length: 50 km
- From Coquelles (France) to Folkestone (England)
- A tunnel for each direction (A) and a service tunnel (B)
- High-Speed-Trains (Eurostar) 160 km/h
- Freight trains 100-120 km/h
- Passenger shuttles (PAX) 140 km/h
- Freight shuttles (HGV) 140 km/h

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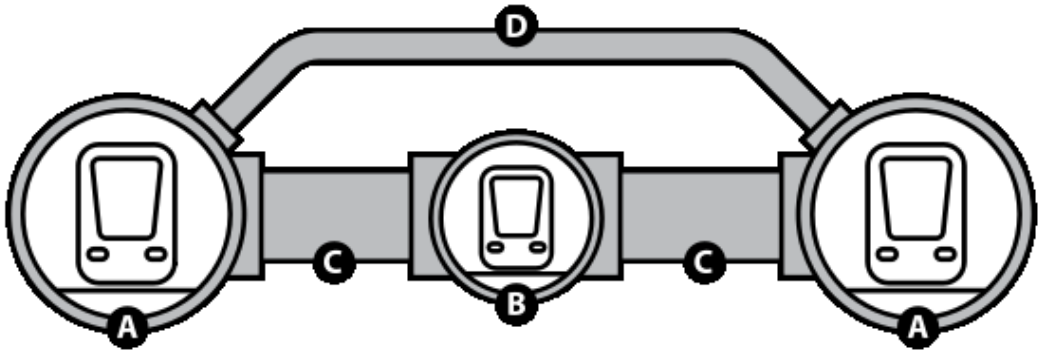

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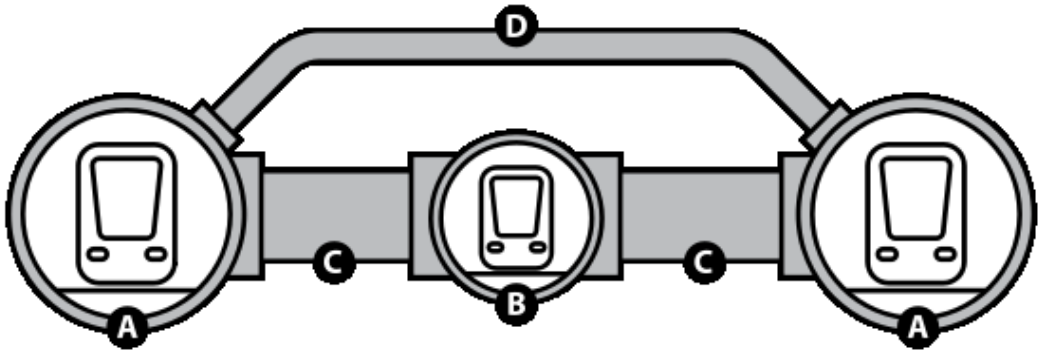

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Objective

Compute one-hour schedule with maximum number of HGV shuttles.

Constraints

- Fixed number of Eurostars, freight trains, and PAX shuttles.
- Security.
- Other constraints.

Output

- $\boldsymbol{d} = \left\{ d_j^{\text{Eur}}, d_j^{\text{Fr}}, d_j^{\text{PAX}}, d_j^{\text{HGV}} \right\}$: scheduled departure times of all trains.

$\rightarrow$ d_j^{A}: jth departure time of train of type A.

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Some other constraints

- Commercial agreements: Eurostars "equally" distributed in the period.
  $\rightarrow$ Eurostars grouped by pairs with departure times at d and $d + 30$ min.
- Loading platforms: impossible to load three HGV or PAX shuttles at the same time.
  $\rightarrow$ at most 2 HGV shuttles in any 12-minute time-window.
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- Discretization: departure times on full minutes.
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- n: number of scheduled HGV shuttles.

Mathematical model

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \left(d_j^{\mathsf{HGV}} - d_j^{\mathsf{PAX}} \right) \\ \text{s.t.} \quad & \boldsymbol{d} \in X \end{aligned}$$

where X = set of constraints

$\rightarrow$ can be expressed with linear constraints

(only non immediate constraint: security headway [Serafini and Ukovich, 1989]).

$\Rightarrow$ Mixed Integer Linear Program

Solved by commercial solver CPLEX.

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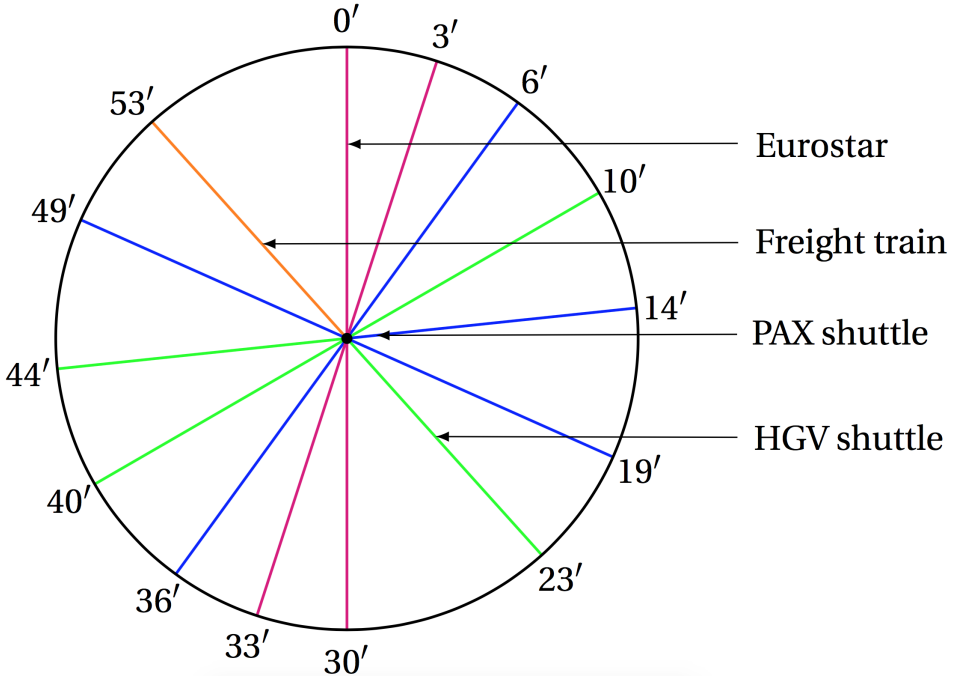

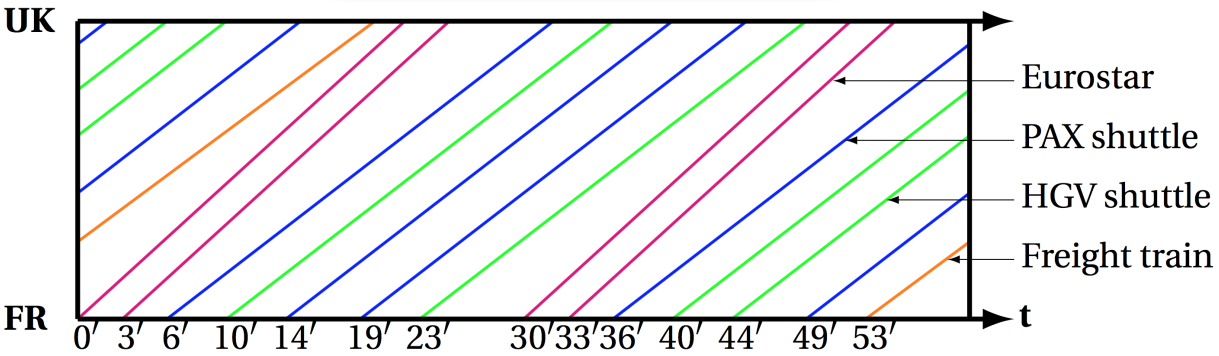

Some schedule improvements

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Constraint relaxed		Instance	Improvement
Length cyclic period	T:	1 h $\rightarrow$ 4 h	Up to 25%
Loading platforms	L:	12 min $\rightarrow$ 0 min	Up to 50%
Full minute discretization	η:	1 min $\rightarrow$ 1 s	Up to 33%
Agreements with Eurostar	C^{Eur}:	30 min $\rightarrow$ [27 min-33 min]	Up to 14%

Some other problems

- Compute one-hour schedules with minimum delays.
- Compute one-hour schedules with maximum number of HGV shuttles and minimum delays.

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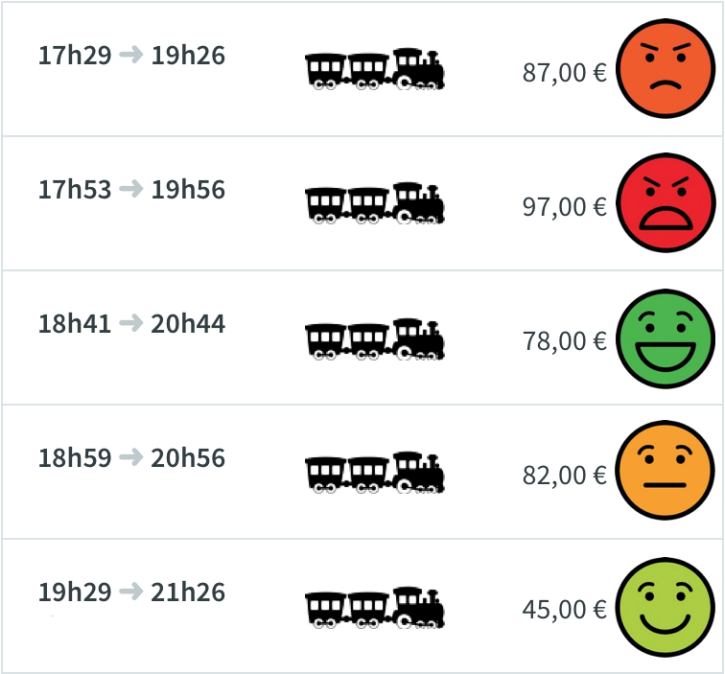

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Objective

Maximize the revenue of the company.

