Asymptotic analysis of large random graphs

Marion Sciauveau
Joint work with J-F. Delmas and J-S. Dhersin

CERMICS (ENPC) and LAGA (Paris 13)

Séminaire des doctorants du CERMICS
ENPC - 13 juin 2018
Motivation:

- Social networks or internet can be represented by large random graphs. Understanding their structure is therefore an important issue in mathematics.

- The theory of graph limits is recent and developed by Lovász and Szegedy (2006) and Borgs et al. (2008).
Motivation:

- Social networks or internet can be represented by **large random graphs**. Understanding their structure is therefore an important issue in mathematics.

- The theory of graph limits is **recent** and developed by Lovász and Szegedy (2006) and Borgs et al. (2008).

Problems:

- How can we describe these large graphs?

- How to characterize the convergence of sequences of graphs when the number of nodes goes to infinity?

- What is the best stochastic model of random graphs to approximate these large graphs?
1. Introduction

2. Convergence of dense graph sequences

3. Main results
Some notations for graphs

A finite graph \( G \) is an ordered pair \((V(G), E(G))\) where

- \( V(G) \) is the set of \( v(G) < +\infty \) vertices.
- \( E(G) \) is the set of \( e(G) \) edges among the collection of \({v(G)}^2 \) unordered pairs of vertices.

- \( G \) is simple if it has no self-loops and no multiple edges.
- \( G \) is dense when the number of edges is close to the maximal number of edges.
- \( G \) can be characterized by its adjacency matrix.

![Graph with 5 vertices, its adjacency matrix and its pixel picture.](image)

**Figure:** A graph with 5 vertices, its adjacency matrix and its pixel picture.
Large random graphs: first example

**Erdős-Rényi graph** $G_n(p)$: random graph such that

- $V(G_n(p)) = [n],$
- edges occur independently with the same probability $p$, $0 < p < 1$. 

Figure: Erdős-Rényi graph with parameter $p = \frac{1}{2}$ and its limit.
Large random graphs: first example

Erdös-Rényi graph $G_n(p)$: random graph such that

- $V(G_n(p)) = [n]$,
- edges occur independently with the same probability $p$, $0 < p < 1$.

(a) $n = 10$
Erdős–Rényi graph $G_n(p)$: random graph such that

- $V(G_n(p)) = [n]$,
- edges occur independently with the same probability $p$, $0 < p < 1$.

(a) $n = 10$  
(b) $n = 100$
Erdős-Rényi graph $G_n(p)$: random graph such that

- $V(G_n(p)) = [n]$,
- edges occur independently with the same probability $p$, $0 < p < 1$.

Figure: Erdős-Rényi graph with parameter $p = \frac{1}{2}$ and its limit

(a) $n = 10$
(b) $n = 100$
(c) $n = 1000$
Large random graphs: first example

**Erdős-Rényi graph** $G_n(p)$: random graph such that

- $V(G_n(p)) = [n]$,
- edges occur independently with the same probability $p$, $0 < p < 1$.

(a) $n = 10$
(b) $n = 100$
(c) $n = 1000$
(d) $W = \frac{1}{2}$

**Figure:** Erdős-Rényi graph with parameter $p = \frac{1}{2}$ and its limit
Large random graphs: second example

Randomly grown uniform attachment graph $GUA_n$: random graph such that

- $V(GUA_n) = [n]$,

- Generation of the random graph:
  - Start with a single node.
  - Create a new node.
  - Connect every pair of nonadjacent nodes with probability $\frac{1}{k}$ where $k$ is the current number of nodes.
Randomly grown uniform attachment graph \( \text{GUA}_n \): random graph such that

- \( V(\text{GUA}_n) = [n] \),

Generation of the random graph:
- Start with a single node.
- Create a new node.
- Connect every pair of nonadjacent nodes with probability \( \frac{1}{k} \) where \( k \) is the current number of nodes.

