Dealing with Uncertainty

in Decision Making Models

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I. A product mix problem
A formulation

A furniture manufacturer must choose \( x_j \geq 0 \), how many dressers of type \( j = 1, \ldots, 4 \) to manufacture so as to maximize profit

\[
\sum_{j=1}^{4} c_j x_j = 12x_1 + 25x_2 + 21x_3 + 40x_4
\]

The constraints: Madir, Izmir, Turkey, Spring 2010

\[
t_{c1}x_1 + t_{c2}x_2 + t_{c3}x_3 + t_{c4}x_4 \leq d_c
\]

\[
t_{f1}x_1 + t_{f2}x_1 + t_{f3}x_3 + t_{f4}x_1 \leq d_f
\]

\( t_{cj} (t_{fj}) \) carpentry (finishing) man-hours: dresser type \( j \)

\( d_c (d_f) = \) total time available for carpentry (finishing)
Solution via linear programming:

$$\max \langle c, x \rangle \text{ so that } Tx \leq d, \; x \in \mathbb{R}_+^n.$$ 

With

$$T = \begin{bmatrix} t_{c1} & t_{c2} & t_{c3} & t_{c4} \\ t_{f1} & t_{f2} & t_{f3} & t_{f4} \end{bmatrix} = \begin{bmatrix} 4 & 9 & 7 & 10 \\ 1 & 1 & 3 & 40 \end{bmatrix}, \quad \begin{bmatrix} d_c \\ d_f \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$$

Optimal: \( x^d = (4000/3, 0, 0, 200/3) \)

Value: $18,667.$
Product mix problem (3)

But . . . “reality” can’t be ignored!

\[ t_{cj} = t_{cj} + \eta_{cj}, \quad t_{fj} = t_{fj} + \eta_{fj} \]

<table>
<thead>
<tr>
<th>entry</th>
<th>possible values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_c + \zeta_c):</td>
<td>5,873 5,967 6,033 6,127</td>
</tr>
<tr>
<td>(d_f + \zeta_f):</td>
<td>3,936 3,984 4,016 4,064</td>
</tr>
</tbody>
</table>

10 random variables, say, 4 possible values each

\[ L = 1,048,576 \text{ possible pairs } (T^l, d^l) \]
Product mix problem (4)

What if $\sum_{j=1}^{4} (t_{cj} + \eta_{cj}) x_j > d_c + \zeta_c$? $\implies$ overtime

With $\xi = (\eta\{,\}, \zeta\{,\})$, recourse: $(y_c(\xi), y_f(\xi)) \@ \text{cost} (q_c, q_f)$.

$$\max \langle c, x \rangle - p_1 \langle q, y^1 \rangle - p_2 \langle q, y^2 \rangle \cdots - p_L \langle q, y^L \rangle$$

s.t. $T^1 x - y^1 \leq d^1$

$T^2 x - y^2 \leq d^2$

$\vdots$

$T^L x - y^L \leq d^L$

$x \geq 0, \quad y^1 \geq 0, \quad y^2 \geq 0, \quad \cdots \quad y^L \geq 0.$

Structured large scale l.p. ($L \approx 10^6$)
Define $\Xi = \{\xi = (\eta, \zeta)\}$, $p_\xi = \text{prob}[\xi = \xi]$

$$Q(\xi, x) = \max \left\{ \langle -q, y \rangle \mid T_\xi x - y \geq d_\xi, y \geq 0 \right\}$$

$$EQ(x) = E\{Q(\xi, x)\} = \sum_{\xi \in \Xi} p_\xi Q(\xi, x)$$

the equivalent deterministic program (DEP):

$$\max \langle c, x \rangle + EQ(x) \text{ so that } x \in IR^n_+.$$ 

a non-smooth convex optimization problem: $EQ$ concave.
Solution of DEP, or large scale l.p.,:

Optimal: $x^* = (257, 0, 665.2, 33.8)$

expected Profit: $18,051$

The solution $x^*$ is robust: it considered all $\approx 10^6$ possibilities.
Solution of DEP, or large scale l.p.,:

Optimal: $x^* = (257, 0, 665.2, 33.8)$

expected Profit: $18,051$

The solution $x^*$ is robust: it considered all $\approx 10^6$ possibilities.

