Equilibrium and Stability

Extended from Chapter 3 of
Sustainable Management of Natural Resources.
Mathematical Models and Methods
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Equilibrium and stability analysis provides insight into factors driving world’s fisheries non-sustainability

Towards sustainability in world fisheries

- Open-access nature of many fisheries
- Common-pool fisheries that are managed non-cooperatively
- Sole-ownership fisheries with high discount rates and/or high price-to-cost ratios ones
- Payment of subsidies by governments to fishers, which generate ‘profits’ even when resources are overfished
Outline of the presentation

1. Equilibrium
2. Sustainable yield and related notions
3. Stability
4. Summary
Outline of the presentation

1 Equilibrium
   • Definition of equilibrium (under constraint)
   • Examples of equilibria

2 Sustainable yield and related notions
   • Sustainable yield for surplus model
   • Maximum sustainable equilibrium
   • Private property equilibrium
   • Common property equilibrium
   • Examples

3 Stability
   • Definition of stability of an equilibrium state
   • Stability for dynamical linear systems
   • Linearized dynamics and stability
   • Examples

4 Summary
Equilibrium is the mathematical concept carrying the idea of sustainability as balance and stationarity.
We consider autonomous dynamical systems, that is, without explicit dependence upon time $t$

- Recall that
  - $x(t) \in \mathbb{X} = \mathbb{R}^n$ represents the state of the system
  - $u(t) \in \mathbb{U} = \mathbb{R}^p$ stands for the control

- In an autonomous dynamical system, the dynamics does not depend directly on time $t$

$$x(t + 1) = \text{Dyn}(x(t), u(t))$$

- Constraints are also time independent

$$\begin{cases} 
  x(t) \in \mathbb{A} \\
  u(t) \in \mathbb{B}(x(t))
\end{cases}$$
Example of non-autonomous system

Monthly plant model

\[ B(t + 1) = R(t)B(t) \]

where the monthly growth rate is periodic, following the seasons.
An equilibrium is a steady state of an autonomous dynamical system

**Equilibrium under constraints**

The state $x_E \in X$ is an equilibrium under constraints if there exists a control $u_E \in U$ satisfying

$$\text{Dyn}(x_E, u_E) = x_E \quad \text{steady state}$$

$x_E \in A \quad \text{admissible state}$

$u_E \in B(x_E) \quad \text{admissible control}$
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4. Summary
Equilibria are trivial for an exhaustible resource

- **Dynamics**
  \[ S(t + 1) = S(t) - h(t) \]
  
  Where:
  - \( S(t + 1) \) is the future stock
  - \( S(t) \) is the current stock
  - \( h(t) \) is the extraction

- The resource stock \( S_E \) and harvesting \( h_E \) are stationary whenever
  \[ S_E = S_E - h_E \]

- The only equilibrium for any \( S_E \) is
  \[ h_E = 0 \]
Flooding in Cuzco Valley, Peru
Mitigation policies for carbon dioxide emissions

- Assuming stationary emissions $E_b$

\[ M(t + 1) = M(t) - \delta(M(t) - M_{-\infty}) + \alpha E_b (1 - a(t)) \]

natural sinks

abatement

- Any equilibrium $(M_E, a_E)$ satisfies

\[ M_E = M_{-\infty} + \frac{\alpha E_b (1 - a_E)}{\delta} \]

that is,

\[ M_E = M_{-\infty} + \frac{\alpha E_b (1 - a_E)}{\delta} \quad \text{with} \quad 0 \leq a_E \leq 1 \]
Populations may be described by abundances at ages

Jack Mackrel abundances (Chilean data) in thousand of individuals

13651022 thousand of age < 1 (recruits)
7495888 thousand of age ∈ [1, 2[
6804151
4191318
4582943
2500338
1139182
523261
269328
166390
95606 thousand of age ≥ 11
Harvested population age-class dynamics

The spawning stock biomass is $SSB(N) = \sum_{a=1}^{A} \gamma_a \mu_a N_a\gamma_a$.

