Information Constraints
and Discretization Puzzles
in Stochastic Optimal Control

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Outline of the presentation

1. From deterministic to stochastic optimal control (SOC)
2. SOC and information constraints
3. SOC and discretization puzzles
4. Conclusion
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1. From deterministic to stochastic optimal control (SOC)
   - Single dam models
   - Interconnected dam models
   - The deterministic optimization problem is well posed
   - In the uncertain framework, the optimization problem is not well posed
   - There are two ways to express the information constraints

2. SOC and information constraints
   - Stochastic programming
   - Dynamic programming
   - Information patterns are determining in SOC

3. SOC and discretization puzzles
   - Working out an example
   - Monte Carlo-based discretization
   - Scenario tree-based discretization
   - Discussion on discretization puzzles
   - A constructive proposal

4. Conclusion
From deterministic to stochastic optimal control (SOC) - Single dam models

At+1

St

Qt

Rt

Qt
A single dam nonlinear dynamical model where turbinating decisions are made before knowing the water inflows

We can model the dynamics of the water volume in a dam by

\[
S_{t+1} = \min\{S^#, \ S_t - Q_t + A_{t+1}\}
\]

- **\( S_t \)**: volume (stock) of water at the beginning of period \([t, t+1]\)
- **\( A_{t+1} \)**: inflow water volume (rain, etc.) during \([t, t+1]\)
- **decision-hazard:**
  - \( A_{t+1} \) is not available at the beginning of period \([t, t+1]\)
- **\( Q_t \)**: turbined outflow volume during \([t, t+1]\)
  - decided at the beginning of period \([t, t+1]\)
  - supposed to depend on the stock \( S_t \) but not on the inflow water \( A_{t+1} \)
  - chosen such that \( 0 \leq Q_t \leq S_t \)
A single dam nonlinear dynamical model in decision-hazard: equivalent formulation

We can model the dynamics of the water volume in a dam by

\[ S_{t+1} = S_t - Q_t - R_{t+1} + A_{t+1} \]

- **\( S_t \)** volume (stock) of water at the beginning of period \([t, t + 1]\)
- **\( A_{t+1} \)** inflow water volume (rain, etc.) during \([t, t + 1]\)
- **\( Q_t \)** turbined outflow volume during \([t, t + 1]\)
  - decided at the beginning of period \([t, t + 1]\)
  - supposed to depend on the stock \( S_t \) but not on the inflow water \( A_{t+1} \)
  - chosen such that \( 0 \leq Q_t \leq S_t \)
- **\( R_{t+1} = \left[ S_t - Q_t + A_{t+1} - S^\#_t \right]_+ \)** the spilled volume
When turbinating decisions are made after knowing the water inflows, we obtain a linear dynamical model

We can model the dynamics of the water volume in a dam by

\[ S_{t+1} = S_t - Q_t - R_t + A_t \]

- \( S_t \) volume (stock) of water at the beginning of period \([t, t+1][\)
- \( A_t \), inflow water volume (rain, etc.) during \([t, t+1][\;
- hazard-decision: \( A_t \) is known and available at the beginning of period \([t, t+1][\;
- \( Q_t \) turbined outflow volume and \( R_t \) spilled volume
  - decided at the beginning of period \([t, t+1][\;
  - supposed to depend on the stock \( S_t \) and on the inflow water \( A_t \)
  - chosen such that

\[ 0 \leq S_t - Q_t + A_t - R_t \leq S^\# \]
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Complexity increases with interconnected dams
Typology of hydro-valleys

- dams in cascade
- converging valleys
- pumping
From deterministic to stochastic optimal control (SOC)  
Interconnected dam models

Sketch of a cascade model with dams $i = 1, \ldots, N$

\[ \begin{align*}
A_{i,t} & : \text{inflow into dam } i \text{ at time } t \text{ (rain, ruissellement)} \\
S_{i,t} & : \text{volume in dam } i \text{ at time } t \text{ (water volume)} \\
Q_{i,t} & : \text{turbined from dam } i \text{ at time } t \text{ (valorized at price } P_{i,t}) \\
R_{i,t} & : \text{spilled volume from dam } i \text{ at time } t \text{ (irrigation…)}
\end{align*} \]
Water inflows historical scenarios
Description of variables: $A_{i,t}$

**Inflow water volume**

$A_{i,t}$: amount of water inflowing **without any control** into the dam

Inflow water volumes $A_{i,t}$ are part of the **problem data**, and may be

- either **deterministic**, in which case the values $(A_{i,0}, \ldots, A_{i,T})$ are unique and are exactly known, for every dam $i$
- or **uncertain**, and many possibilities can be considered
  - inflows chronicles may be available: $(A_{i,0}^s, \ldots, A_{i,T}^s)$ where $s$ belongs to a set $S$ of **historical scenarios**
  - in addition, the **probability** $\pi^s$ of scenario $s$ may be known
  - more generally, the sequence $(A_{i,0}, \ldots, A_{i,T})$ may be considered as a **random process** with known **probability distribution**