The problem (2/2)

Each customer i

- has a preferred departure time: random variable χ_i
- belongs to economic class b_i (e.g. business, tourist, low-cost, ...)
- "value of time" v_b for economic class b

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Discrete choice model

- Each customer i and product j (departure d_j at price p_j)

$\rightarrow$ Random utility $U_{ij}(\boldsymbol{d}, \boldsymbol{p})$ representing satisfaction.

The model (1/2)

- Denote by $\boldsymbol{\xi}$ vector representing uncertainty.

- Revenue of company $\rightarrow R(\boldsymbol{d}, \boldsymbol{p}, \boldsymbol{\xi})$ where $\boldsymbol{\xi}$ has been revealed.

$\rightarrow$ Easy to compute (simulation, Linear Programming).

- Objective function $\rightarrow f(\boldsymbol{d}, \boldsymbol{p}) = \mathbb{E}\left[R(\boldsymbol{d}, \boldsymbol{p}, \boldsymbol{\xi})\right]$.

$\rightarrow$ Remark: $f(\boldsymbol{d}, \boldsymbol{p})$ well defined (for all $(\boldsymbol{d}, \boldsymbol{p}) \in \mathcal{X}$, $R(\boldsymbol{d}, \boldsymbol{p}, \cdot)$ measurable and $\mathbb{E}\left[R(\boldsymbol{d}, \boldsymbol{p}, \boldsymbol{\xi})\right] < \infty$) and $\mathrm{Var}\left[R(\boldsymbol{d}, \boldsymbol{p}, \boldsymbol{\xi})\right] < \infty$.

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$$\mathrm{Max}_{(\boldsymbol{d},\boldsymbol{p})\in X}\ f(\boldsymbol{d},\boldsymbol{p}) = \mathbb{E}\left[R(\boldsymbol{d},\boldsymbol{p},\boldsymbol{\xi})\right]$$

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Compute $\boldsymbol{d}$ and $\boldsymbol{p}$ without knowing $\boldsymbol{\xi}$!

Sample Average Approximation

- $(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_\Omega)$ of Ω independent and identically distributed realizations

$\Rightarrow$ ξ_ω is not random variable!

- We approximate objective function $f(d, p) = \mathbb{E}\left[R(d, p, \xi)\right]$ by

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We denote by

$\rightarrow$ $v^* = \mathrm{Max}_{(\boldsymbol{d},\boldsymbol{p}) \in X} \ f(\boldsymbol{d}, \boldsymbol{p})$

$\rightarrow$ $\hat{v}_\Omega = \mathrm{Max}_{(\boldsymbol{d},\boldsymbol{p}) \in X} \ \hat{f}_\Omega(\boldsymbol{d}, \boldsymbol{p})$

SAA properties

Proposition, Shapiro et al., 2009

We have

(i) $\mathbb{E}[\hat{f}_\Omega(\boldsymbol{d}, \boldsymbol{p})] = f(\boldsymbol{d}, \boldsymbol{p})$, for all $(\boldsymbol{d}, \boldsymbol{p}) \in X$,

(ii) $\hat{f}_\Omega(\boldsymbol{d}, \boldsymbol{p})$ converges to $f(\boldsymbol{d}, \boldsymbol{p})$ w.p. 1, for all $(\boldsymbol{d}, \boldsymbol{p}) \in X$,

(iii) $\mathbb{E}[\hat{v}_\Omega] \geq v^*$, and

(iv) $\hat{v}_\Omega$ converges to v^* w.p. 1.

- $\mathbb{E}[\hat{v}_\Omega]$ is an upper bound on v^* (iii).
  - $\Rightarrow$ With the Central Limit Theorem, we can compute $(1-\alpha)$-confidence interval.
- For any solution $(\bar{\boldsymbol{d}}, \bar{\boldsymbol{p}}) \in X$, $f(\bar{\boldsymbol{d}}, \bar{\boldsymbol{p}}) \leq v^* \left(= \text{Max}_{(\boldsymbol{d},\boldsymbol{p}) \in X} f(\boldsymbol{d}, \boldsymbol{p})\right)$
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A first heuristic: a sequential heuristic

→ Try to mimic natural way of scheduling and then fixing prices.

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2. Compute prices $\boldsymbol{p}$ with previous two-stage recourse program (with fixed departure times $\tilde{\boldsymbol{d}}$).

$$\mathrm{Max}_{\boldsymbol{p} \in X_p} \ \hat{f}_\Omega(\tilde{\boldsymbol{d}}, \boldsymbol{p})$$

A second heuristic: a Gauss-Seidel heuristic

1. Initialize $(\boldsymbol{d}, \boldsymbol{p})$ to some $(\boldsymbol{d}^0, \boldsymbol{p}^0)$.
2. Let $m \in \mathbb{Z}_+$ and $J_1 \cup J_2 \cup \cdots \cup J_m$ be a partition of $\{1, \ldots, S\}$.
3. Generate sequence of feasible solutions $(\boldsymbol{d}^k, \boldsymbol{p}^k)$:

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The results

Academic instance:

- Realistic distribution for preferred departure times.
- Gumbel distribution for random part of utilities.

Instance			Lower bounds (20 min)		
Q	S	C	Frontal SAA	Seq. heur.	G.-S. heur.
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Other heuristic: Lagrangian relaxation

$\Rightarrow$ does not improve Frontal SAA for big instances.

Conclusion

- Joint scheduling and pricing: higher revenues.
- Upper bounds: not very precise...

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- Huge waiting lines during peak hours.

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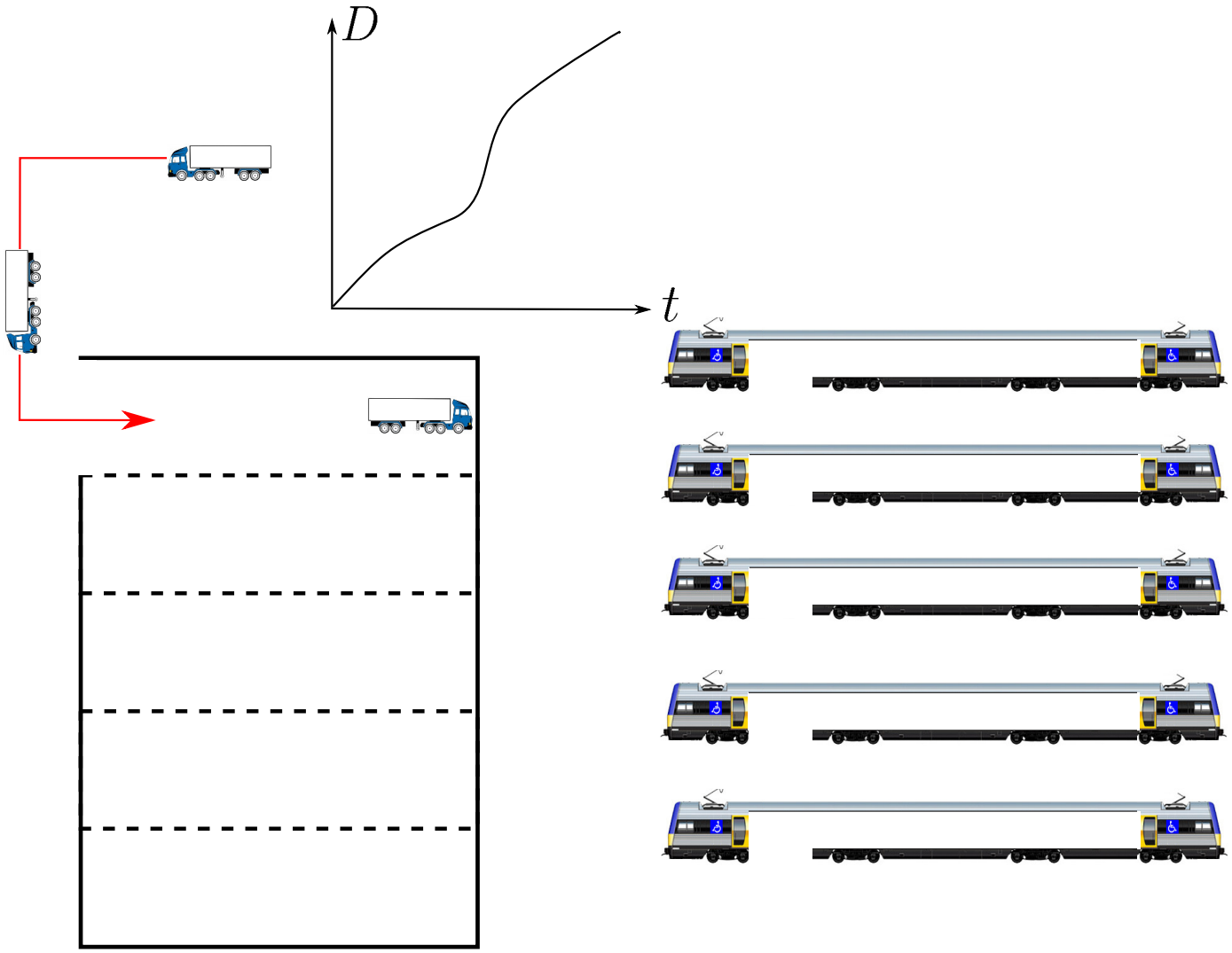