- Spawning biomass: $N_1(t+1) = \frac{S}{R} \left( \frac{SSB(N(t))}{SSB(N(t))} \right)$ recruitment
- $N_2(t+1) = e^{-(M_1 + \lambda(t)F_1)}N_1(t)$
- $N_a(t+1) = e^{-(M_{a-1} + \lambda(t)F_{a-1})}N_{a-1}(t)$ for $a = 2, \ldots, A - 1$
- $N_A(t+1) = e^{-(M_{A-1} + \lambda(t)F_{A-1})}N_{A-1}(t) + \pi e^{-(M_A + \lambda(t)F_A)}N_A(t)$ plus group
An $A$ age-classes equilibrium is solution of an algebraic system of $A$ equations

\[
\begin{pmatrix}
S/R(\text{SSB}(N)) \\
N_1 \exp (- (M_1 + \lambda F_1)) \\
N_2 \exp (- (M_2 + \lambda F_2)) \\
\vdots \\
N_{A-2} \exp (- (M_{A-2} + \lambda F_{A-2})) \\
N_{A-1} \exp (- (M_{A-1} + \lambda F_{A-1}))
\end{pmatrix}
= 
\begin{pmatrix}
N_1 \\
N_2 \\
\vdots \\
N_{A-1} \\
N_A
\end{pmatrix}
\]
$A - 1$ equations display a chained structure

The computation of an equilibrium $N_E(\lambda)$, for $\lambda \geq 0$, gives

$$N_{a,E}(\lambda) = s_a(\lambda) N_{1,E}(\lambda)$$

for $a = 1, \ldots, A$, where

$$s_a(\lambda) = \exp\left(- (M_1 + \cdots + M_{a-1} + \lambda (F_1 + \cdots + F_{a-1}))\right)$$

is the proportion of equilibrium recruits which survive up to age $a$ ($a = 2, \ldots, A$) while $s_1(\lambda) = 1$
Exercise: compute the equilibrium when the stock-recruitment is a Beverton-Holt function

When the stock-recruitment is a Beverton-Holt function

\[ \frac{S}{R}(B) = \frac{B}{\alpha + \beta B} \]

the equilibrium recruits are either \( N_{1,E}(\lambda) = 0 \) or

\[ N_{1,E}(\lambda) = \frac{\text{spr}(\lambda) - \alpha}{\beta \text{spr}(\lambda)} \quad \text{when} \quad \text{spr}(\lambda) > \alpha \]

where \( \text{spr} \) is the equilibrium spawners per recruits

\[ \text{spr}(\lambda) = \sum_{a=1}^{A} \gamma_a \mu_a s_a(\lambda) \]

Exercice: case \( A = 2 \) and interpret the condition \( \frac{\text{spr}(\lambda)}{\alpha} > 1 \)
Summary

- An equilibrium is solution of an algebraic system of equations
- The number of equations is equal to the dimension of the state
- For control systems, equilibria are parameterized by the controls
- Constraints can restrict the number of admissible equilibria
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   - Maximum sustainable equilibrium
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   - Common property equilibrium
   - Examples

3. Stability
   - Definition of stability of an equilibrium state
   - Stability for dynamical linear systems
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4. Summary
The sustainable yield for the Schaefer model is the harvest attached to a biomass equilibrium

- The Schaefer model builds upon a biological dynamics

\[ B(t + 1) = \text{Biol}(B(t) - h(t)) \text{ with } 0 \leq h(t) \leq B(t) \]

- At equilibrium, we have that

\[ B_E = \text{Biol}(B_E - h_E) \text{ and } 0 \leq h_E \leq B_E \]

- The stationary harvest \( h_E \) solution to that equilibrium equation is the so-called sustainable yield

\[ h_E = \text{Sust}(B_E) \]
The sustainable yield is a surplus production that can be harvested in perpetuity without altering the stock level.

The relation $\text{Biol}(B_E - h_E) = B_E$ may also be written as

$$h_E = \text{Biol}(B_E - h_E) - (B_E - h_E) \geq 0$$

meaning that

* a surplus production exists that can be harvested in perpetuity without altering the stock level

(Clark, 1990)
Sustainable yield curve for the blue whale

\[ K = 400\,000 \text{ BWU} \]
\[ R \approx 1.05 \]

Beverton-Holt
Exercise: compute the equilibrium when the biological dynamics is a Beverton-Holt function

- Biological Beverton-Holt dynamics $\text{Biol}(B) = \frac{RB}{1+bB}$
- Any equilibrium $(B_E, h_E)$ satisfies
  $$B_E = \frac{R(B_E - h_E)}{1 + b(B_E - h_E)} \quad \text{with} \quad 0 \leq h_E \leq B_E$$

- Particular equilibria are
  - $(B_E, h_E) = (0, 0)$
  - $(B_E, h_E) = (K, 0)$ where $K = \frac{R-1}{b}$ is the carrying capacity

- The sustainable yield is
  $$\text{Sust}(B) = B - \frac{B}{R - bB} \quad \text{for} \quad 0 \leq B \leq K$$
Three equilibria are worth being distinguished: MSE, PPE, CPE
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4. Summary
The sustainable yield has a hump-shape form as a function of biomass.
The maximum sustainable yield is the maximum surplus production that can be harvested in perpetuity without altering the stock level.