(a) \( n = 10 \)
Large random graphs: second example

Randomly grown uniform attachment graph $\text{GU\!A}_n$: random graph such that

- $V(\text{GU\!A}_n) = [n]$,

- Generation of the random graph:
  - Start with a single node.
  - Create a new node.
  - Connect every pair of nonadjacent nodes with probability $\frac{1}{k}$ where $k$ is the current number of nodes.

Figure: Randomly grown uniform attachment graph and its limit

(a) $n = 10$  (b) $n = 100$
Large random graphs: second example

Randomly grown uniform attachment graph $\text{GU}_A_n$: random graph such that

- $V(\text{GU}_A_n) = [n],$

- Generation of the random graph:
  - Start with a single node.
  - Create a new node.
  - Connect every pair of nonadjacent nodes with probability $\frac{1}{k}$ where $k$ is the current number of nodes.

Figure: Randomly grown uniform attachment graph ant its limit

(a) $n = 10$  
(b) $n = 100$  
(c) $n = 1000$
Large random graphs: second example

Randomly grown uniform attachment graph $\text{GUA}_n$: random graph such that

- $V(\text{GUA}_n) = [n]$,
- Generation of the random graph:
  - Start with a single node.
  - Create a new node.
  - Connect every pair of nonadjacent nodes with probability $\frac{1}{k}$ where $k$ is the current number of nodes.

Figure: Randomly grown uniform attachment graph and its limit

(a) $n = 10$  
(b) $n = 100$  
(c) $n = 1000$  
(d) $W(x, y) = 1 - \max(x, y)$
From graphs to graphons

<table>
<thead>
<tr>
<th>Graph $F$</th>
<th>Graphon $W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertex set $V(F)$</td>
<td>$\sigma$-finite measure space $([0, 1], \mathcal{B}([0, 1]), \lambda)$</td>
</tr>
<tr>
<td>Adjacency matrix $A : V(F) \times V(F) \to {0, 1}$</td>
<td>Symmetric, measurable function $W : [0, 1]^2 \to [0, 1]$</td>
</tr>
</tbody>
</table>

**Figure:** Exponential graphon: $W(x, y) = \frac{e^{x+y}}{1+e^{x+y}}$

We denote by $\mathcal{W}$ the space of all graphons.
1 Introduction

2 Convergence of dense graph sequences

3 Main results
We consider a sequence of dense finite graphs \((G_n : n \in \mathbb{N}^*)\) i.e. such that \(\lim_{n \to +\infty} \frac{2e(G_n)}{v(G_n)(v(G_n)-1)} > 0\).

Questions:

- When can we say that this sequence is convergent?
- How can we characterize the convergence?
- What is the limit?
- How can we generate a sequence of graphs whose limit is a given graphon?
Let $F$ and $G$ be two simple graphs and let $\varphi$ be a map from $V(F)$ to $V(G)$. We define several types of homomorphism:

| Set $F$ | Definition | Density $t(F, G) = \frac{|\text{Hom}(F, G)|}{v(G)v(F)}$ |
|-----------------|-----------------|-----------------|
| Hom$(F, G)$ | $\varphi$ such that $\{i, j\} \in E(F) \Rightarrow \{\varphi(i), \varphi(j)\} \in E(G)$ | $t_{\text{inj}}(F, G) = \frac{|\text{Inj}(F, G)|}{A_v(F)}$ |
| Inj$(F, G)$ | $\varphi$ injective such that $\{i, j\} \in E(F) \Leftrightarrow \{\varphi(i), \varphi(j)\} \in E(G)$ | $t_{\text{ind}}(F, G) = \frac{|\text{Ind}(F, G)|}{A_v(F)}$ |

**Figure**: $\varphi$ is as injective homomorphism but not an induced homomorphism.
How to characterize the convergence of sequence of dense graphs?