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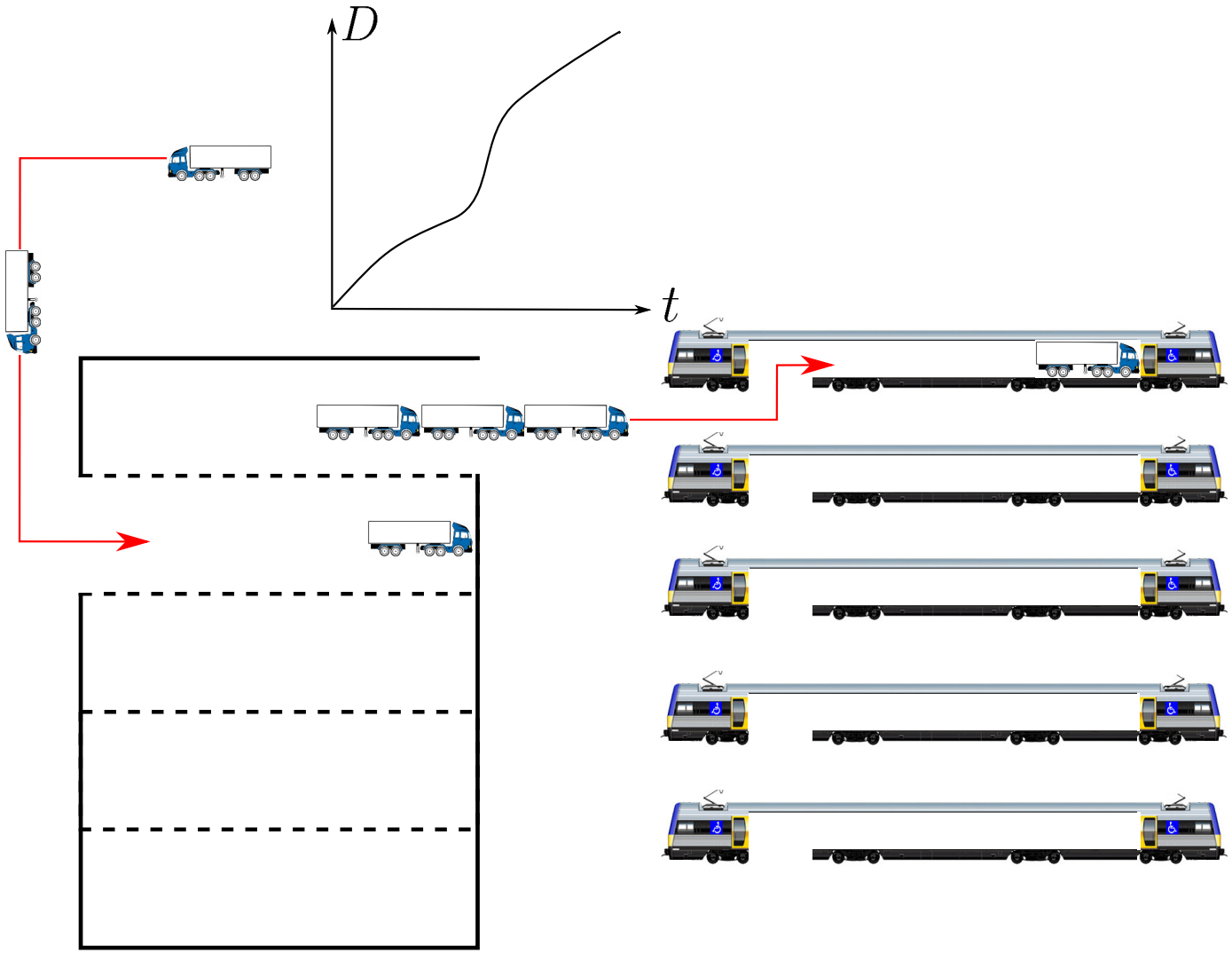

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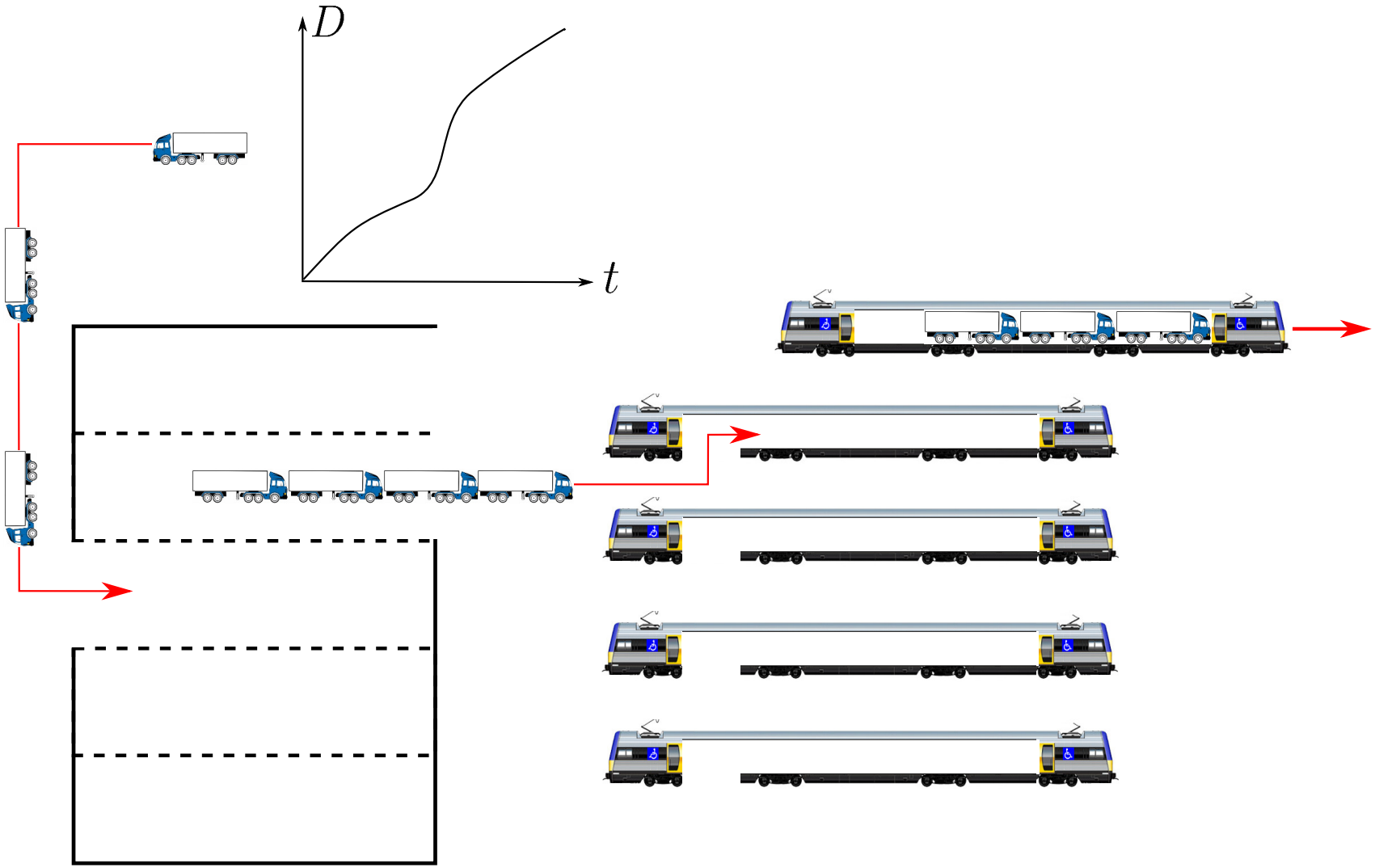

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Objectives

- Minimize maximum waiting time of users $\rightsquigarrow$ Problem $\mathrm{P}^{\max}$
- Minimize average waiting time of users $\rightsquigarrow$ Problem $\mathrm{P}^{\mathrm{ave}}$

$\rightarrow$ Waiting time: time between arrival time on terminal and departure of shuttle.

The model (1/3)

Variables

- d_j: departure time jth shuttle.
- y_j: cumulative loads for shuttles 1 to j.

Parameters

- S: number of shuttles.
- C: capacity.
- ν: loading rate (loading x users takes a time νx).
- T: time horizon.
- $D : [0, T] \to \mathbb{R}_+$: cumulative demand known *a priori* (oracle)

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The model (2/3)

Arrival time of user $y \to$ function $\tau(\,\cdot\,)$, pseudo-inverses of $D(\,\cdot\,)$:

$$\tau(y) = \inf\left\{t \in [0,\, T] \colon D(t) \geq y\right\}.$$

Objective function

$$g^{\text{ave}}(\boldsymbol{d}, \boldsymbol{y}) = \frac{1}{D(T)} \sum_j \int_{y_{j-1}}^{y_j} \left(d_j - \tau(y)\right) dy$$

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Similar function $g^{\mathsf{max}}(\boldsymbol{d}, \boldsymbol{y})$ for maximum waiting time.