Recall: $x^d = (1, 333.33, 0, 0, 66.67)$

expected “profit” relying on $x^d = 16,942$. 
Solution of DEP, or large scale l.p.,:

Optimal: \( x^* = (257, 0, 665.2, 33.8) \)

expected Profit: $ 18,051

The solution \( x^* \) is robust: it considered all \( \approx 10^6 \) possibilities.

Recall: \( x^d = (1, 333.33, 0, 0, 66.67) \)

expected “profit” relying on \( x^d = $ 16,942. \)

- \( x^d \) is not close to optimal
- \( x^d \) isn’t pointing in the right direction
Stochastic Programming relies on:

- linear, non-linear, mixed-integer programming
- large scale: decomposition methods, structured programs, grid computing
- Variational Analysis: non-smooth, duality, epi-convergence (approximations), etc.
- Probability: stochastic processes, asymptotic laws
- Statistics: estimation, lack of data issues
- Functional Analysis, Combinatorial Geometry, etc.
II. Modeling, modeling & modeling!
Uncertain parameters

Deterministic Optimization problem:

$$\min f_0(x) \text{ so that } x \in S \subset \mathbb{R}^n$$
Deterministic Optimization problem:

\[ \min f_0(x) \text{ so that } x \in S \subset \mathbb{R}^n \]

Uncertain parameters: \( \xi \in \Xi \subset \mathbb{R}^N \),

\[ \min f_0(\xi, x) \text{ so that } x \in S(\xi) \subset \mathbb{R}^n \]
Deterministic Optimization problem:
\[ \min f_0(x) \text{ so that } x \in S \subset \mathbb{R}^n \]

Uncertain parameters: \( \xi \in \Xi \subset \mathbb{R}^N \),
\[ \min f_0(\xi, x) \text{ so that } x \in S(\xi) \subset \mathbb{R}^n \]

Wait-and-see solution ??
\[ x(\xi) \in \text{argmin} \left\{ f_0(\xi, x) \mid x \in S(\xi) \right\} \]

What’s needed: a here-and-now solution.
The News Vendor Problem

- $\xi \in \Xi \subset \mathbb{R}_+^+$ demand for a (perishable) good
e.g., plant capacity, overbooking, etc.
- $x \geq 0$ quantity ordered @ unit cost: $c = 10$
- $y \geq 0$ quantity sold, per unit profit $r = 15$

Total revenue (possibly negative):

$$-cx + (c + r)y \quad \text{where} \quad 0 \leq x,$$
$$0 \leq y \leq \min \{x, \xi\}$$

Find optimal $x^*$!
The “deterministic” approach

Pick $\hat{\xi} \in \Xi$ (guessing the future) and solve

$$\min f_0(\hat{\xi}, x) \text{ so that } x \in S(\hat{\xi}) \subset \mathbb{R}^n$$
The “deterministic” approach

Pick $\hat{\xi} \in \Xi$ (guessing the future) and solve

$$\min f_0(\hat{\xi}, x) \text{ so that } x \in S(\hat{\xi}) \subset \mathbb{R}^n$$

**NewsVendor**: $\Xi = [0, 150]$, pick $\hat{\xi} = 75$,

$$\max -cx + (c + r)y$$

$$x \geq 0, \quad 0 \leq y \leq \min\{x, \hat{\xi}\}$$

Solution: $x^o = y^o = \hat{\xi}$, obj. value = $r\hat{\xi} = 1125$

But doesn’t tell much about “profit” if $\xi \neq 75$!
Pick $\xi^1, \ldots, \xi^L$ (scenarios), and for each $\xi^l$ find:

$$x^l \in \arg\min \{ f_0(\xi^l, x) \mid x \in S(\xi^l) \}$$

and “reconcile” the solutions to obtain $x^o$.  