**Prices** $P_{i,t}$ can also be part of the problem data

In the same way, the prices of turbined volumes can be part of the problem data, with an uncertain or deterministic status
Description of variables: $Q_{i,t}$

Turbined water volume

$Q_{i,t}$: amount of water turbined to produce electricity

- Turbined volumes $Q_{i,t}$ are control variables, the possible values of which are in the hands of the decision-maker to achieve different goals.
- In the deterministic framework, one looks for a single sequence of values $(Q_{i,0}, \ldots, Q_{i,T-1})$ for each dam $i$.
- In the uncertain framework, the situation is more complex.
  - One has to specify the online available information upon which decisions are made.
  - The controls $Q_{i,t}$ are also uncertain variables.
  - And they can be obtained by means of feedback policies feeding on online information.
Description of variables:  $R_{i,t}$

Spilled water volume

$R_{i,t}$: amount of water spilled from the dam with or without control

The status of the spilled volumes depends on the context, and $R_{i,t}$ can represent

- an amount taken in the dam (for irrigation purposes, for instance), and can be a data (deterministic or uncertain)
- a volume deliberately taken in the dam and released without being turbined, and it is then a control variable
- the amount of water which overflows in case of excess

$$R_{i,t+1} = \left[ S_{i,t} + A_{i,t+1} + Q_{i-1,t} - Q_{i,t} - S^\#_i \right] +$$

and it is then an output variable which results from a calculation based upon other variables.
The dynamics describes the temporal evolution of the water stock volume in the dam

The water volume in the dam evolves when time goes by and it is given by

\[ S_{i,t+1} = S_{i,t} + A_{i,t+1} + Q_{i-1,t} - Q_{i,t} \]

A general form is

\[ S_{i,t+1} = F_{i,t}(S_{i,t}, Q_{i,t}, A_{i,t+1}, Q_{i-1,t}) \]
The payoffs are decomposed in instantaneous and final

- The payoff obtained by turbining the volume $Q_{i,t}$ is
  $$P_{i,t} Q_{i,t}$$

- We shall use the general form
  $$L_i(t)(S_{i,t}, Q_{i,t}, A_{i,t}, P_{i,t})$$

- To avoid emptying the dam at the end of the optimization process, we introduce a “water value” term bearing on the final stock volume
  $$K_i(S_{i,T})$$
The management of the dams cascade can be formulated as an intertemporal optimization problem

- The payoff attached to the dam cascade is
  \[
  \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L_{i,t}(S_{i,t}, Q_{i,t}, A_{i,t}, P_{i,t}) + K_{i}(S_{i,T}) \right)
  \]

- This payoff must be optimized
  - under constraints of dynamics
    \[
    S_{i,t+1} = F_{i,t}(S_{i,t}, Q_{i,t}, A_{i,t+1}, Q_{i-1,t}), \quad i \in [1, N], \quad t \in [0, T-1]
    \]
  - and under bounds constraints
    \[
    Q_{i,t} \in \left[ Q_{i}, \overline{Q}_{i} \right], \quad i \in [1, N], \quad t \in [0, T-1]
    \]
    \[
    S_{i,t} \in \left[ S_{i}, \overline{S}_{i} \right], \quad i \in [1, N], \quad t \in [0, T]
    \]
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The deterministic optimization problem

- In the deterministic case, inflows $A_i := (A_{i,0}, \ldots, A_{i,T})$ and prices $P_i := (P_{i,0}, \ldots, P_{i,T})$ are unique and are exactly known, for every dam $i$.
- With $Q_i := (Q_{i,0}, \ldots, Q_{i,T-1})$ and $S_i := (S_{i,0}, \ldots, S_{i,T})$, the optimization problem is

$$
\max_{(Q_i,S_i)_{i=1,\ldots,N}} \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L_{i,t}(S_{i,t}, Q_{i,t}, A_{i,t}, P_{i,t}) + K_{i}(S_{i,T}) \right)
$$

under constraints of dynamics

$$
S_{i,0} \text{ given, } i \in [1, N]
$$

$$
S_{i,t+1} = F_{i,t}(S_{i,t}, Q_{i,t}, A_{i,t+1}, Q_{i-1,t}), \quad i \in [1, N], \ t \in [0, T-1]
$$

and under bounds constraints

$$
Q_{i,t} \in [\underline{Q}_i, \overline{Q}_i], \ i \in [1, N], \ t \in [0, T-1]
$$

$$
S_{i,t} \in [\underline{S}_i, \overline{S}_i], \ i \in [1, N], \ t \in [0, T]
$$
The deterministic optimization problem is well posed

- If, to make things simple, we assume all variables to be scalar, the optimization must be made with respect to all control variables \((\in \mathbb{R}^{N(T-0)})\) and to all “state” variables \((\in \mathbb{R}^{N(T-1)})\).

- The optimization yields for every dam trajectories \((Q_{i,0}, \ldots, Q_{i,T-1})\) and \((S_{i,0}, \ldots, S_{i,T})\) of the controls and of the “states”, as well as the optimal payoff.
Here are some characteristics of the deterministic optimization problem

- The resolution of the deterministic optimization problem can be made in the framework of mathematical programming, that is, the maximization of a function over \( \mathbb{R}^n \) under equality and inequality constraints.