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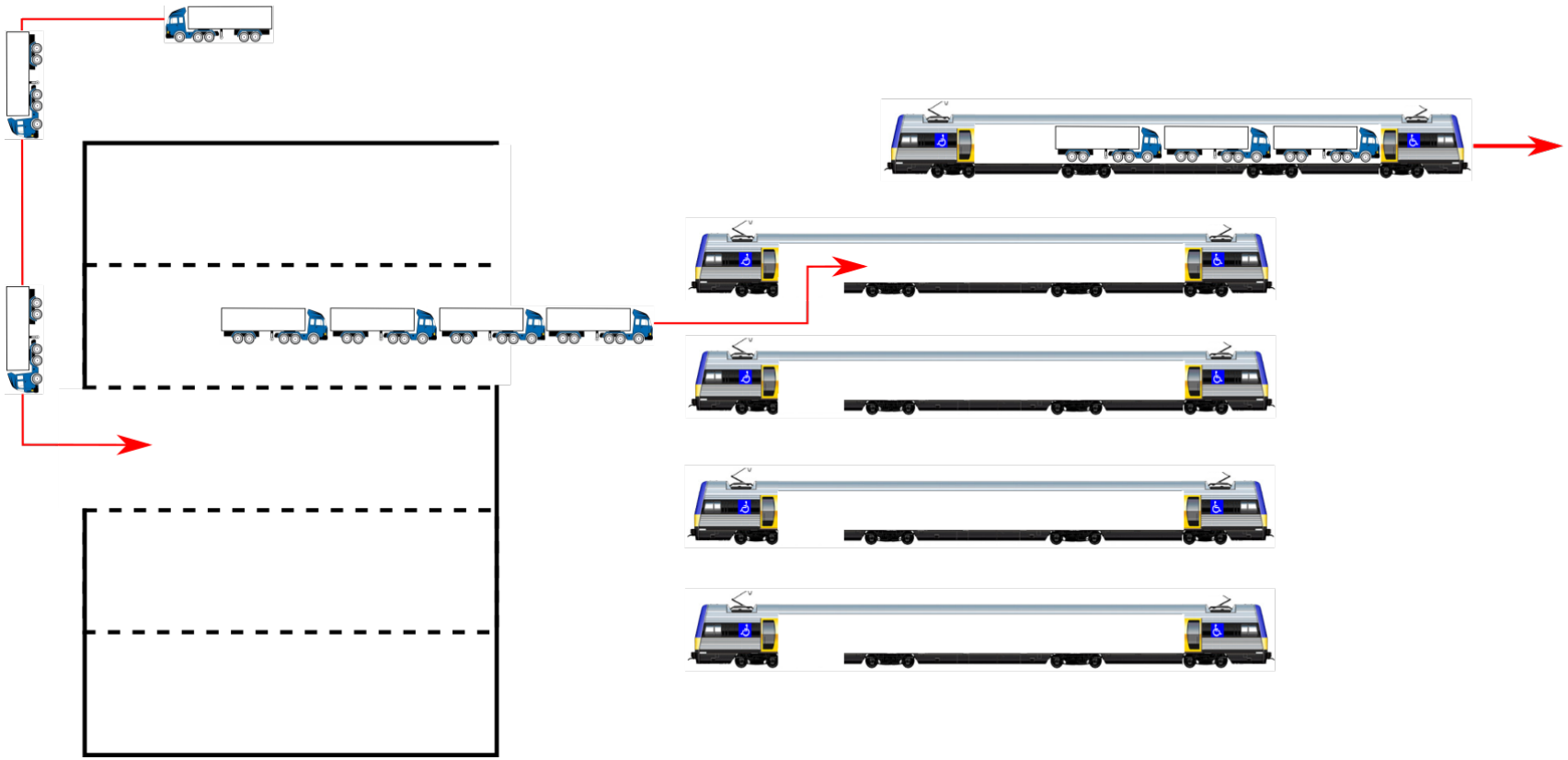
When demand is a step function

Theorem

Assume that $D(\cdot)$ is a step function defined with K discontinuities, and that $\nu = 0$. There is an algorithm computing an optimal solution of $\mathrm{P}^{\mathrm{ave}}$ in $O(K^2 S)$.

Precision on $\nu = 0$



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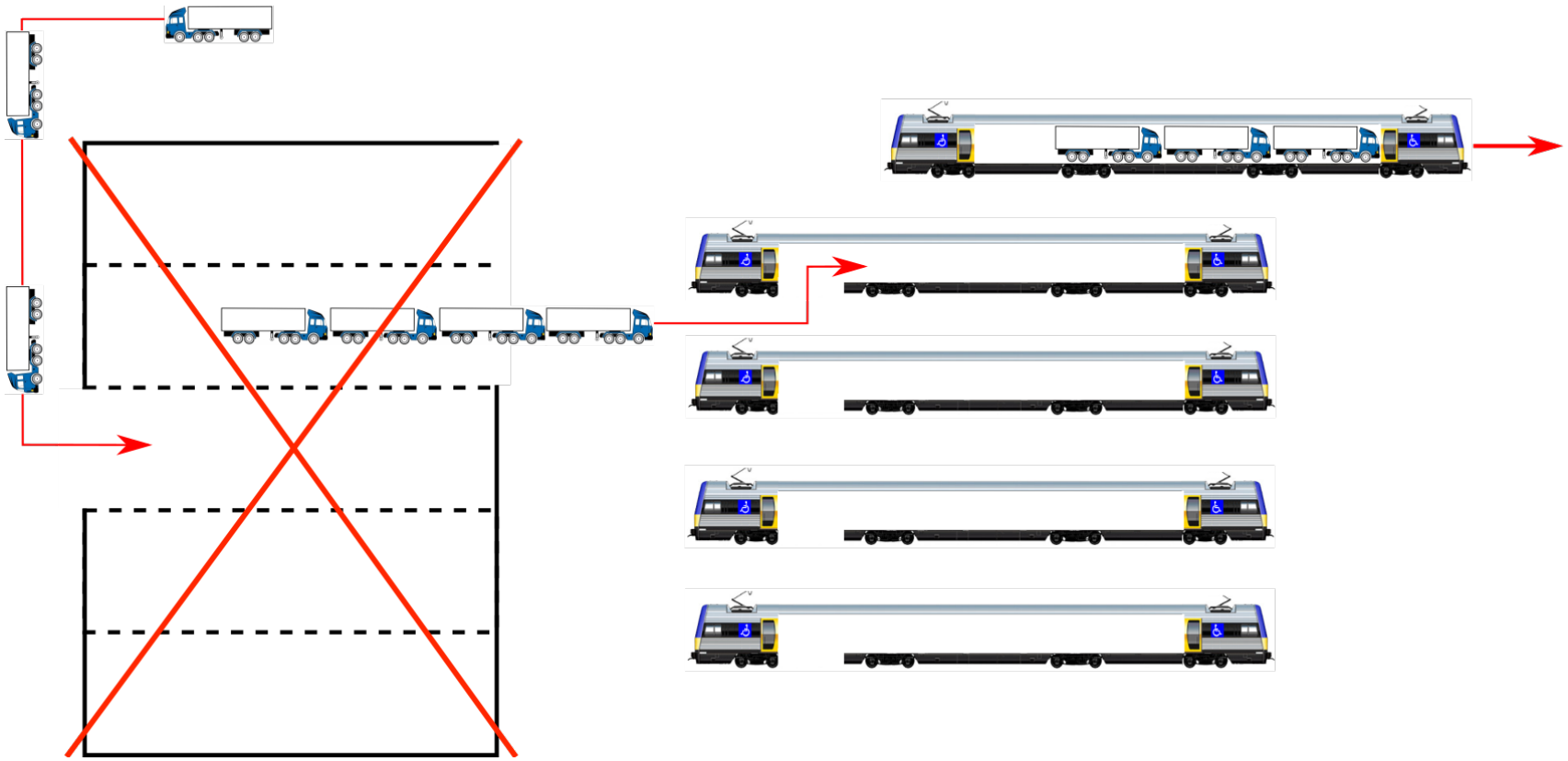