Scenario Analysis

Pick $\xi^1, \ldots, \xi^L$ (scenarios), and for each $\xi^l$ find:

$$x^l \in \arg\min \left\{ f_0(\xi^l, x) \mid x \in S(\xi^l) \right\}$$

and “reconcile” the solutions to obtain $x^o$.

**NewsVendor:** pick $\xi^1 = 10, \xi^2 = 20, \ldots, \xi^{15} = 150$,

$$(x^l, y^l) \in \arg\max \left\{ -cx + (c + r)y \mid y \leq \min[\xi^l, x] \right\}$$

**Wait-and-see sol’ns:** $x^l = \xi^l$. “Reconciliation”? **No** help in choosing $x^o$ the quantity to order.
\( \xi: \) Estimated Density \( h \)

\[
\xi \text{ log-normal: } h(z) = \left( z \tau \sqrt{2\pi} \right)^{-1} e^{-\frac{(\ln z - \theta)^2}{2\tau^2}} \\
\theta = 4.43, \tau = 0.38; \quad H(z) = \int_0^z h(s) \, ds \\
\text{from data, expert(s), all information available}
\]

\[
\begin{array}{c}
\text{lognormal density} \\
\text{Expectation} = 90 \\
\text{Strd.Dev.} = 36
\end{array}
\]

might affect choice of \( \hat{\xi} \), scenarios: \( \xi^1, \ldots \)
Maximize Expected Return

\[
\max -cx + E\{(c + r)y_\xi\}
\]
so that \(x \geq 0, \ 0 \leq y_\xi \leq \min[\xi, x]\)

The *equivalent deterministic program*:

\[
\max_{x \geq 0} -cx + EQ(x), \quad EQ(x) = E\{Q(\xi, x)\}
\]

where \(Q(\xi, x) = \begin{cases} (c + r)\xi & \text{if } \xi \leq x, \\ (c + r)x & \text{if } \xi \geq x \end{cases}\)

\[
EQ(x) = (c + r) \left( \int_0^x \xi H(d\xi) + \int_x^\infty x H(d\xi) \right)
\]
Optimal: Expected Profit

\[ x^* = H^{-1}\left( \frac{r}{c + r} \right) = H^{-1}(0.6) = 99.2 \]

for \( c = 10, r = 15 \).
NewsVendor’s Objective

\[ E_w = 90, \text{ Stdev}=36 \]
\[ \text{Cost}=10, \text{ Revenue}=15 \]
\[ \text{Opt. sol'n: 99.2, } E[\text{profit}] = 1,084 \]
\[ \text{Sol'n guess: 75, } E[\text{profit}] = 1,000 \]
... but is maximum expected return the “real” objective?
The Returns’ Densities

Return Densities

- $x = 72.2$
- $x = 92.2$
- $x = 112.2$
- $x = 132.2$
Choosing the Returns’ Distribution

Distribution Functions

- $x = 72.2$
- $x = 92.2$
- $x = 112.2$
- $x = 132.2$
- $x = 152.2$
Reduction the choice of a distribution function to the choice of a “number”

- maximize expected return (scaled?),
- max. E{return} & minimize customers lost,
- minimize Value-at-Risk (VaR),
- minimize the probability of any loss,
- minimizing a “Safeguarding” Measure
- variants & combinations of the above
"generic" stochastic optimization problem:
\[
\max E\{f_0(\xi, x)\} \quad \text{such that} \quad f_i(\xi, x) \leq 0, \quad i = 1, \ldots, m,
\]

Risk-averse or risk-seeking \(\Rightarrow\) utility function

von Neuman-Morgenstern: under "rationality" (axiomatics) there exists a utility function \(u\) such that \(\bar{x} \in \arg\max E\{u(f_0(\xi, x))\}\) (subject to the constraints) identifies the preferred return's distribution

Modeling hurdle: no blueprint for \(u\)'s design!
Robust Optimization

“generic” optimization problem: \[ \max \gamma \]
\[ \text{so that } \gamma - f_0(\xi, x) \leq 0, \]
\[ f_i(\xi, x) \leq 0, \ i = 1, \ldots, m, \]