- The volumes \( S_{i,t} \) are intermediary variables, completely determined by the choice of the \( Q_{i,t} \).
  (however, such intermediary variables may be used to compute gradients by means of an adjoint state)

- The optimization problem is often solved after formulating it as a linear problem and with softwares such as CPLEX.

- In the nonlinear case, the optimization problem can be difficult to solve, due to its large size, but it is known how to apply decomposition and coordination techniques.
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From deterministic to stochastic optimal control (SOC)

In the uncertain framework, the optimization problem is not well posed

Figure: Stock volumes in a dam following an optimal strategy, with a final value of water
In the uncertain framework, two additional questions must be answered

Question

How are the uncertainties taken into account in the payoff criterion and in the constraints?

Question

Upon which online information are turbinating decisions made?

In practice, one assumes

- more or less precise knowledge of the past (online information)
- statistical assumptions about the future (offline information)
How are the uncertainties taken into account in the payoff criterion and in the constraints?

In a probabilistic setting, where uncertainties are random variables, a classical answer is

- to take the **mathematical expectation** of the payoff (risk-neutral approach)

\[
\mathbb{E}\left( \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L_{i,t}(S_{i,t}, Q_{i,t}, A_{i,t}, P_{i,t}) + K_{i}(S_{i,T}) \right) \right)
\]

- and to satisfy all (physical) constraints **almost surely** that is, practically, for all possible issues of the uncertainties (robust approach)

\[
P\left( Q_{i,t} \in [Q_{i}, \bar{Q}_{i}] \quad , \quad i \in [1, N] , \quad t \in [0, T-1] \right)
\]
\[
P\left( S_{i,t} \in [S_{i}, \bar{S}_{i}] \quad , \quad i \in [1, N] , \quad t \in [0, T] \right) = 1
\]
Upon which online information are turbinating decisions made?

- On the one hand, it is suboptimal to restrict oneself, as in the deterministic case, to open-loop controls depending only upon time.
- On the other hand, it is impossible to suppose that we know in advance what will happen for all times: clairvoyance is impossible.

In our case, the decision $Q_{i,t}$ will be looked after as a
- fonction of past uncertainties,
- that is, of the water inflows and the prices $(A_{j,\tau}, P_{j,\tau})$ for $j \in [1, N]$ and $\tau \in [0, t]$. 
On-line information feeds turbinating decisions

As a consequence, the control $Q_{i,t}$ is looked after under the form

$$Q_{i,t} = \phi_{i,t}(A_{1,0}, P_{1,0}, \ldots, A_{N,0}, P_{N,0}, \ldots, A_{1,\tau}, P_{1,\tau}, \ldots, A_{N,\tau}, P_{N,\tau}, \ldots A_{1,t}, P_{1,t}, \ldots, A_{N,t}, P_{N,t})$$

Denote the uncertainties at time $t$ by

$$W_t = (A_{1,t}, P_{1,t}, \ldots, A_{N,t}, P_{N,t})$$

When uncertainties are considered as random variables (measurable mappings), the above formula for $Q_{i,t}$ expresses the measurability of the control variable $Q_{i,t}$ with respect to the past uncertainties, also written as

$$\sigma(Q_{i,t}) \subset \sigma(W_0, \ldots, W_t) \iff Q_{i,t} \preceq \sigma(W_0, \ldots, W_t)$$
On-line information structure can reflect decentralized information

When the control $Q_{i,t}$ is looked after under the form

$$Q_{i,t} = \phi_{i,t}(A_{i,0}, P_{i,0}, \ldots, A_{i,\tau}, P_{i,\tau}, \ldots, A_{i,t}, P_{i,t})$$

this expresses that each dam is managed with local information
The stochastic optimization problem

In a probabilistic setting, where uncertainties are random variables and where a probability $\mathbb{P}$ is given, a possible formulation is

$$\max_{(Q_i, S_i)_{i=1,\ldots,N}} \mathbb{E} \left( \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L_{i,t}(S_i,t, Q_i,t, A_i,t, P_i,t) + K_i(S_i,T) \right) \right)$$

- under constraints of dynamics

$$S_{i,0} \text{ given, } \quad i \in [1, N]$$

$$S_{i,t+1} = F_{i,t}(S_i,t, Q_i,t, A_i,t+1, Q_{i-1,t}), \quad i \in [1, N], \quad t \in [0, T-1]$$

- under bounds constraints

$$\mathbb{P} \left( \begin{array}{l}
Q_{i,t} \in \left[ Q_i, \bar{Q}_i \right], \quad i \in [1, N], \quad t \in [0, T-1] \\
S_{i,t} \in \left[ S_i, \bar{S}_i \right], \quad i \in [1, N], \quad t \in [0, T] 
\end{array} \right) = 1$$

- and under measurability constraints

$$Q_{i,t} \preceq \sigma(W_0, \ldots, W_t), \quad i \in [1, N], \quad t \in [0, T-1]$$
The outputs of a stochastic optimization problem are random variables.