- The maximum sustainable equilibrium (MSE) is the biomass $B_{\text{MSE}}$ solution of

$$\text{Sust}(B_{\text{MSE}}) = h_{\text{MSE}} = \max_{B \geq 0, \ h = \text{Sust}(B)} h = \max_{B \geq 0} \text{Sust}(B)$$

- The maximum catch $h_{\text{MSE}}$ is called the maximum sustainable yield (MSY)
Sustainable yield and related notions

Maximum sustainable equilibrium

MSY is criticized for its simplistic assumptions

- **MSY was developed in the early 1930s, and adopted in the 1950s by several international organizations and individual countries**

- **Larkin PA (1977) An epitaph for the concept of maximum sustained yield outlines deficiencies of MSY**
  - monospecific dynamical model (no trophic relationships)
  - scalar dynamical model
  - no spatial variability
  - no age structure
  - no reproductive status
  - steady state approach
  - only benefits, no costs
Though being accused of having led to the collapse of many fisheries, the MSY remains the reference till now.

- Accused of having led to the collapse of many fisheries
- Following the World Summit on Sustainable Development (Johannesburg, 2002), the signatory States undertook to restore and exploit their stocks at MSY
- About a quarter of world fisheries are overexploited in the sense that stocks are less than the MSE, the stock size that supports MSY.
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We introduce the rent or profit, which depends on price and costs.

The economic rent or profit $\text{Rent}(h, B)$ is

$$\text{Rent}(h, B) = \underbrace{p}_{\text{price}} \underbrace{h}_{\text{harvest}} - \underbrace{\text{Cost}(h, B)}_{\text{harvest costs}}$$

where

- $p$ price per unit of harvested biomass ("price-taker")
- $\text{Cost}(h, B)$ harvesting costs
The private property equilibrium (PPE) maximizes the profit

- Among all sustainable equilibria on the curve $h = \text{Sust}(B)$
- the private property equilibrium (PPE) is the one achieving the maximal rent

$$\text{Rent}(h_{\text{PPE}}, B_{\text{PPE}}) = \max_{B \geq 0, h = \text{Sust}(B)} \text{Rent}(h, B)$$
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The common property equilibrium (CPE) makes the profit zero

- The assumption is that, when a resource is in open access, any positive rent is dissipated as newcomers exhaust it.
- Among all sustainable equilibria on the curve $h = \text{Sust}(B)$
- the common property equilibrium (CPE) is the one that makes the rent zero

$$\text{Rent}(h_{CPE}, B_{CPE}) = 0 \text{ and } h_{CPE} = \text{Sust}(B_{CPE})$$
Let us consider the case where the cost is linear in the effort

- Catches are supposed to be linear in an effort variable

\[ \text{catch} = q \cdot E \cdot \text{effort} \]

- Assume harvest costs of the form

\[ \text{Cost}(h, B) = \frac{ch}{qB} = c \cdot \frac{h}{qB} \]

  - \( q \) catchability coefficient
  - \( c \) unit cost of effort

\[ \text{Rent}(h, B) = ph - \text{Cost}(h, B) = \left( p - \frac{c}{qB} \right) h \]
We derive the common property equilibrium (CPE) in the linear effort costs case

\[
\text{Rent}(h, B) = \left( p - \frac{c}{qB} \right) h
\]

\[
\text{Rent}(h_{\text{CPE}}, B_{\text{CPE}}) = 0 \quad \text{and} \quad h_{\text{CPE}} = \text{Sust}(B_{\text{CPE}})
\]

\[
\Downarrow
\]

\[
B_{\text{CPE}} = \frac{c}{pq} \quad \text{and} \quad h_{\text{CPE}} = \text{Sust}(B_{\text{CPE}})
\]

leaving aside the trivial equilibrium \((B, h) = (0, 0)\)
High price-to-cost ratio drives biomass to collapse

\[ B_{\text{CPE}} = \frac{c}{pq} = \frac{1/q}{p/c} \propto \frac{1}{\text{price / cost}} \]

