Asymptotic analysis of large random graphs

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Introduction/ Motivation

Motivation:

- Social networks or internet can be represented by **large random graphs**. Understanding their structure is therefore an important issue in mathematics.
- The theory of graph limits is **recent** and developped by Lovász and Szegedy (2006) and Borgs et al. (2008).

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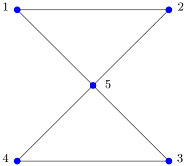
Problems:

- How can we describe these large graphs ?
- How to characterize the convergence of sequences of graphs when the number of nodes goes to infinity ?
- What is the best stochastic model of random graphs to approximate these large graphs ?

Some notations for graphs

A finite graph G is an ordered pair $(V(G), E(G))$ where

- $V(G)$ is the set of $v(G) < +\infty$ vertices.
- $E(G)$ is the set of $e(G)$ edges among the collection of $\binom{v(G)}{2}$ unordered pairs of vertices.
- G is simple if it has no self-loops and no multiple edges.
- G is dense when the number of edges is close to the maximal number of edges.
- G can be caracterized by its adjacency matrix.



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

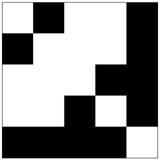


Figure: A graph with 5 vertices, its adjacency matrix and its pixel picture.

Large random graphs: first example

Erdös-Rényi graph $\mathbf{G_n}(\mathfrak{p})$: random graph such that

- $V(G_n(\mathfrak{p})) = [n]$,
- edges occur independently with the same probability $\mathfrak{p}$, $0 < \mathfrak{p} < 1$.

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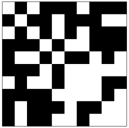

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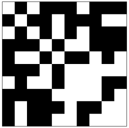

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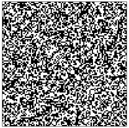

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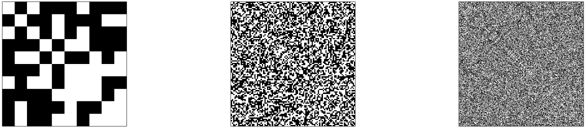

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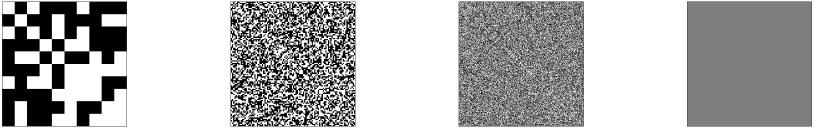

(a) $n = 10$ (b) $n = 100$ (c) $n = 1000$ (d) $W = \frac{1}{2}$

Figure: Erdös-Rényi graph with parameter $\mathfrak{p} = \frac{1}{2}$ ant its limit

Large random graphs: second example

Randomly grown uniform attachment graph $\mathbf{GUA_n}$: random graph such that

- $V(GUA_n) = [n]$,
- Generation of the random graph:
  - Start with a single node.
  - Create a new node.
  - Connect every pair of nonadjacent nodes with probability $\frac{1}{k}$ where k is the current number of nodes.

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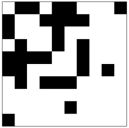

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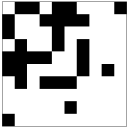

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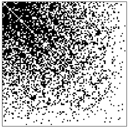

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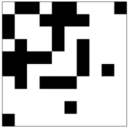

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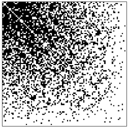

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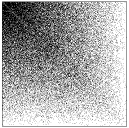

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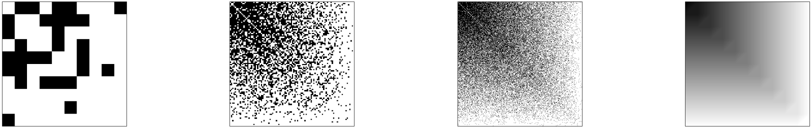


(a) $n = 10$ (b) $n = 100$ (c) $n = 1000$ (d) $W(x, y) = 1 - max(x, y)$

Figure: Randomly grown uniform attachment graph ant its limit

From graphs to graphons

Graph F	**Graphon** W
Vertex set $V(F)$	σ-finite measure space $([0,1], \mathcal{B}([0,1]), \lambda)$
Adjacency matrix $A : V(F) \times V(F) \to \{0,1\}$	Symmetric, measurable function $W : [0,1]^2 \to [0,1]$

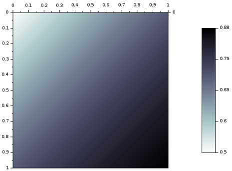


Figure: Exponential graphon: $W(x,y) = \frac{e^{x+y}}{1+e^{x+y}}$

We denote by $\mathcal{W}$ the space of all graphons.