**Theorem [Lovász, Szegedy (2006)]**

A sequence of simple graphs \((G_n : n \in \mathbb{N}^*)\) is called convergent if the sequence \((t_{\text{inj}}(F, G_n) : n \in \mathbb{N}^*)\) has a limit for every simple graph \(F\). The limit can be represented as a graphon.

\[
\lim_{n \to +\infty} t_{\text{inj}}(F, G_n) = t(F, W)
\]

where

\[
t(F, W) = \int_{[0,1]} v(F) \prod_{\{i,j\} \in E(F)} W(x_i, x_j) \prod_{k \in V(F)} dx_k.
\]
How to characterize the convergence of sequence of dense graphs?

**Theorem [Lovász, Szegedy (2006)]**

A sequence of simple graphs \((G_n : n \in \mathbb{N}^*)\) is called convergent if the sequence \((t_{inj}(F, G_n) : n \in \mathbb{N}^*)\) has a limit for every **simple graph** \(F\). The limit can be represented as a **graphon**.

**Theorem [Lovász, Szegedy (2006)]**

A sequence of graphs \((G_n : n \in \mathbb{N}^*)\) is said to converge to a graphon \(W\) if for every **finite simple graph** \(F\), we have

\[
\lim_{n \to +\infty} t_{inj}(F, G_n) = t(F, W),
\]

where

\[
t(F, W) = t_{inj}(F, W) = \int_{[0,1]^v(F)} \prod_{\{i,j\} \in E(F)} W(x_i, x_j) \prod_{k \in V(F)} dx_k.
\]
How to understand $t(F, W)$?

Let $F$ be a finite simple graph with $p$ vertices.

- let $G$ be a finite simple graph with $n$ vertices then we have the density of $F$ in $G$:

$$t(F, G) = \frac{1}{n^p} \sum_{\beta \in S_{n,p}} \prod_{\{i, j\} \in E(F)} 1_{\{\beta_i, \beta_j\} \in E(G)}.$$

- let $W$ be a graphon then we have the density of $F$ in $W$:

$$t(F, W) = \frac{1}{\lambda([0, 1])^p} \int_{[0,1]^p} \prod_{k \in V(F)} \prod_{\{i, j\} \in E(F)} W(x_i, x_j).$$
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, $W$, we construct a $W$-random generated $G_n(W)$, as follows:
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, $W$, we construct a $W$-random generated $G_n(W)$, as follows:

- vertex set $\{n\}$

Figure: $G_4(W)$
Given a graphon, $W$, we construct a $W$-random generated $G_n(W)$, as follows:

- vertex set $[n]$

$$X = (X_i : i \in \mathbb{N}^*)$$ i.i.d r.v. uniform on $[0, 1]$

Figure: $G_4(W)$
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, \( W \), we construct a \( W \)-random generated \( G_n(W) \), as follows:

- **Vertex set** \( [n] \)

\[ X = (X_i : i \in \mathbb{N}^*) \text{ i.i.d r.v. uniform on } [0, 1] \]

- \( \{i, j\} \) is an edge in \( G_n(W) \) with probability \( W(X_i, X_j) \).

**Figure: \( G_4(W) \)**
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, \( W \), we construct a \( W \)-random generated \( G_n(W) \), as follows:

- vertex set \([n]\)

\[
X = (X_i : i \in \mathbb{N}^*) \text{ i.i.d r.v. uniform on } [0, 1]
\]

\( \{i, j\} \) is an edge in \( G_n(W) \) with probability \( W(X_i, X_j) \).

\[\begin{array}{cccc}
0 & X_1 & X_4 & X_2 & X_3 & 1 \\
\end{array}\]

\( \text{Figure: } G_4(W) \)
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, $W$, we construct a $W$-random generated $G_n(W)$, as follows:

- vertex set $[n]$

- $X = (X_i : i \in \mathbb{N}^*)$ i.i.d r.v. uniform on $[0, 1]$

- $\{i, j\}$ is an edge in $G_n(W)$ with probability $W(X_i, X_j)$.