Histogram of the optimal payoff with a final value of water.
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There are two ways to express the information constraints

- **Functional approach** \( Q_{i,t} = \phi_t(W_0, \ldots, W_t) \)

  In the special case of the **Markovian framework with white noise**, it can be shown that there is **no loss of optimality** to look for solutions under the form

  \[
  Q_{i,t} = \phi_{i,t}(S_{1,t}, \ldots, S_{N,t})
  \]

- **Measurability constraints** \( Q_{i,t} \preceq \sigma(W_0, \ldots, W_t) \)
  
  In the special case of the **scenario tree approach**, uncertainties are supposed to be observed, and all possible scenarios \( (W_{0}^s, \ldots, W_{T}^s), s \in S \), are organized in a tree, and controls \( Q_{i,t} \) are indexed by nodes on the tree.

  Equivalently, \( Q_{i,t} \) are indexed by \( s \in S \) with the constraint that if two scenarios coincide up to time \( t \), so must do the controls at time \( t \)

  \[
  (W_{0}^s, \ldots, W_{t}^s) = (W_{0}^{s'}, \ldots, W_{t}^{s'}) \Rightarrow Q_{i,t}^s = Q_{i,t}^{s'}
  \]
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Stochastic Optimal Control (SOC)

Standard problem

\[
\begin{align*}
\min_{X,U} & \quad \mathbb{E} \left( \sum_{t=0}^{T-1} L_t (X_t, U_t, W_{t+1}) + K (X_T) \right) \\
\text{s.t.} & \quad X_{t+1} = f_t (X_t, U_t, W_{t+1}), \quad \forall t = 0, \ldots, T - 1 \\
\end{align*}
\]

objective function

dynamics

\[
X_0 = W_0
\]

\(\mathbb{P}\text{-a.s. constraints}\) (e.g. bounds on \(X_t\) and \(U_t\))

\(U_t\) is \(\sigma\{W_0, \ldots, W_t\}\)-measurable \((\text{nonanticipativity constraint})\)
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Classical methods

Stochastic Programming (1)

- Discretize random inputs of the problem using scenario trees
- Write the constraints and objective function on this structure
- Use standard numerical methods from mathematical programming

Complexity of multistage stochastic programs

- The amount of scenarios needed to achieve a given precision grows exponentially w.r.t. the number of time steps
Classical methods

Stochastic Programming (2)

Information pattern

The noise variables $W_0, \ldots, W_t$ are recalled and observed at time $t$
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Classical methods
Dynamic Programming (1)

Markovian framework
Noise variables $W_0, \ldots, W_T$ are independent

Information pattern
The state $X_t$ is observed at time $t$
or equivalently (see Pierre Carpentier’s lecture)
the noise variables $W_0, \ldots, W_t$ are recalled and observed at time $t$

Cost-to-go function, Bellman function, value function

$$V_t(x) := \min_U \mathbb{E} \left( \sum_{s=t}^{T-1} L_s(X_s, U_s, W_{s+1}) + K(X_T) \mid X_t = x \right)$$

subject to constraints of the original problem
Classical methods

Dynamic Programming (2)

The Dynamic Programming (DP) principle

There is no loss of optimality in looking for the optimal strategy as feedback functions on $X_t$ only

$$U_t = \phi_t(X_t)$$

Moreover, DP provides a way to compute Bellman functions and optimal feedbacks backwards

$$V_T(x) = K(x)$$

and, for all $t = t_0, \ldots, T - 1$

$$V_t(x) = \min_{u \in U_t} \mathbb{E} \left( L_t(x, u, W_{t+1}) + V_{t+1}(f_t(x, u, W_{t+1})) \right)$$
Classical methods
Dynamic Programming (3)

- Most of the time, the DP equation cannot be solved analytically
- Numerical methods have to be used

Curse of dimensionality
The complexity associated with the resolution of the DP equations by discretization grows exponentially w.r.t. the dimension of the state space

Some attempts of breaking the curse are through functional approximations
- Approximate Dynamic Programming
- Stochastic Dual Dynamic Programming
- Dual Approximate Dynamic Programming
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A simple linear quadratic control problem

- State equations
  \[ X_1 = X_0 + U_0 \]
  \[ X_2 = X_1 - U_1 \]

- Optimization problem
  \[ \min_{U_0, U_1} \mathbb{E} \left( k^2 U_0^2 + X_2^2 \right) \]

- Observations
  \[ Y_0 = X_0 \]
  \[ Y_1 = X_1 + W_1 \]
Changing information pattern can turn an easy problem into an open problem

- Observations
  \[ Y_0 = X_0 \text{ and } Y_1 = X_1 + W_1 \]

- The classical LQG problem
  \[
  \min_{U_0 \preceq Y_0, U_1 \preceq (Y_0, Y_1)} \mathbb{E} \left( k^2 U_0^2 + X_2^2 \right)
  \]
  has optimal (linear) solution \( U_0^\# = 0 \) and \( U_1^\# = X_0 = X_0 + U_0^\# = X_1 \)