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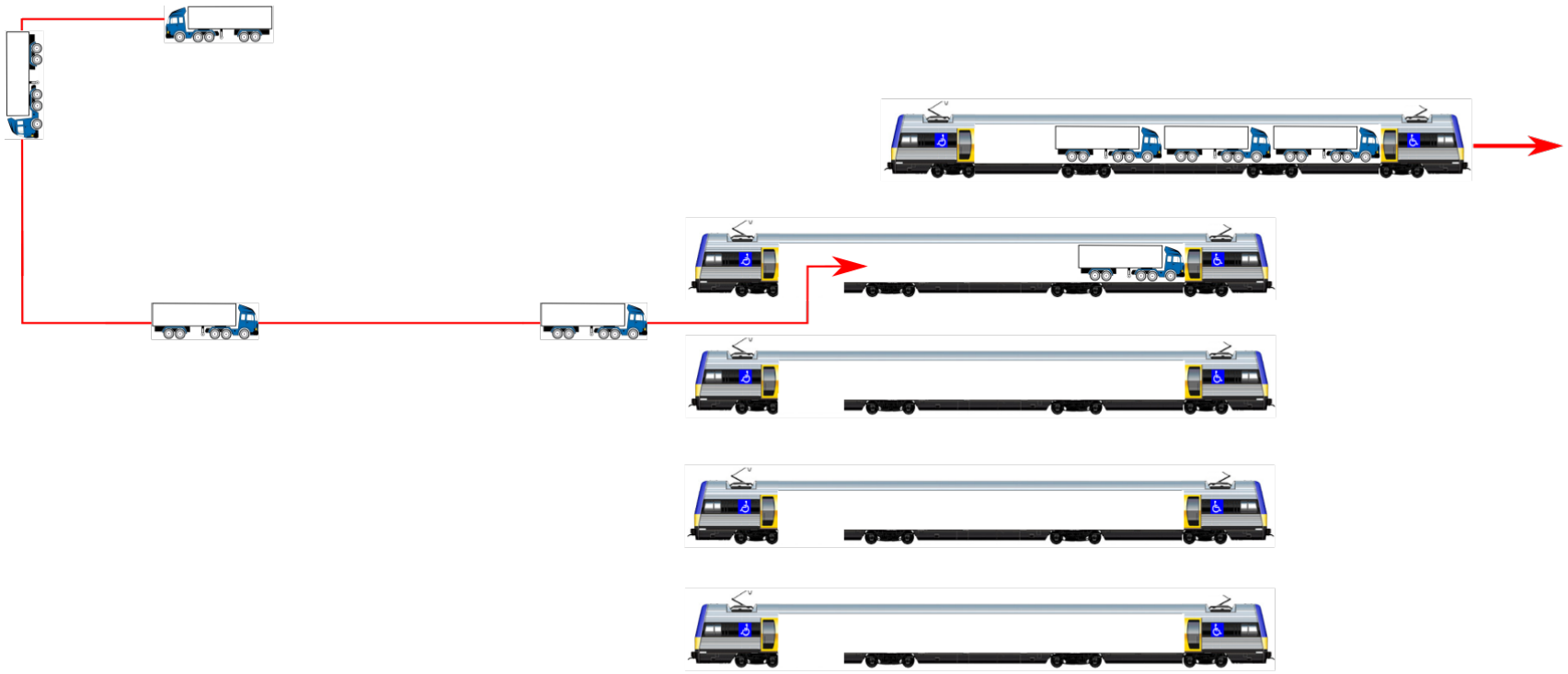
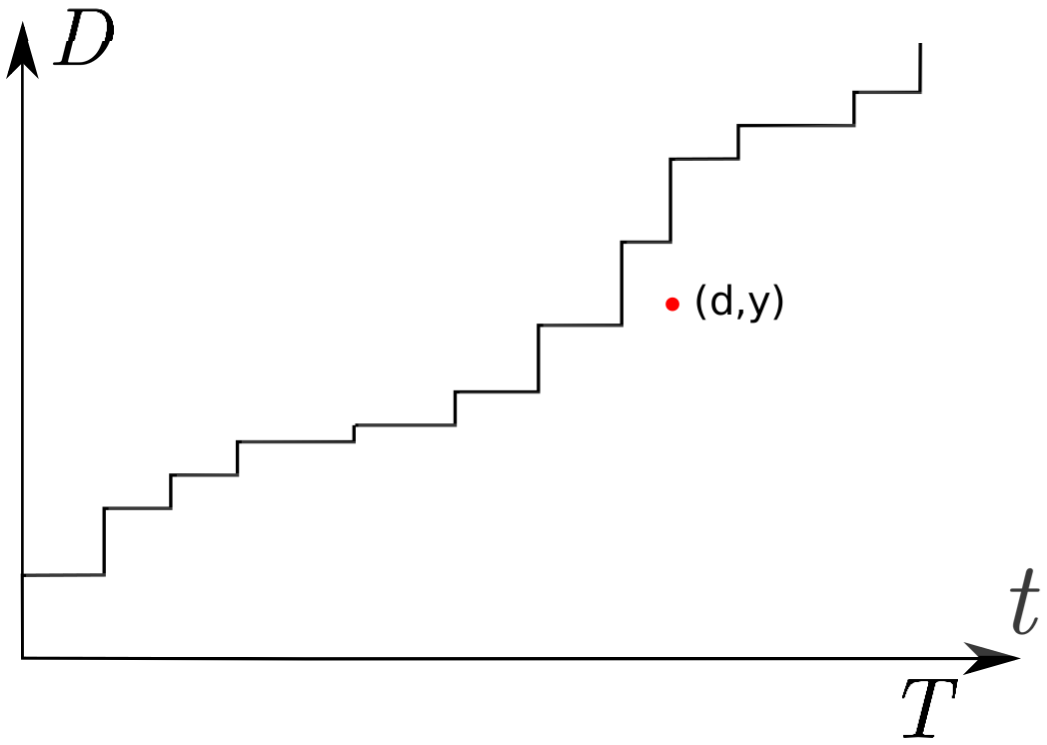


An algorithm for $\mathrm{P}^{\mathrm{ave}}$ and $C = \infty$



- Shortest path in S arcs minimizing sum on arcs in directed acircuitic graph with $K+1$ vertices
  $\Rightarrow$ complexity $O(K^2 S)$.
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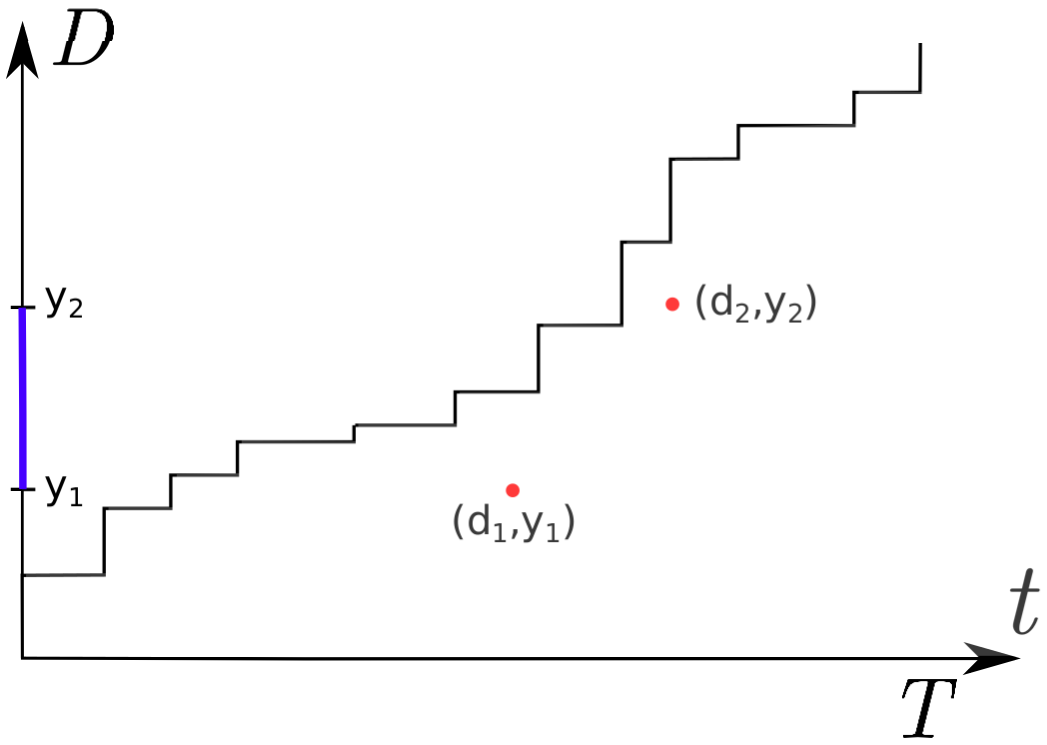

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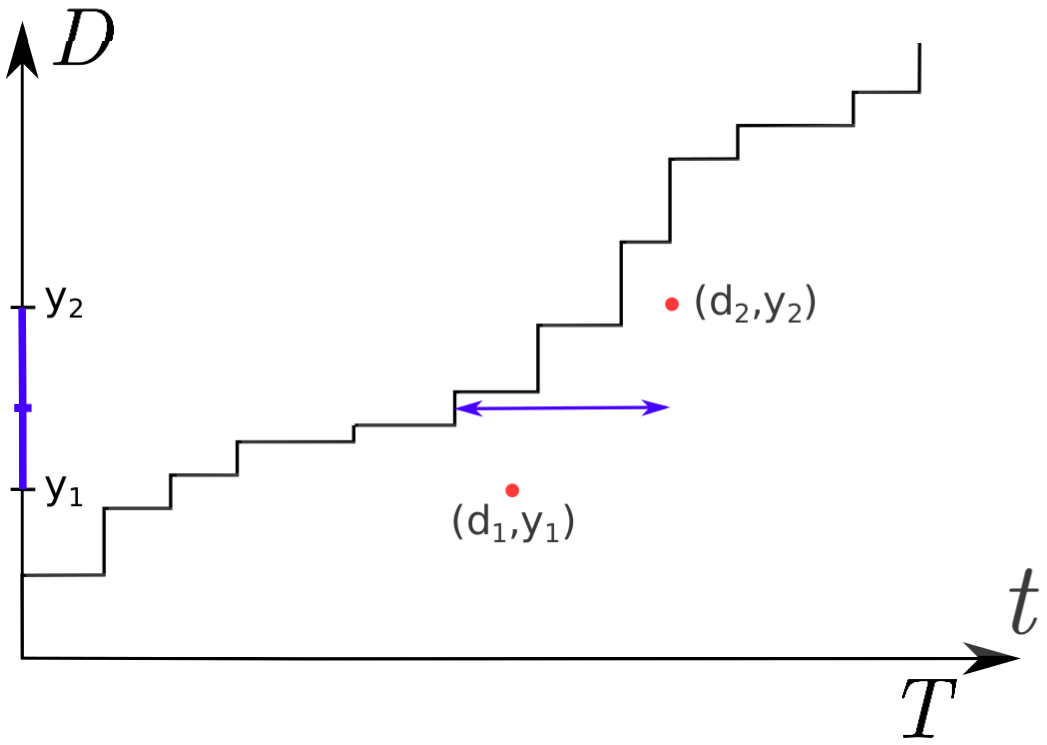