Figure: $G_4(W)$
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, \( W \), we construct a \( W \)-random generated \( G_n(W) \), as follows:

- **vertex set** \([n]\)

\[
X = (X_i : i \in \mathbb{N}^*) \text{ i.i.d r.v. uniform on } [0, 1]
\]

- \{i, j\} is an edge in \( G_n(W) \) with probability \( W(X_i, X_j) \).

**Figure:** \( G_4(W) \)
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, $W$, we construct a $W$-random generated $G_n(W)$, as follows:

- vertex set $[n]$

$$X = (X_i : i \in \mathbb{N}^*)$$ i.i.d r.v. uniform on $[0, 1]$

- $\{i, j\}$ is an edge in $G_n(W)$ with probability $W(X_i, X_j)$.

Figure: $G_4(W)$
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, $W$, we construct a $W$-random generated $G_n(W)$, as follows:

- vertex set $[n]$

$$X = (X_i : i \in \mathbb{N}^*) \text{ i.i.d r.v. uniform on } [0, 1]$$

- $\{i, j\}$ is an edge in $G_n(W)$ with probability $W(X_i, X_j)$.

**Figure:** $G_4(W)$
Generating dense graphs with a given number of vertices from a graphon

Given a graphon, $W$, we construct a $W$-random generated $G_n(W)$, as follows:

- vertex set $[n]$

$$X = (X_i : i \in \mathbb{N}^*) \text{i.i.d r.v. uniform on } [0, 1]$$

- $\{i, j\}$ is an edge in $G_n(W)$ with probability $W(X_i, X_j)$.

Figure: $G_4(W)$
1 Introduction

2 Convergence of dense graph sequences

3 Main results
A preliminary result

Let
- \( W \in \mathcal{W} \) be a graphon
- \( (G_n(W) : n \in \mathbb{N}^*) \) be the associated \( W \)-random graphs.

Proposition

For all finite simple graph \( F \) and \( n \geq v(F) \), we have

\[
\mathbb{E} \left[ t_{\text{inj}}(F, G_n(W)) \right] = t(F, W).
\]

Proof:

\[
\mathbb{E} \left[ t_{\text{inj}}(F, G_n(W)) \right] = \frac{1}{A_n^p} \sum_{\beta \in S_{n,p}} \mathbb{E} \left[ \prod_{\{i,j\} \in E(F)} 1 \{ \{\beta_i, \beta_j\} \in E(G_n(W)) \} \right]
\]

\[
= \frac{1}{A_n^p} \sum_{\beta \in S_{n,p}} \mathbb{E} \left[ \prod_{\{i,j\} \in E(F)} 1 \{ \{\beta_i, \beta_j\} \in E(G_n(W)) \} \bigg| X \right] \]

\[
= \frac{1}{A_n^p} \sum_{\beta \in S_{n,p}} \mathbb{E} \left[ \prod_{\{i,j\} \in E(F)} W(X_i, X_j) \right]
\]

\[
= t(F, W).
\]
An overview of existing results

Let $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of $W$-random graphs.

Questions:

What is the limit of this sequence?

Theorem [Borgs, Chayes, Lovász, Sós and Vesztergombi (2008)]

We have almost surely, for every finite simple graph $F$,

$$\lim_{n \to +\infty} t_{\text{inj}}(F, G_n(W)) = t(F, W).$$

What are the fluctuations associated to this convergence?

Theorem [Féray, Méliot and Nikeghbali (2017)]

We have the following convergence in distribution:

$$\sqrt{n} \left( t_{\text{inj}}(F, G_n(W)) - t(F, W) \right) \overset{L}{\to} \mathcal{N}(0, \sigma(F)^2)$$
An overview of existing results

Let $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of $W$-random graphs.

Questions:

What is the limit of this sequence?

Theorem [Borgs, Chayes, Lovász, Sós and Vesztergombi (2008)]

We have almost surely, for every finite simple graph $F$,

$$ \lim_{n \to +\infty} t_{\text{inj}}(F, G_n(W)) = t(F, W). $$

What are the fluctuations associated to this convergence?