- In contrast, the so-called Witsenhausen Counterexample (1968)
  \[
  \min_{U_0 \preceq Y_0, U_1 \preceq Y_1} \mathbb{E} \left( k^2 U_0^2 + X_2^2 \right)
  \]
  is known to have an optimal (tricky) solution, not known till today
Summary

- Information constraints may be expressed
  - algebraically, by measurability constraints (like in the scenario tree approach)
  - functionally, by looking after feedback solutions (like in the Markovian setting)

- Changing information pattern can turn an easy problem (LQG with classical pattern) into an open problem (Witsenhausen counterexample)

- We shall now discuss, by working out an example, issues in the discretization of information constraints
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A simple SOC problem

\[ \min_{U \preceq W_0} \mathbb{E}(\varepsilon U^2 + (W_0 + U + W_1)^2) \]

- The noises \( W_0 \) and \( W_1 \) are two independent random variables, each with a uniform probability distribution over \([-1, 1]\) (zero mean, variance 1/3)
- The decision variable \( U \) may only use the observation of \( W_0 \) (which we view as the initial state \( X_0 \)):
  \[ U \preceq W_0 \]

- The final state \( X_1 \) is equal to \( W_0 + U + W_1 \)
- The goal is to minimize \( \mathbb{E}(\varepsilon U^2 + X_1^2) \), where \( \varepsilon \) is a given “small” positive number (“cheap control”)
We simplify the expression of the expected cost

We have that

\[
\mathbb{E}(\varepsilon U^2 + (W_0 + U + W_1)^2) =
\mathbb{E}\left(\frac{W_0^2}{1/3} + \frac{W_1^2}{1/3} + (1 + \varepsilon)U^2 + 2UW_0 + 2UW_1 + 2W_0W_1 \right)
\]

- The last two terms in the right-hand side yield zero in expectation since \(W_0\) and \(W_1\) are centered independent random variables and since \(U\) is measurable with respect to \(W_0\).
- The first two terms yield twice the variance 1/3 of the noises.

Therefore, we remain with the problem of minimizing

\[
\frac{2}{3} + \mathbb{E}\left((1 + \varepsilon)U^2 + 2UW_0 \right)
\]

by choosing \(U\) as a measurable function of \(W_0\).
We find the optimal control and the exact value of the optimal cost, namely 1/3

• The problem is minimizing

\[ \frac{2}{3} + \mathbb{E}((1 + \varepsilon)U^2 + 2UW_0) \]

by choosing \( U \) as a measurable function of \( W_0 \)

• One can prove that the optimal solution is

\[ U^\# = -\frac{W_0}{1 + \varepsilon} \]

• The corresponding optimal cost is readily calculated to be

\[ J^\# = \frac{1}{3} \times \frac{1 + 2\varepsilon}{1 + \varepsilon} \approx \frac{1}{3} \]
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4. Conclusion
We now proceed to some discretization based upon Monte Carlo simulations.

To that purpose, we first consider \( N \) Monte Carlo samplings \((w_0^i, w_1^i)\), \( i = 1, \ldots, N \), of the noises which are \( N \) sample realizations of a two-dimensional vector \((W_0, W_1)\) with uniform probability distribution over \([-1, 1]^2\).

Those samples will serve to approximate the cost expectation by a usual Monte Carlo averaging.
We must translate the measurability constraint on the decision in the discretized setting.

- We consider $N$ corresponding realizations $\{u^i\}_{i=1,\ldots,N}$ of the random decision variable $U$.

- But we must keep in mind that the random variable $U$ should be measurable with respect to the first component $W_0$ of the previous vector:

$$U \preceq W_0$$

- To that purpose, we impose the constraint

$$\forall i, j, \quad u^i = u^j \quad \text{whenever} \quad w_0^i = w_0^j$$

which prevents $U$ from taking different values whenever $W_0$ assumes the same value in any two sample trajectories.
In practice, the measurability constraint is not effective and the optimization problems splits into $N$ problems

- For each sample $i$, the cost term $\mathbb{E}(\varepsilon U^2 + (W_0 + U + W_1)^2)$ is
  $$\varepsilon (u^i)^2 + (w_0^i + u^i + w_1^i)^2 = (\varepsilon + 1)(u^i)^2 + 2(w_0^i + w_1^i)u^i + (w_0^i + w_1^i)^2$$

- This expression must be minimized in $u^i$ for every $i = 1, \ldots, N$, under the constraint
  $$\forall i, j, \quad u^i = u^j \quad \text{whenever} \quad w_0^i = w_0^j$$

- However, since the $N$ sample trajectories are produced by a Monte Carlo sampling with the uniform probability distribution over the square $[-1, 1]^2$, then, with probability 1, $w_0^i \neq w_0^j$ for any couple $(i, j)$ with $i \neq j$

- Therefore, the above constraint will practically not be effective, that is, the above cost expression can be minimized for each value of $i$ independently
Something strange is happening:
we obtain a cost far below (!) the exact minimal cost

- This yields the optimal value

\[ u^i = -\frac{w^i_0 + w^i_1}{1 + \varepsilon} \]

and the evaluation of the cost \( \varepsilon (w^i_0 + w^i_1)^2 / (1 + \varepsilon) \)

- This is of order \( \varepsilon \), and so is the average over \( N \) samples

\[ \frac{1}{N(1 + \varepsilon)} \sum_{i=1}^{N} \varepsilon (w^i_0 + w^i_1)^2 \approx \frac{2}{3} \varepsilon \]

- This is far below (!) the exact optimal cost

\[ 0 + O(\varepsilon) < J^\# = \frac{1}{3} + O(\varepsilon) \]

\[ \text{optimal cost} \]
What is the real value of this “naive Monte Carlo solution”?