“robust” counterpart: \[ \max \gamma \]
\[ \text{so that } \gamma - f_0(\xi, x) \leq 0, \ \forall \xi \in U \subset \Xi \]
\[ f_i(\xi, x) \leq 0, \ i = 1, \ldots, m, \ \forall \xi \in U, \]

Challenges:
- formulate a computationally tractable robust counterpart
- specify reasonable uncertainty for set \( U \)
Reliability: Chance Constraints

Satisfy constraints with probability $\alpha \in (0, 1]$

$$\min f_0(x) \text{ so that } \text{prob. } [x \in S(\xi)] \geq \alpha$$

**Variant:**

$$\min f_0(x) \text{ so that } \text{prob. } [f_i(\xi, x) \leq 0] \geq \alpha_i, \ i \in I$$

$\alpha_i$ dictated by

- contractual obligations
- company policy, guess, etc.
\[
\max -cx + (c + r)y \quad \text{so that} \quad x \geq 0, \quad y \in [H^{-1}(\alpha), x]
\]

Infeasible if \( x < H^{-1}(\alpha) \). Profit? later
When \( \alpha = 0.9 : \hat{x} = 135.9; \quad \alpha = 0.75 : \hat{x} = 108.1. \)
Let $F(s; x) = \text{prob}\left[ -cx + Q(\xi, x) \leq s \right]$

**Value-at-Risk (VaR) for $\alpha \in (0, 1)$:**

$$\text{VaR}(\alpha; x) = F^{-1}(\alpha; x) \quad (= \sup\{v \mid v \in F^{-1}(\alpha; x)\})$$

Objective: find $x$ that maximizes $\text{VaR}(\alpha; x)$
Let $F(s; x) = \text{prob } [-cx + Q(\xi, x) \leq s]$

**Value-at-Risk (VaR) for $\alpha \in (0, 1)$:**

$\text{VaR}(\alpha; x) = F^{-1}(\alpha; x) \quad (= \sup \{v \mid v \in F^{-1}(\alpha; x)\})$

**Objective:** find $x$ that maximizes $\text{VaR}(\alpha; x)$

**Challenge:** $x \mapsto \text{VaR}(\alpha; x)$ isn’t concave.

**Heuristic:** $F$ is $\mathcal{N}(\mu(x), \sigma(x)^2)$ and

$$\text{VaR}(\alpha; x) = \mathcal{N}^{-1}(\alpha; \mu(x), \sigma(x)^2)$$
**VaR: NewsVendor Problem**

![Graph of VaR (Value at Risk)](image)

- **x=72.2**
- **x=92.2**
- **x=112.2**
- **x=132.2**
- **x=152.2**

**VaR (Value @ Risk)**

- 10% line

Dealing with Uncertainty – p. 28/33
Stochastic Dominance: \( D_x(s) \leq D_{\hat{x}}(s) \), \( \forall s \)

\( \Rightarrow \) probability of the return to be \( \leq s \)
always smaller when choosing \( x \) rather than \( \hat{x} \)

Unfortunately unusual
Second order Stochastic Dominance

\[ D^2(s) = \int_{-\infty}^{s} D(\xi) \, d\xi = E\{(s - \xi)_+\} \]

\( D^2 \): the expected shortfall function
Stochastic Dominance Constraint

NewsVendor problem

$$\max rx, \quad x \geq 0$$

such that $$D_x^2(s) \leq G^2(s), \quad s \in [\alpha, \beta]$$

given a “desirable” distribution function $$G$$

$$D_x$$: distribution of actual return, decision $$x$$

$$rx$$ when $$\xi \leq x$$ and $$(c + r)\xi - cx$$ when $$\xi < x$$

leads to a semi-infinite optimization problem

used in portfolio optimization, for example
Perturbing the Probability Measure

Newsboy problem
Objective function
$Ew = 90$, $Stdev = 36$
$Cost = 10$, $Revenue = 15$
Optimal sol’n: 99.2

102.4 Optimal sol’n
$Ew = 100$, $Stdev = 40$
"loss": 0.15%

stress testing via distribution contamination
A few references