Theorem [Féray, Méliot and Nikeghbali (2017)]

We have the following convergence in distribution:

$$ \sqrt{n} \left( t_{\text{inj}}(F, G_n(W)) - t(F, W) \right) \overset{L}{\to} \mathcal{N}(0, \sigma(F)^2) $$
An overview of existing results

Let $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of $W$-random graphs.

Questions:

What is the limit of this sequence?

Theorem [Borgs, Chayes, Lovász, Sós and Vesztergombi (2008)]

We have almost surely, for every finite simple graph $F$,

$$\lim_{n \to +\infty} t_{\text{inj}}(F, G_n(W)) = t(F, W).$$
An overview of existing results

Let $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of $W$-random graphs.

Questions:

What is the limit of this sequence?

Theorem [Borgs, Chayes, Lovász, Sós and Vesztergombi (2008)]

We have almost surely, for every finite simple graph $F$,

$$
\lim_{n \to +\infty} t_{\text{inj}}(F, G_n(W)) = t(F, W).
$$

What are the fluctuations associated to this convergence?
An overview of existing results

Let $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of $W$-random graphs.

Questions:

What is the limit of this sequence?

Theorem [Borgs, Chayes, Lovász, Sós and Vesztergombi (2008)]
We have almost surely, for every finite simple graph $F$,

$$\lim_{n \to +\infty} t_{\text{inj}}(F, G_n(W)) = t(F, W).$$

What are the fluctuations associated to this convergence?

Theorem [Féray, Méliot and Nikeghbali (2017)]
We have the following convergence in distribution:

$$\sqrt{n} \left( t_{\text{inj}}(F, G_n(W)) - t(F, W) \right) \xrightarrow{n \to \infty} \mathcal{N} \left( 0, \sigma(F)^2 \right).$$
What happens when $W \equiv p$?

- $F$ is a finite simple graph with $p$ vertices and $e$ edges
- $W \equiv p$ with $0 < p < 1$

We have:

$$
t(F, W) = \int_{[0,1]^{v(F)}} \prod_{\{i,j\} \in E(F)} W(x_i, x_j) \prod_{k \in V(F)} dx_k = p^e.
$$

and

$$
\sigma(F)^2 = 0.
$$
What happens when $W \equiv p$?

- $F$ is a finite simple graphs with $p$ vertices and $e$ edges
- $W \equiv p$ with $0 < p < 1$

We have:

$$t(F, W) = \int_{[0,1]^{v(F)}} \prod_{\{i,j\} \in E(F)} W(x_i, x_j) \prod_{k \in V(F)} dx_k = p^e.$$ 

and

$$\sigma(F)^2 = 0.$$ 

**Theorem** [Nowicki (1989) or Nowicki and Janson (1991)]

We have the following convergence in distribution:

$$\sqrt{n} \left( t_{inj}(F, G_n(W)) - p^e \right) \xrightarrow{\mathcal{L}} N(0, 0).$$
What happens when \( W \equiv p \)?

- \( F \) is a finite simple graphs with \( p \) vertices and \( e \) edges
- \( W \equiv p \) with \( 0 < p < 1 \)

We have:

\[
t(F, W) = \int_{[0,1]^{v(F)}} \prod_{\{i,j\} \in E(F)} W(x_i, x_j) \prod_{k \in V(F)} d x_k = p^e.
\]

and

\[
\sigma(F)^2 = 0.
\]

**Theorem [Nowicki (1989) or Nowicki and Janson (1991)]**

We have the following convergence in distribution:

\[
n \left( t_{\text{inj}}(F, G_n(W)) - p^e \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \tilde{\sigma}(F)^2 \right).
\]
Generalization for partially labeled graphs

- \( F \) and \( G \) are two simple graphs and \( \varphi \) is a map from \( V(F) \) to \( V(G) \)
- \( k \in [v(F)] \)
- \( \ell = (\ell_1, \ldots, \ell_k) \in S_{v(F), k} \) s.t. \( \ell_i \in [v(F)] \) and \( \ell_1, \ldots, \ell_k \) are all distinct
- \( \alpha = (\alpha_1, \ldots, \alpha_k) \in S_{v(G), k} \) s.t. \( \alpha_i \in [v(G)] \) and \( \alpha_1, \ldots, \alpha_k \) are all distinct.