- However, any admissible solution (any $U$ such that $U \preceq W_0$) cannot achieve a cost $\approx \varepsilon$ which is better than the optimal cost $J^\# \approx 1/3$
- The value of order $\varepsilon$ is just a “fake” cost estimation, because we have not produced an admissible solution
- The resolution of the discretized problem derived from the previous Monte Carlo procedure yielded an optimal value $u^i = -\frac{w^i_0 + w^i_1}{1 + \varepsilon}$ associated with each sample noise trajectory represented by a point $(w^i_0, w^i_1)$ in the square $[-1, 1]^2$
- Before trying to evaluate the cost associated with this “solution”, we must first derive from it an admissible solution for the original problem
Before trying to evaluate a cost, we must derive an admissible solution

- Before trying to evaluate the cost associated with this “solution”
- we must first derive from it an admissible solution for the original problem, that is, a random variable $U$ over $\Omega = [-1, 1]^2$
- but with constant value along every vertical line of this square (since the abscissa represents the first component $W_0$ of the 2-dimensional noise $(W_0, W_1)$)
We can translate the measurability constraint by constructing an admissible control constant on vertical stripes

- We first renumber the $N$ sample points so that the first component $w_0^i$ is increasing with $i$.
- Then, we divide the square into $N$ vertical strips by drawing vertical lines in the middle of segments $[w_0^i, w_0^{i+1}]$.
- That is, the $i$-th strip is $[a^{i-1}, a^i] \times [-1, 1]$ with $a^i = (w_0^i + w_0^{i+1})/2$ for $i = 2, \ldots, N - 1$, $a^0 = -1$, and $a^N = 1$. 

We first renumber the $N$ sample points so that the first component $w_0^i$ is increasing with $i$.
We construct an admissible solution constant on vertical stripes

- We define the solution $U$ as the function of $(w_0, w_1)$ which is piecewise constant over the square divided into those $N$ strips, using of course the optimal value $u^i = -\frac{w_0^i + w_1^i}{1 + \varepsilon}$ in strip $i$
- That is, we consider the random variable

$$U(w_0, w_1) = \sum_{i=1}^{N} u^i 1_{[a_i-1, a_i] \times [-1, 1]}(w_0, w_1)$$

where $(w_0, w_1)$ ranges in the square $[-1, 1]^2$ and $1_A(\cdot)$ is the function which takes the value 1 in $A$ and 0 elsewhere
We evaluate the expected cost attached to this admissible control

- Since \( U = \sum_{i=1}^{N} u^i 1_{[a_i^{-1}, a_i] \times [-1, 1]} \) is an admissible solution for the original (continuous) problem, the corresponding cost value \( E(\epsilon U^2 + (W_0 + U + W_1)^2) \) can be evaluated.
- Here, the expectation is over the argument \((w_0, w_1)\) considered as a random variable over the square with uniform distribution (the calculation of this expectation is done analytically).
- This expected cost is easily evaluated as

\[
E(\epsilon U^2 + (W_0 + U + W_1)^2) = \frac{2}{3} + \sum_{i=1}^{N} \left( (1 + \epsilon)(u^i)^2 \frac{a_i - a_i^{-1}}{2} + u^i \frac{(a_i)^2 - (a_i^{-1})^2}{2} \right)
\]
The expected cost depends on the Monte Carlo samplings

Recall that

\[ u^i = -\frac{w^i_0 + w^i_1}{1 + \varepsilon} \quad \text{for} \quad i = 1, \ldots, N, \]
\[ a^i = \frac{(w^i_0 + w^i_1)}{2} \quad \text{for} \quad i = 2, \ldots, N - 1, \quad a^0 = -1, \quad \text{and} \quad a^N = 1 \]

are functions of the Monte Carlo samplings \((w^i_0, w^i_1), i = 1, \ldots, N,\)

Therefore, the expected cost

\[
\frac{2}{3} + \sum_{i=1}^{N} \left( (1 + \varepsilon)(u^i)^2 \frac{a^i - a^{i-1}}{2} + u^i \frac{(a^i)^2 - (a^{i-1})^2}{2} \right)
\]

depends on the Monte Carlo samplings
The true cost provided by the naive Monte Carlo method is not easy to calculate analytically.

- In order to assess the value of this estimate, we must compute the expectation when considering that the $w_0^i$'s are realizations of $N$ independent Monte Carlo samples $W_0^i$, each uniformly distributed over $[-1, 1]$
- This calculation is not straightforward because the expression of the $a^i$'s as functions of the $w_0^i$'s is meaningful as long as the $w_0^i$'s have been reordered into an increasing sequence
- Therefore, although those $N$ random numbers are the result of independent drawings, the calculation of expectations is made somewhat tricky by this reordering
The true cost provided by the naive Monte Carlo method is numerically estimated at $2/3$.