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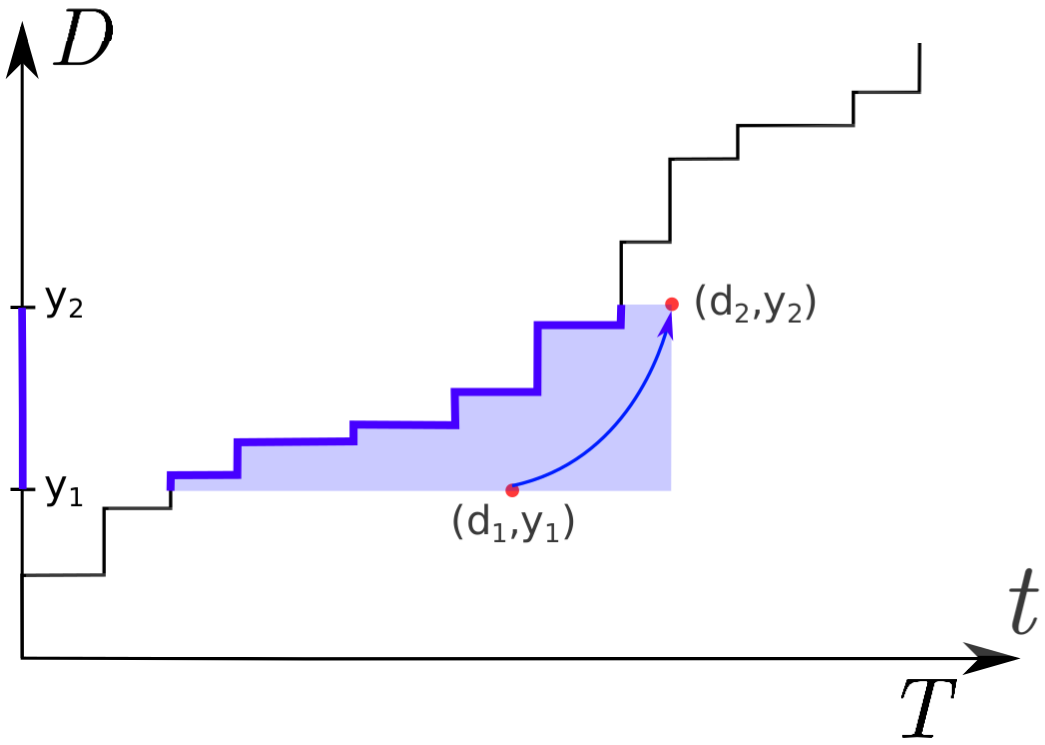

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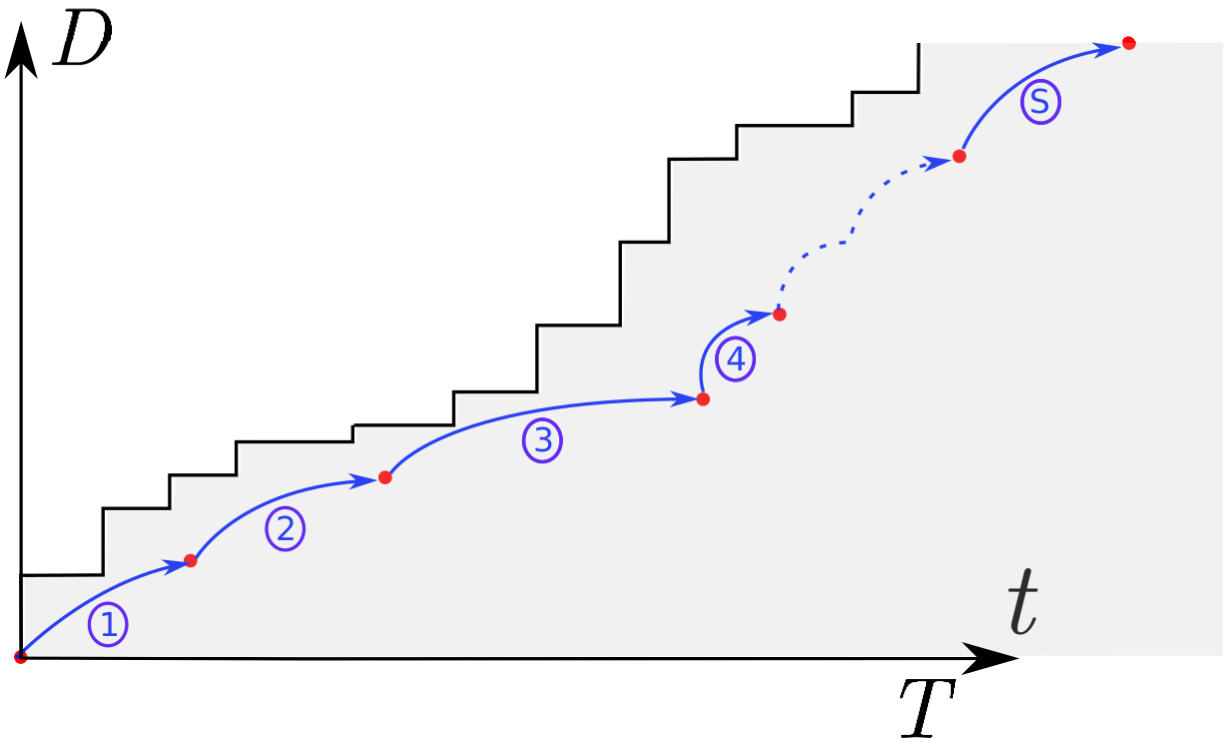

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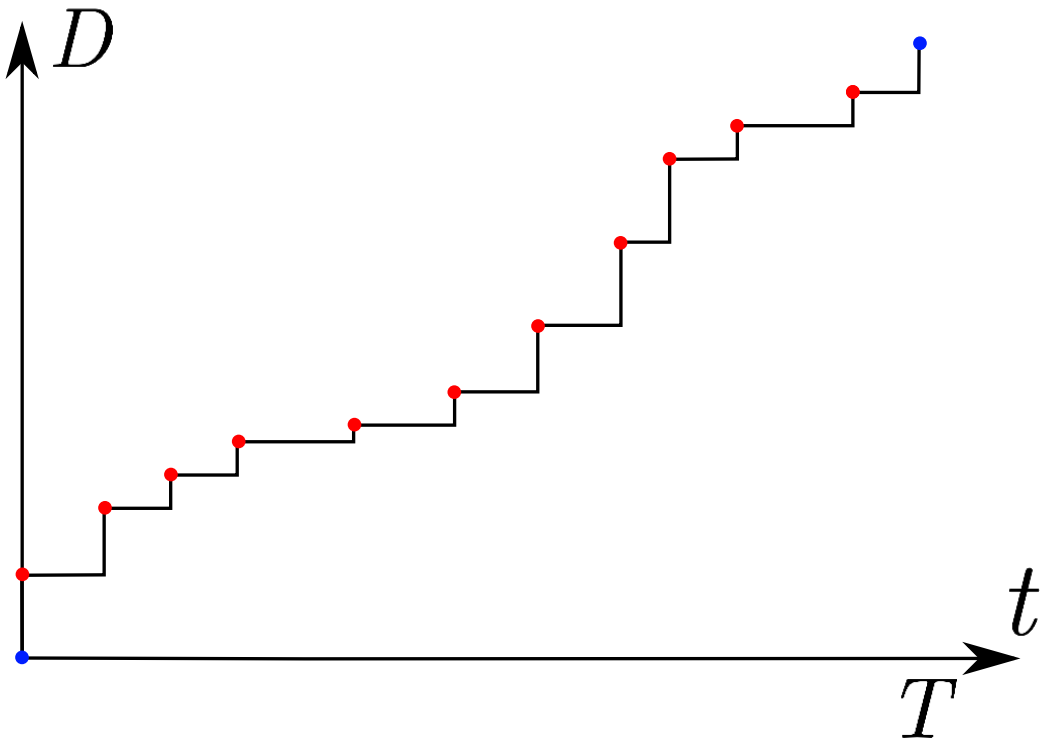

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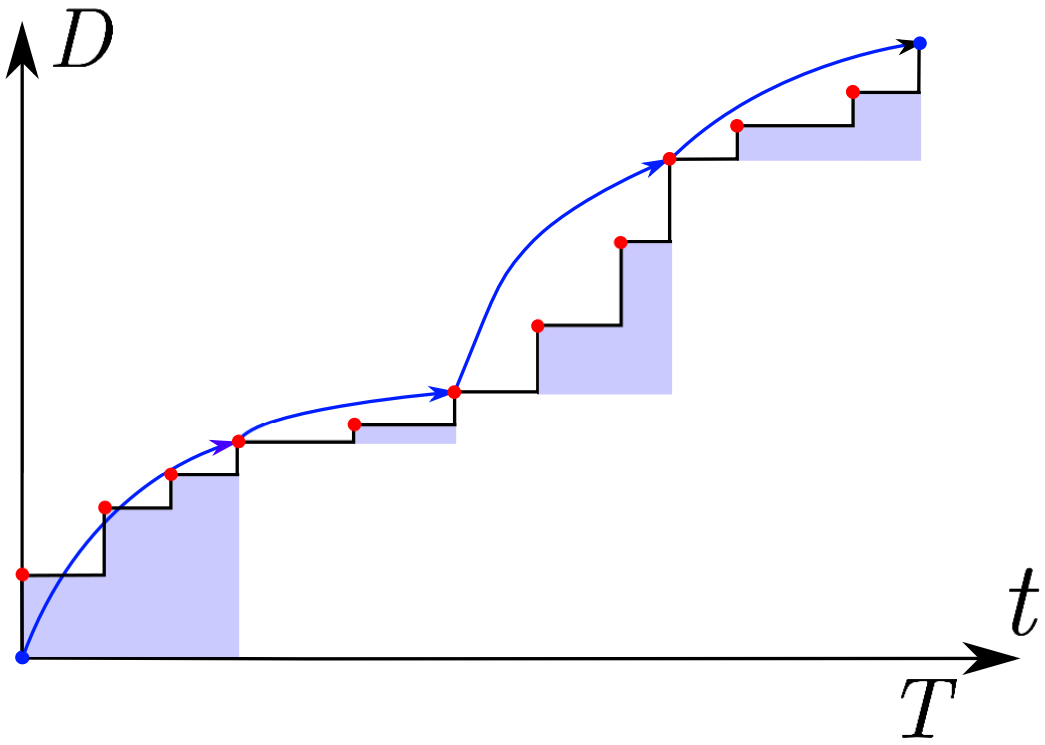