<table>
<thead>
<tr>
<th>Set ( F )</th>
<th>Definition</th>
<th>Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Inj}(F^\ell, G^\alpha) )</td>
<td>( \varphi \in \text{Inj}(F, G) ) such that ( \varphi(\ell_i) = \alpha_i ), for all ( i \in [k] )</td>
<td>( t_{\text{Inj}}(F^\ell, G^\alpha) = \frac{</td>
</tr>
</tbody>
</table>
Our results

- $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of $W$-random graphs
- $d \geq 1$ and $F = (F_m : 1 \leq m \leq d)$ a sequence of finite simple graphs with the same number of nodes $p$
- $k \in [p]$ and $\ell \in S_{n,p}$

We define the random probability measure $\Gamma_{n,\ell}^F$ by,

$$
\Gamma_{n,\ell}^F = t_{\text{inj}}(F^\ell, G_n^\alpha(W))
$$
Our results

- $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of $W$-random graphs
- $d \geq 1$ and $F = (F_m : 1 \leq m \leq d)$ a sequence of finite simple graphs with the same number of nodes $p$
- $k \in [p]$ and $\ell \in S_{n,p}$

We define the random probability measure $\Gamma_{n}^{F,\ell}$ by, for all $g \in \mathcal{B}^+([0,1]^d)$:

$$
\Gamma_{n}^{F,\ell}(g) = g\left(t_{\text{inj}}(F^\ell, G_n(W))\right)
$$
Our results

- \( W \in \mathcal{W} \) and \( (G_n(W) : n \in \mathbb{N}^*) \) the associated sequence of \( W \)-random graphs
- \( d \geq 1 \) and \( F = (F_m : 1 \leq m \leq d) \) a sequence of finite simple graphs with the same number of nodes \( p \)
- \( k \in [p] \) and \( \ell \in S_{n,p} \)

We define the random probability measure \( \Gamma_{n,\ell}^{F,\ell} \) by, for all \( g \in B^+([0, 1]^d) \):

\[
\Gamma_{n,\ell}^{F,\ell}(g) = \frac{1}{A_n^k} \sum_{\alpha \in S_{n,k}} g \left( t_{\text{inj}}(F_\ell, G_n^{\alpha}(W)) \right)
\]
Our results

- $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of $W$-random graphs
- $d \geq 1$ and $F = (F_m : 1 \leq m \leq d)$ a sequence of finite simple graphs with the same number of nodes $p$
- $k \in [p]$ and $\ell \in S_{n,p}$

We define the random probability measure $\Gamma_{n,\ell}^{F}$ by, for all $g \in \mathcal{B}^+([0, 1]^d)$:

$$\Gamma_{n,\ell}^{F}(g) = \frac{1}{A_{n}^k} \sum_{\alpha \in S_{n,k}} g\left(t_{\text{inj}}(F^{\ell}, G_{n}(W))\right)$$

Theorem

- **Invariance principle**: $(\Gamma_{n,\ell}^{F} : n \in \mathbb{N}^*)$ converges a.s. for the weak topology towards a deterministic probability measure $\Gamma_{\ell}^{F}$
Our results

- \( W \in \mathcal{W} \) and \( (G_n(W) : n \in \mathbb{N}^*) \) the associated sequence of \( W \)-random graphs
- \( d \geq 1 \) and \( F = (F_m : 1 \leq m \leq d) \) a sequence of finite simple graphs with the same number of nodes \( p \)
- \( k \in [p] \) and \( \ell \in \mathcal{S}_{n,p} \)