**Figure:** Cost provided by the naive Monte Carlo method as a function of the number $N$ of samples.
The solution we have produced with our naive Monte-Carlo approach is not better than the open-loop solution!

- Remember that the true optimal cost is close to 1/3!
- Moreover, it is readily checked that the optimal open-loop solution, that is, the optimal \( U \) which is measurable w.r.t. the trivial \( \sigma \)-algebra \( \{\emptyset, \Omega\} \), is equal to 0 and that the corresponding cost is also 2/3

\[
\min_{U \leq \{\emptyset, \Omega\}} \frac{2}{3} + \mathbb{E}\left((1 + \varepsilon)U^2 + 2UW_0\right) = \min_{u \in \mathbb{R}} \left(\frac{2}{3} + (1 + \varepsilon)u^2 + 2u\mathbb{E}(W_0)\right) = \frac{2}{3}
\]

Hence the solution we have produced with our naive Monte-Carlo approach (and especially the naive way of handling the information structure of the problem) is not better than the open-loop solution!
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4. Conclusion
Making the information constraint active: the scenario trees approach

We consider scenarios

\[ \{(w_0^j, w_1^{jk})\}_{j=1, \ldots, N_0}^{k=1, \ldots, N_1} \]

Notice that the first noise discretization \( w_0^j \) only depends on \( j = 1, \ldots, N_0 \)

Whereas the second noise discretization \( w_1^{jk} \) “hangs” from \( j = 1, \ldots, N_0 \), and then depends on \( k = 1, \ldots, N_1 \)
We consider $N_0 \times N_1$ scenarios $\{(w_0^j, w_1^{jk})\}_{j=1,\ldots,N_0}^{k=1,\ldots,N_1}$

- To fix notations, we consider scenarios $\{(w_0^j, w_1^{jk})\}_{j=1,\ldots,N_0}^{k=1,\ldots,N_1}$
- We introduce the following additional symbol

$$
\bar{w}_1^j = \frac{1}{N_1} \sum_{k=1}^{N_1} w_1^{jk}
$$

which can be interpreted as an estimate of the conditional expectation of $W_1$ knowing that $W_0 = w_0^j$

- Likewise, the symbol

$$
(\bar{\sigma}_1^j)^2 = \frac{1}{N_1} \sum_{k=1}^{N_1} (w_1^{jk})^2
$$

can be interpreted as an estimate of the conditional second order moment
The original cost is $\mathbb{E}\left(\varepsilon U^2 + (W_0 + U + W_1)^2\right)$

The cost of the discretized problem is

$$\frac{1}{N_0} \sum_{j=1}^{N_0} \left(\varepsilon(u^j)^2 + \frac{1}{N_1} \sum_{k=1}^{N_1} (u^j + w_0^j + w_1^j)^2\right)$$

The minimizer is

$$u^j = -\frac{w_0^j + \overline{w}_1}{1 + \varepsilon} \quad \text{where} \quad \overline{w}_1 = \frac{1}{N_1} \sum_{k=1}^{N_1} w_{1k}^j$$

to be compared with the naive Monte Carlo method minimizer

$$u^j = -\frac{w_0^j + w_1^j}{1 + \varepsilon}$$
From the scenario tree approach minimizer
From the scenario tree approach minimizer we build a random variable on the square $[-1, 1]^2$
The scenario tree approach optimal cost gets close to 1/3

This yields the optimal cost

$$\frac{1}{N_0(1 + \varepsilon)} \sum_{j=1}^{N_0} \left( \varepsilon(w_j^0)^2 + 2\varepsilon w_j^0 w_j^1 - (w_j^1)^2 + (1 + \varepsilon)(\sigma_j^1)^2 \right)$$

If we assume that the estimates $w_j^1$ and $(\sigma_j^1)^2$ converge towards their right values (respectively, 0 and 1/3) as $N_1$ goes to infinity, then the scenario tree approach optimal cost gets close to

$$\frac{1}{N_0(1 + \varepsilon)} \sum_{j=1}^{N_0} \left( \varepsilon(w_j^0)^2 + \frac{1 + \varepsilon}{3} \right) \approx \frac{1}{3}$$

to be compared with

- the naive Monte Carlo method optimal cost $\varepsilon$
- the exact optimal cost $J^\# \approx 1/3$
The limit as $N$ goes to infinity seems now to be the correct value of the optimal cost, namely $1/3$.

**Figure:** Cost provided by the use of a stochastic tree as a function of the number $N_0$ of pieces of the piecewise constant $U(\cdot)$ ($N_0^2$ scenarios)
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4. Conclusion
Where do we stand?

- **The naive Monte Carlo method**
  - discretizes the noise
  - subordinates *naively* the discretization of the measurability constraint to the noise discretization
  - yields an “optimal” cost not better than the open-loop solution!