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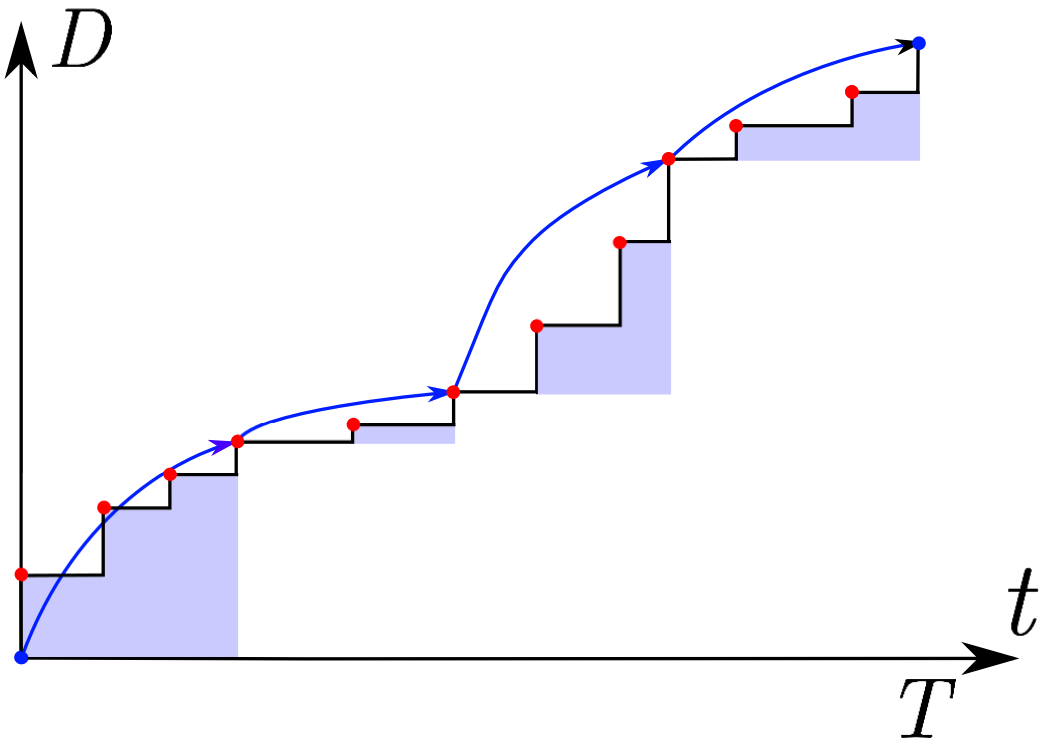

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An approximation scheme for P^{ave}

Theorem

Suppose that $D(\cdot)$ admits right derivatives everywhere (denoted $D'_+(t)$) and $\inf_{t\in[0,T)} D'_+(t)$ is positive. Then, for any positive integer M, a feasible solution of value

$$SOL \leq OPT + O\left(\frac{S^2}{M}\right)$$

can be computed in $O\left(SM^3\right)$.

Some elements of proof (1/2)

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Lemma

There exists a collection of problems (P^η), with parameter η, providing:

(i) lower bound LB^η of OPT.

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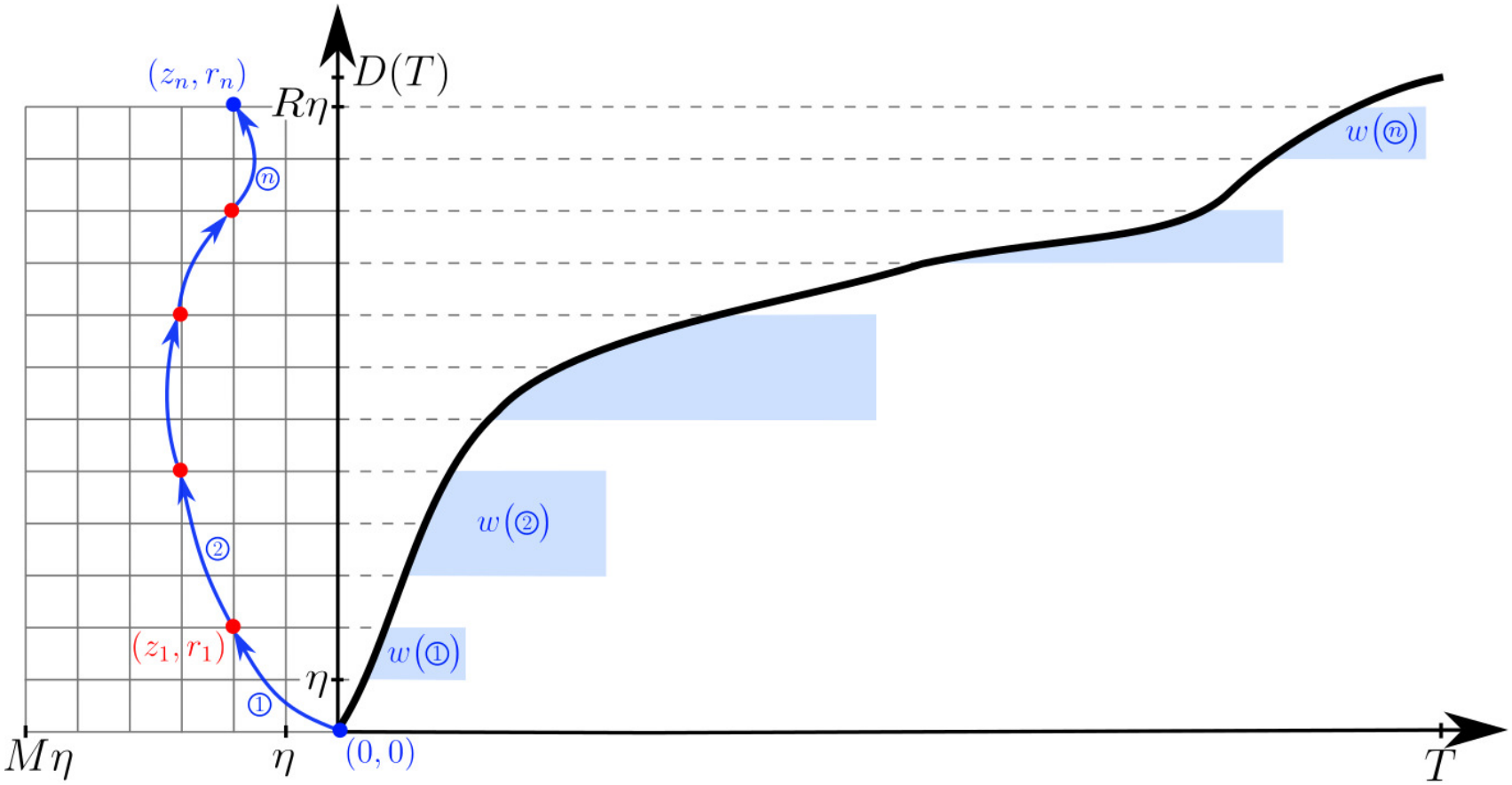
→ η depends on M.

→ Problems (P^η): Demand is discretized with step η
$\Rightarrow$ Shortest path in directed acircuitic graph.

Elements of proof (2/2)



Existence of optimal solution of P^{ave}?

- $\tau(\cdot)$ lower semicontinuous $\rightarrow$ set of constraints is closed.

$\Rightarrow$ set of constraints is compact!

- $g^{ave}(d, y)$ continuous.

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\mathrm{Min}_{\boldsymbol{d},\boldsymbol{y}} \quad & g^{\mathsf{ave}}(\boldsymbol{d},\boldsymbol{y}) = \frac{1}{D(T)} \sum_j \int_{y_{j-1}}^{y_j} \left(d_j - \tau(y)\right) dy \\
\text{s.t.} \quad & y_j - y_{j-1} \leq C \\
& y_{j-1} \leq y_j \\
& d_{j-1} \leq d_j \\
& y_S = D(T) \\
& d_j \geq \tau(y_j) + \nu(y_j - y_{j-1}) \\
& y_0 = 0 \\
& d_j \leq T + \nu C.
\end{aligned}
$$

- $\tau(\,\cdot\,)$ lower semicontinuous $\rightarrow$ set of constraints is closed.

$\Rightarrow$ set of constraints is compact!