We define the random probability measure \( \Gamma^{F,\ell}_n \) by, for all \( g \in \mathcal{B}^+([0,1]^d) \):

\[
\Gamma^{F,\ell}_n(g) = \frac{1}{A_n^k} \sum_{\alpha \in \mathcal{S}_{n,k}} g\left(t_{\text{inj}}(F^\ell, G_n^\alpha(W))\right)
\]

Theorem

- **Invariance principle**: \( (\Gamma^{F,\ell}_n : n \in \mathbb{N}^*) \) converges a.s. for the weak topology towards a deterministic probability measure \( \Gamma^{F,\ell} \)
- and its fluctuations: for all \( g \in \mathcal{C}^2([0,1]^d) \), we have:

\[
\sqrt{n} \left( \Gamma^{F,\ell}_n(g) - \Gamma^{F,\ell}(g) \right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^{F,\ell}(g)^2\right)
\]
Let $\mathcal{F}$ be the set of all simple finite graphs.

**Theorem**

We have the following convergence of finite-dimensional distributions:

$$\left(\sqrt{n}(t_{\text{inj}}(F, G_n) - t(F, W)) : F \in \mathcal{F}\right) \xrightarrow{(fdd)} \Theta,$$

where $\Theta = (\Theta_F : F \in \mathcal{F})$ is a centered Gaussian process.
Let $F$ be a finite simple graph with $p = v(F)$ and $W \in \mathcal{W}$ a graphon.

Because

$$|t_{\text{inj}}(F, G) - t(F, G)| \leq \frac{1}{n} \binom{p}{2},$$

we have the following convergence in distribution:

**Theorem**

$$\sqrt{n} \left( t(F, G_n) - t(F, W) \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \sigma(F)^2 \right),$$

Because

$$t_{\text{ind}}(F, G) = \sum_{F' : F \leq F'} \left( -1 \right)^{e(F') - e(F)} t_{\text{inj}}(F', G),$$

we have the following convergence in distribution:

**Theorem**

$$\sqrt{n} \left( t_{\text{ind}}(F, G_n) - t_{\text{ind}}(F, W) \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \sigma_{\text{ind}}(F)^2 \right).$$
Let $W$ be a graphon. We define its normalized degree function $D$ by:

$$D(x) = \int_{[0,1]} W(x,y) \, dy, \quad \forall x \in [0,1].$$

Let $y \in (0,1)$. Recall

$$\Gamma_{F,\ell}^{n} (g) = \frac{1}{A_n^k} \sum_{\alpha \in S_{n,k}} g \left( \text{t}_{\text{inj}} (F^\ell, G_n^\alpha (W)) \right)$$

Then with $F = K_2$, $\ell \in \{1, 2\}$, $k = 1$ and $g(x) = 1_{[0,D(y)]}(x)$, we get that

$$\Gamma_{F,\ell}^{n} (g) = \frac{1}{n} \sum_{i=1}^{n} 1 \left\{ D_i^{(n)} \leq D(y) \right\},$$

where $D_i^{(n)} = \text{t}_{\text{inj}} (K_2^\bullet, G_n^i (W))$, for all $i \in [n]$. 
For \( y \in (0, 1) \), the empirical CDF of the degrees of the graph \( G_n \) is:

\[
\Lambda_n(y) = \frac{1}{n} \sum_{i=1}^{n} 1\{D_i^{(n)} \leq D(y)\}.
\]

**Theorem**

Under regularity conditions on \( W \), we have the following convergence of finite-dimensional distributions:

\[
\left( \sqrt{n} \left( \Lambda_n(y) - y \right) : y \in (0, 1) \right) \xrightarrow{(fdd)} \chi,
\]

where \( \chi = (\chi(y) : y \in (0, 1)) \) is a centered Gaussian process.