- **The scenario tree approach**
  - discretizes jointly the noise and the measurability constraint
  - yields the optimal cost
Quantization = encoding + decoding
In a scenario tree discretization, groups of samples are aligned vertically.
However, we can construct another quantization from the same sample set, like the Voronoi tessellation.

The Voronoi tessellation minimizes the mean quadratic error among finite random variables having prescribed values.
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4. Conclusion
An alternative approach consists in independent discretizations of noise and information.
An alternative approach consists in independent discretizations of noise and information.
An alternative approach consists in independent discretizations of noise and information.
This alternative approach does not necessarily lead to a tree structure
Formulation of the discretized optimization problem

- Denoting
  \[ j(u, w_0, w_1) = \varepsilon u^2 + (w_0 + u + w_1)^2 \]
- The discretized optimization problem is
  \[
  \min_{\{u^k\}} \sum_{k \in \{a, \ldots, e\}} \sum_{i=1}^{8} p^{ik} j(u^k, w^i_0, w^i_1)
  \]
- in which \( p^{ik} \) is the probability weight of the cell \( ik \)
  \((i = 1, \ldots, 8 \text{ and } k \in \{a, \ldots, e\}) \)
A general convergence result

\[ V(W, \mathcal{F}) := \min_{U \in L^2(\Omega, \mathcal{A}, P; U)} \mathbb{E}[j(U, W)] \]

subject to \( U \) is \( \mathcal{F} \)-measurable

Theorem

Let \( 1 \leq q < +\infty \). Under the assumptions

- \( \mathcal{H}_1 \) the sequence \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \) strongly converges to \( \mathcal{F} \), and \( \mathcal{F}_n \subset \mathcal{F} \)
- \( \mathcal{H}_2 \) the sequence \( \{W_n\}_{n \in \mathbb{N}} \) converges in norm to \( W \) in \( L^q(\Omega, \mathcal{A}, P; W) \)
- \( \mathcal{H}_3 \) the normal integrand \( j \) is such that

\[
\forall (u, u') \in U^2, \forall (w, w') \in W^2, \quad |j(u, w) - j(u', w')| \leq \alpha \|u - u'\|_U^r + \beta \|w - w'\|_W^q
\]

the convergence of the approximated optimal costs holds true

\[
\lim_{n \to +\infty} V(W_n, \mathcal{F}_n) = V(W, \mathcal{F})
\]

Conditions for epi-convergence ensure convergence of the optimal solutions
Summary

- In the discretization of a SOC problem, there are two issues
  - noise discretization,
  - information discretization

- Independent discretization of noise and information offers
  - greater latitude (than joint discretization) to select discretization schemes
  - proper convergence results

- The naive Monte Carlo discretization scheme provides a too weak noise convergence (in distribution, but not in probability)
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4. Conclusion
In the way from deterministic to stochastic control problems, there are two issues:
- Noise
- Information

which have both theoretical and numerical importance

**Theoretical issues**
- The classical approaches to SOC — namely, Stochastic Programming and Dynamic Programming — rely upon the so-called classical information pattern: perfect memory of past noises.
- Changing information pattern can turn an easy problem (LQG with classical pattern) into an open problem (Witsenhausen counterexample).

**Numerical issues**
- Naive joint discretizations of noise and information can lead to puzzles.
- Independent discretization of noise and information offers:
  - Greater latitude (than joint discretization) to select discretization schemes.
  - Proper convergence results.
More on information patterns: the dual effect in Witsenhausen counterexample

\[
\begin{align*}
\min_{U'_0 \preceq Y_0, U'_1 \preceq Y_1} & E \left( k^2 U'^2_0 + X_2^2 \right) \\
= & \min_{U'_0 \preceq Y_0} E \left( k^2 (X_0 - U'_0)^2 + \text{Var}(W_1 \mid U'_0 + W_1) \right)
\end{align*}
\]

\[
\left\{
\begin{array}{l}
a_{U'_0}(y_0, y_1) = \exp \left( - (y_1^2 + U'_0(y_0)^2 - 2y_1 U'_0(y_0))/2 \right) \\
b_{U'_0}(y_0, y_1) = \frac{a_{U'_0}(y_0, y_1)}{\int_{-\infty}^{+\infty} a_{U'_0}(y'_0, y_1) \, dy'_0} \\
c_{U'_0}(y_1) = \int_{-\infty}^{+\infty} U'_0(y'_0)^2 b_{U'_0}(y'_0, y_1) \, dy'_0 - \left( \int_{-\infty}^{+\infty} U'_0(y'_0) b_{U'_0}(y'_0, y_1) \, dy'_0 \right)^2
\end{array}\right.
\]

\[
\text{Var} \left( W_1 \mid U'_0 + W_1 \right)(\omega) = c_{U'_0}(U'_0(\omega) + W_1(\omega)), \quad \forall \omega \in \Omega
\]
In this lecture, we
- discretized a SOC problem
- then optimized the discretized version

Whereas, in the next lecture given by Pierre Carpentier, he will
- display optimality conditions for a SOC problem
- then discretize those conditions

To display first-order optimality conditions, the classical information pattern
with perfect memory of past noises will be assumed

to avoid having to differentiate with respect to a control
in a conditioning term