- $g^{\mathsf{ave}}(\boldsymbol{d},\boldsymbol{y})$ continuous.

Existence of optimal solution of $\mathsf{P}^{\mathsf{ave}}$?

$$
\begin{aligned}
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Table of results

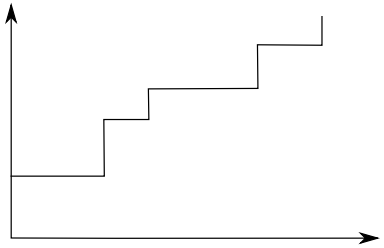
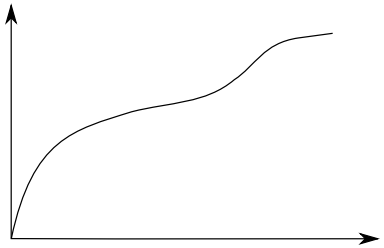
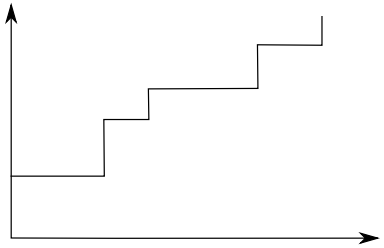
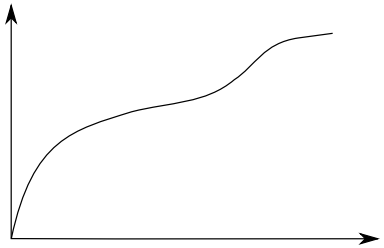
	p^{max}	p^{ave}	
		Exact algorithm $O(K^2S)$	
		Approx. algorithm $O\left(SM^3\right)$	

Table of results

	p^{max}	p^{ave}
	Exact algorithm $O(K^2S)$	Exact algorithm $O(K^2S)$
		Approx. algorithm $O\left(SM^3\right)$

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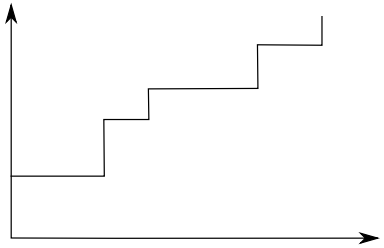
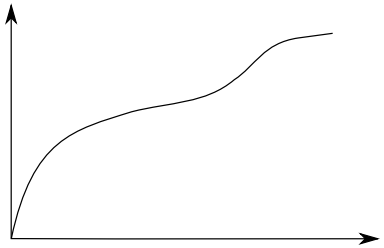
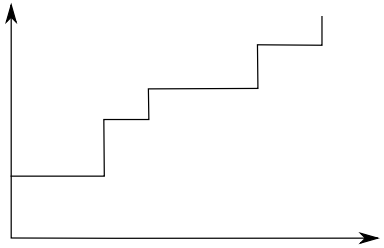
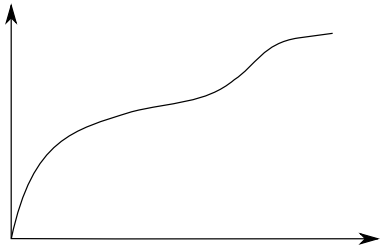
	p^{max}	p^{ave}
	Exact algorithm $O(K^2S)$	Exact algorithm $O(K^2S)$
	Approx. algorithm $O\left(S \log \frac{S}{\varepsilon}\right)$	Approx. algorithm $O\left(SM^3\right)$

Table of results

	P^{max}	P^{ave}	P^{max}_{return}	P^{ave}_{return}
	Exact algorithm $O(K^2S)$	Exact algorithm $O(K^2S)$		
	Approx. algorithm $O\left(S \log \frac{S}{\varepsilon}\right)$	Approx. algorithm $O\left(SM^3\right)$		

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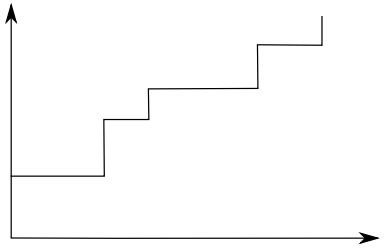
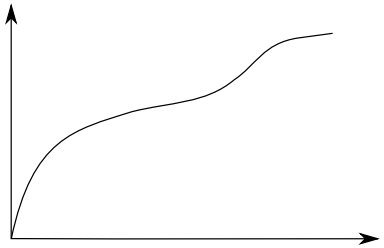
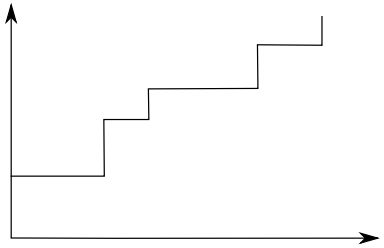
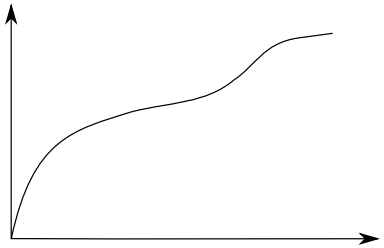
	Exact algorithm $O(K^2S)$	Exact algorithm $O(K^2S)$		
	Approx. algorithm $O\left(S\log\frac{S}{\varepsilon}\right)$	Approx. algorithm $O\left(SM^3\right)$	Approx. algorithm $O\left(\beta^{3S}M^{2S+1}\right)$	

Table of results

	P^{max}	P^{ave}	P^{max}_{return}	P^{ave}_{return}
	Exact algorithm $O(K^2S)$	Exact algorithm $O(K^2S)$	?	?
	Approx. algorithm $O\left(S \log \frac{S}{\varepsilon}\right)$	Approx. algorithm $O\left(SM^3\right)$	Approx. algorithm $O\left(\beta^{3S} M^{2S+1}\right)$	?

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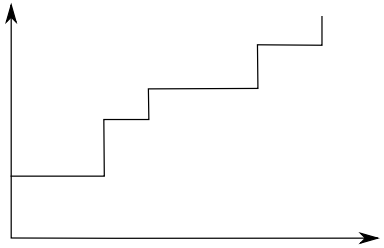
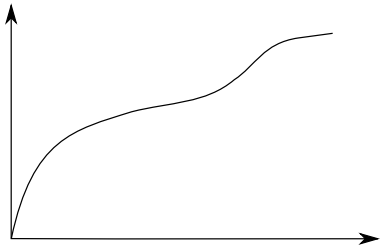
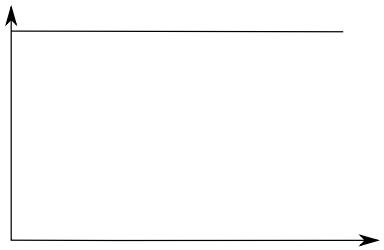
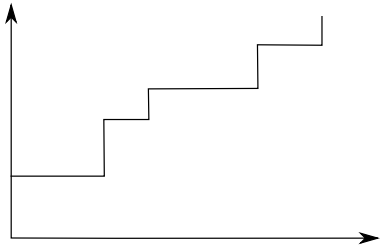
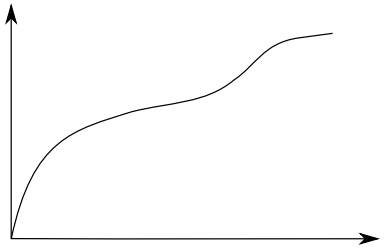
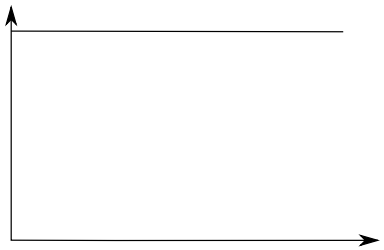
	P^{max}	P^{ave}	P^{max}_{return}	P^{ave}_{return}
	Exact algorithm $O(K^2S)$	Exact algorithm $O(K^2S)$	?	?
	Approx. algorithm $O\left(S\log\frac{S}{\varepsilon}\right)$	Approx. algorithm $O\left(SM^3\right)$	Approx. algorithm $O\left(\beta^{3S}M^{2S+1}\right)$	?
	Closed-form expression $O(1)$	Closed-form expression $O(1)$	Closed-form expression $O(1)$	Exact algorithm $O(S)$

Table of results

	P^{max}	P^{ave}	P^{max}_{return}	P^{ave}_{return}
	Exact algorithm $O(K^2S)$	Exact algorithm $O(K^2S)$	?	?
	Approx. algorithm $O\left(S\log\frac{S}{\varepsilon}\right)$	Approx. algorithm $O\left(SM^3\right)$	Approx. algorithm $O\left(\beta^{3S}M^{2S+1}\right)$	?
	Closed-form expression $O(1)$	Closed-form expression $O(1)$	Closed-form expression $O(1)$	Exact algorithm $O(S)$

A conjecture for P^{ave}_{return}

Conjecture

There exists an optimal solution of P^{ave}_{return}.

A conjecture for P^{ave}_{return}

Conjecture

There exists an optimal solution of P^{ave}_{return}.

$\rightarrow$ If such conjecture true, similar theorem than for P^{max}_{return}.

General conclusion

- Three main problems:
  - → Operational scheduling problem with maximum number of shuttles.
  - → Prospective scheduling and pricing problem with maximum revenue.
  - → Theoretical scheduling problem with minimum waiting time.
- Various methods:
  - → Mixed Integer Linear Programming.
  - → Stochastic Optimization and Sample Average Approximation.
  - → Lagrangian relaxation.
  - → Heuristics.
  - → Exact algorithms and approximation schemes.

Thank you for your attention.