Motivation

We consider a sequence of dense finite graphs $(G_n : n \in \mathbb{N}^*)$ i.e. such that $\lim_{n\to+\infty} \frac{2e(G_n)}{v(G_n)(v(G_n)-1)} > 0$.

Questions:

- When can we say that this sequence is convergent ?
- How can we caracterize the convergence ?
- What is the limit ?
- How can we generate a sequence of graphs whose limit is a given graphon ?

A key object: homomorphism densities

Let F and G be two simple graphs and let φ be a map from $V(F)$ to $V(G)$. We define several types of homomorphism:

Set F	Definition	Density
$\mathrm{Hom}(F,G)$	φ such that $\{i,j\} \in E(F) \Rightarrow \{\varphi(i),\varphi(j)\} \in E(G)$	$t(F,G) = \frac{\lvert\mathrm{Hom}(F,G)\rvert}{v(G)^{v(F)}}$
$\mathrm{Inj}(F,G)$	$\varphi \in \mathrm{Hom}(F,G)$ injective	$t_{\mathrm{inj}}(F,G) = \frac{\lvert\mathrm{Inj}(F,G)\rvert}{A_{v(G)}^{v(F)}}$
$\mathrm{Ind}(F,G)$	φ injective such that $\{i,j\} \in E(F) \Leftrightarrow \{\varphi(i),\varphi(j)\} \in E(G)$	$t_{\mathrm{ind}}(F,G) = \frac{\lvert\mathrm{Ind}(F,G)\rvert}{A_{v(G)}^{v(F)}}$

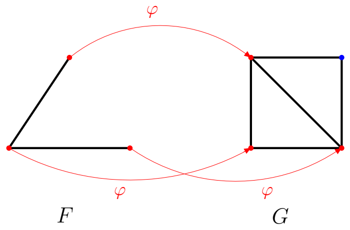
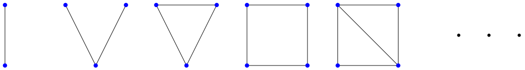


Figure: φ is as injective homomorphism but not an induced homomorphism.

How to charaterize the convergence of sequence of dense graphs ?

Theorem [Lovász, Szegedy (2006)]

A sequence of simple graphs $(G_n : n \in \mathbb{N}^*)$ is called convergent if the sequence $(t_{\text{inj}}(F, G_n) : n \in \mathbb{N}^*)$ has a limit for **every simple graph** F. The limit can be represented as a graphon.



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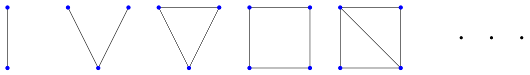


Theorem [Lovász, Szegedy (2006)]

A sequence of graphs $(G_n : n \in \mathbb{N}^*)$ is said to converge to a graphon W if for **every finite simple graph** F, we have

$$\lim_{n \to +\infty} t_{inj}(F, G_n) = t(F, W),$$

where

$$t(F, W) = t_{\mathsf{inj}}(F, W) = \int_{[0,1]^{v(F)}} \prod_{\{i,j\} \in E(F)} W(x_i, x_j) \prod_{k \in V(F)} dx_k.$$

How to understand $t(F, W)$?

Let F be a finite simple graph with p vertices.

- let G be a finite simple graph with n vertices then we have the density of F in G:

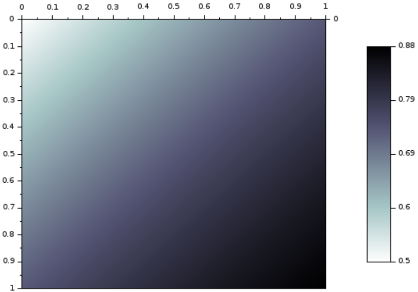
$$t(F, G) = \frac{1}{n^p} \sum_{\beta \in \mathcal{S}_{n,p}} \prod_{\{i,j\} \in E(F)} \mathbf{1}_{\{\{\beta_i, \beta_j\} \in E(G)\}}.$$

- let W be a graphon then we have the density of F in W:

$$t(F, W) = \frac{1}{\lambda([0,1])^p} \int_{[0,1]^p} \prod_{k \in V(F)} \prod_{\{i,j\} \in E(F)} W(x_i, x_j).$$

Generating dense graphs with a given number of vertices from a graphon

Given a graphon, W, we construct a W-random generated $G_n(W)$, as follows:



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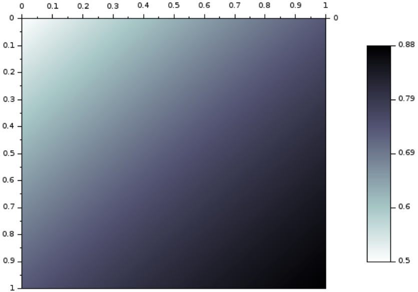

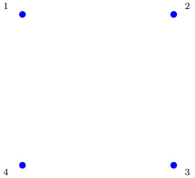

Figure: $G_4(W)$

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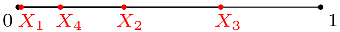

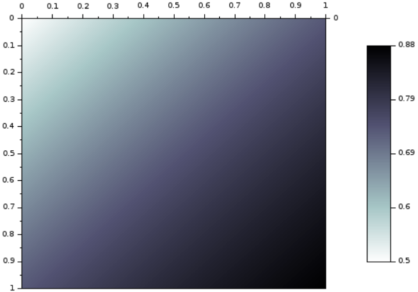

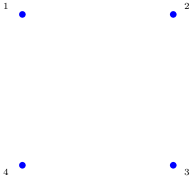

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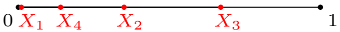

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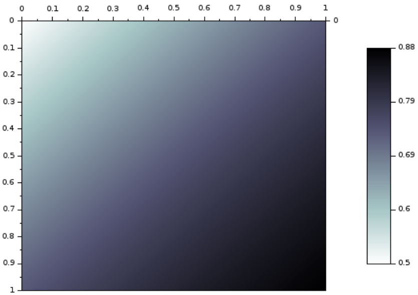

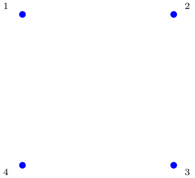

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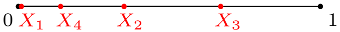

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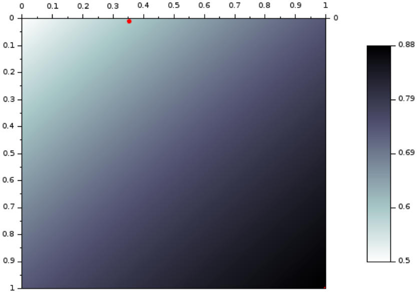

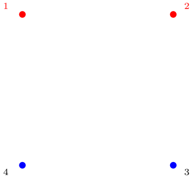

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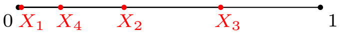

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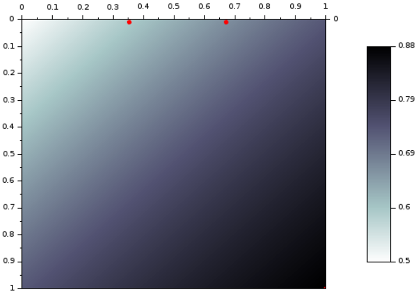

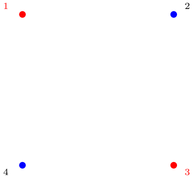

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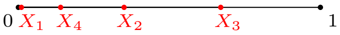

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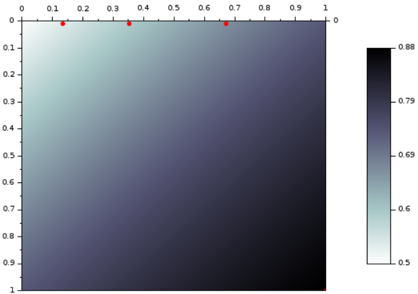

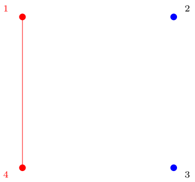

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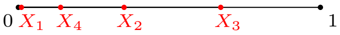

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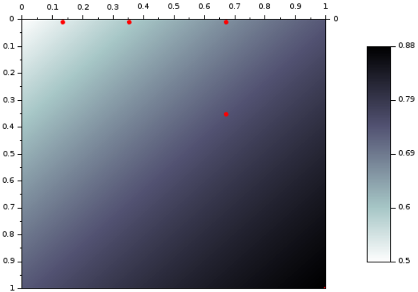

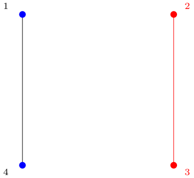

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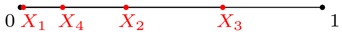

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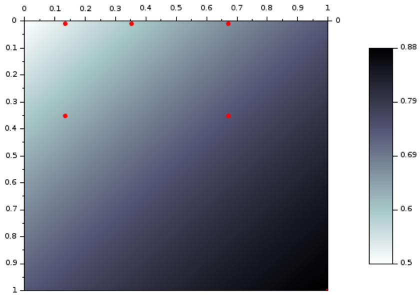

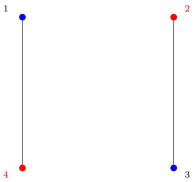

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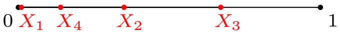
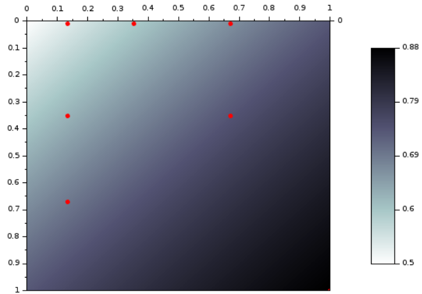


- $\{i, j\}$ is an edge in $G_n(W)$ with probability $W(X_i, X_j)$.



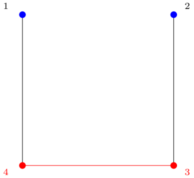


Figure: $G_4(W)$

A preliminary result

Let

- $W \in \mathcal{W}$ be a graphon
- $(G_n(W) : n \in \mathbb{N}^*)$ be the associated W-random graphs.

Proposition

For all finite simple graph F and $n \geq v(F)$, we have

$$\mathbb{E}\left[t_{\text{inj}}(F, G_n(W))\right] = t(F, W).$$

Proof:

$$\begin{aligned}
\mathbb{E}\left[t_{\text{inj}}(F, G_n(W))\right] &= \frac{1}{A_n^p} \sum_{\beta \in \mathcal{S}_{n,p}} \mathbb{E}\left[\prod_{\{i,j\} \in E(F)} \mathbf{1}_{\{\{\beta_i, \beta_j\} \in E(G_n(W))\}}\right] \\
&= \frac{1}{A_n^p} \sum_{\beta \in \mathcal{S}_{n,p}} \mathbb{E}\left[\mathbb{E}\left[\prod_{\{i,j\} \in E(F)} \mathbf{1}_{\{\{\beta_i, \beta_j\} \in E(G_n(W))\}} \middle| X\right]\right] \\
&= \frac{1}{A_n^p} \sum_{\beta \in \mathcal{S}_{n,p}} \mathbb{E}\left[\prod_{\{i,j\} \in E(F)} W(X_i, X_j)\right] \\
&= t(F, W).
\end{aligned}$$

An overview of existing results

Let $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of W-random graphs.

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Theorem [Féray, Méliot and Nikeghbali (2017)]

We have the following convergence in distribution:

$$\sqrt{n}\,(t_{\mathsf{inj}}(F, G_n(W)) - t(F, W)) \xrightarrow[n \to \infty]{\mathcal{L}} \mathcal{N}\left(0, \sigma(F)^2\right).$$

What happens when $W \equiv \mathfrak{p}$?

- F is a finite simple graphs with p vertices and e edges
- $W \equiv \mathfrak{p}$ with $0 < \mathfrak{p} < 1$

We have:

$$t(F,W) = \int_{[0,1]^{v(F)}} \prod_{\{i,j\} \in E(F)} \underbrace{W(x_i, x_j)}_{=\mathfrak{p}} \prod_{k \in V(F)} dx_k = \mathfrak{p}^e.$$

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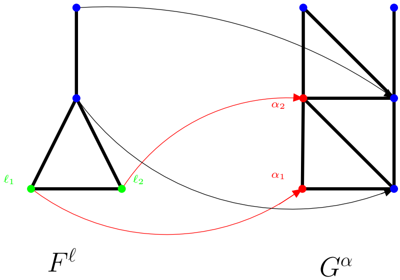
Generalization for partially labeled graphs

- F and G are two simple graphs and φ is a map from $V(F)$ to $V(G)$
- $k \in [v(F)]$
- $\ell = (\ell_1, \dots, \ell_k) \in \mathcal{S}_{v(F),k}$ s.t. $\ell_i \in [v(F)]$ and $\ell_1, \dots, \ell_k$ are all distinct
- $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{S}_{v(G),k}$ s.t. $\alpha_i \in [v(G)]$ and $\alpha_1, \dots, \alpha_k$ are all distinct.

Set F	Definition	Density
$\mathrm{Inj}(F^\ell, G^\alpha)$	$\varphi \in \mathrm{Inj}(F, G)$ such that $\varphi(\ell_i) = \alpha_i$, for all $i \in [k]$	$t_{\mathrm{Inj}}(F^\ell, G^\alpha) = \frac{\left\lvert \mathrm{Inj}(F^\ell, G^\alpha) \right\rvert}{A^{v(F)-k}_{v(G)-k}}$



Our results

- $W \in \mathcal{W}$ and $(G_n(W) : n \in \mathbb{N}^*)$ the associated sequence of W-random graphs
- $d \geq 1$ and $F = (F_m : 1 \leq m \leq d)$ a sequence of finite simple graphs with the same number of nodes p
- $k \in [p]$ and $\ell \in \mathcal{S}_{n,p}$

We define the random probability measure $\Gamma_n^{F,\ell}$ by,

$$\boxed{\Gamma_n^{F,\ell} \quad = \quad t_{\text{inj}}(F^\ell, G_n^\alpha(W))}$$

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Theorem

- *Invariance principle:* $(\Gamma_n^{F,\ell} : n \in \mathbb{N}^*)$ *converges a.s. for the weak topology towards a deterministic probability measure* $\Gamma^{F,\ell}$

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Theorem

- *Invariance principle:* $(\Gamma_n^{F,\ell} : n \in \mathbb{N}^*)$ *converges a.s. for the weak topology towards a deterministic probability measure* $\Gamma^{F,\ell}$
- *and its fluctuations: for all* $g \in \mathcal{C}^2([0,1]^d)$, *we have:*

$$\sqrt{n}\left(\Gamma_n^{F,\ell}(g) - \Gamma^{F,\ell}(g)\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \sigma^{F,\ell}(g)^2\right)$$

A limiting gaussian process

Let $\mathcal{F}$ be the set of all simple finite graphs.

Theorem

We have the following convergence of finite-dimensional distributions:

$$\left(\sqrt{n}\,(t_{\mathit{inj}}(F, G_n) - t(F, W)) : F \in \mathcal{F}\right) \xrightarrow[n \to +\infty]{(fdd)} \Theta,$$

where $\Theta = (\Theta_F : F \in \mathcal{F})$ *is a centered Gaussian process.*

Central limit theorem for other homomorphism densities

Let F be a finite simple graph with $p = v(F)$ and $W \in \mathcal{W}$ a graphon.

1. Because

$$|t_{\text{inj}}(F, G) - t(F, G)| \leq \frac{1}{n}\binom{p}{2},$$

we have the following convergence in distribution:

Theorem

$$\sqrt{n}\,(t(F, G_n) - t(F, W)) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \sigma(F)^2\right),$$

2. Because

$$t_{\text{ind}}(F, G) = \sum_{F' : F \leq F'} (-1)^{e(F')-e(F)} t_{\text{inj}}(F', G),$$

we have the following convergence in distribution:

Theorem

$$\sqrt{n}\,(t_{\textit{ind}}(F, G_n) - t_{\textit{ind}}(F, W)) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \sigma_{\textit{ind}}(F)^2\right).$$

Asymptotics for the empirical degrees CDF (1)

Let W be a graphon. We define its normalized degree function D by:

$$D(x) = \int_{[0,1]} W(x, y) dy, \quad \forall x \in [0, 1].$$

Let $y \in (0, 1)$. Recall

$$\Gamma_n^{F,\ell}(g) = \frac{1}{A_n^k} \sum_{\alpha \in \mathcal{S}_{n,k}} g\left(t_{\mathsf{inj}}(F^\ell, G_n^\alpha(W))\right)$$

Then with $F = K_2$, $\ell \in \{1, 2\}$, $k = 1$ and $g(x) = \mathbf{1}_{[0,D(y)]}(x)$, we get that

$$\Gamma_n^{F,\ell}(g) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\left\{D_i^{(n)} \leq D(y)\right\}},$$

where $D_i^{(n)} = t_{\mathsf{inj}}(K_2^\bullet, G_n^i(W))$, for all $i \in [n]$.

Asymptotics for the empirical degrees CDF (2)

For $y \in (0,1)$, the empirical CDF of the degrees of the graph G_n is:

$$\Lambda_n(y) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\left\{ D_i^{(n)} \leq D(y) \right\}}.$$

Theorem

Under regularity conditions on W, we have the following convergence of finite-dimensional distributions:

$$\left(\sqrt{n} \left(\Lambda_n(y) - y \right) : y \in (0,1) \right) \xrightarrow[n \to +\infty]{(fdd)} \chi,$$

where $\chi = (\chi(y) : y \in (0,1))$ is a centered Gaussian process.

Thank you for your attention !