- The common property equilibrium \( B_{\text{CPE}} \) does not depend upon the biological dynamics \( \text{Biol} \)
- Conservation problem if high price-to-cost ratio:
  cost \( c \downarrow \) (sonar, satellite) or price \( p \uparrow \) (scarcity) \( \Rightarrow B_{\text{CPE}} \downarrow 0 \)
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Exercise: compute MSE and PPE when the biological dynamics is a Beverton-Holt function

Consider the biological Beverton-Holt dynamics $Biol(B) = \frac{RB}{1+bB}$

The sustainable yield is

$$Sust(B) = B - \frac{B}{R - bB} \quad \text{for} \quad 0 \leq B \leq K = \frac{R - 1}{b}$$

The $\text{MSE}$ achieves the maximum of $Sust$, that is,

$$\left. \frac{d Sust(B)}{dB} \right|_{B=B_{MSE}} = 0 \iff 1 - \frac{R}{(R - bB)^2} = 0$$

The $\text{MSE}$ biomass is

$$B_{MSE} = \frac{R - \sqrt{R}}{b}$$
Here stands the Beverton-Holt PPE

- The sustainable yield is

\[ \text{Sust}(B) = B - \frac{B}{R - bB} \quad \text{for } 0 \leq B \leq K = \frac{R - 1}{b} \]

- The PPE achieves the maximum of the rent, that is,

\[ \frac{d}{dB} \bigg|_{B=B_{\text{PPE}}} \left( (p - \frac{c}{qB}) \text{Sust}(B) \right) = 0 \]

- Show that the sign of the above derivative is positive at \( B = B_{\text{MSE}} \)

- The PPE is

\[ B_{\text{PPE}} = \frac{R - \sqrt{R - \frac{b}{q} \frac{p}{c}}}{b} \]
There is more biomass at equilibrium under private property equilibrium than under maximum sustainable equilibrium

\[ B_{PPE} = \frac{R - \sqrt{R - \frac{b}{p} q / c}}{b} > B_{MSE} = \frac{R - \sqrt{R}}{b} \]

- \( B_{PPE} > B_{MSE} \implies \) more biomass at equilibrium
  - under private property equilibrium
  - than under maximum sustainable equilibrium

To the first-order, we have that

\[ B_{PPE} - B_{MSE} \propto \frac{1}{p} \frac{c}{c} = \frac{c}{p} = \text{cost} / \text{price} \]
Sustainable yield and related notions

Examples

MSE, PPE, CPE for blue whale

\[ K = 400,000 \text{ BWU} \]
\[ q = 0.0016 \text{ per whale catcher year} \]
\[ c = 600,000 \text{ $ per whale catcher year} \]
\[ R \approx 1.05 \]
\[ p = 7,000 \text{ $ per BWU} \]
The Aboré marine reserve dilapidation illustrates the “Tragedy of the Commons” / open access

- In August 1993, in New Caledonia the Aboré marine reserve has been re-opened to fishing
- The reserve had been closed during three years
- The benefits of these three years were dissipated in a few weeks
- Resources in open access suffer from over-exploitation
Open access dynamics displays CPE equilibrium

We consider the open access dynamics

\[
\begin{align*}
B(t+1) &= \text{Biol}(B(t) - \text{Catch}(E(t), B(t))) \\
E(t+1) &= E(t) + \alpha \text{Rent}(\text{Catch}(E(t), B(t)), E(t))
\end{align*}
\]

Peru’s fishing overcapacity highlights the point that this model assumes a too flexible effort adaptation

Gordon’s bionomic equilibrium CPE is \( B_{\text{CPE}} = \frac{c}{pq} \) when

- catches \( \text{Catch}(E, B) = qEB \)
- rent \( \text{Rent}(h, E) = ph - cE \)
The maximum sustainable yield (MSY) is the maximum surplus production that can be harvested in perpetuity without altering the stock level.

Though being accused of having led to the collapse of many fisheries, the MSY remains the reference till now.

The private property equilibrium (PPE) maximizes the profit.

The common property equilibrium (CPE) makes the profit zero.

Open access dynamics displays CPE equilibrium.
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Open loop stability of an equilibrium

The dynamics around an equilibrium $x_E$ with the fixed equilibrium decision $u_E$ is

$$
\begin{cases}
  x(t+1) = \text{Dyn}(x(t), u_E) \\
  x(t_0) = x_0 \approx x_E
\end{cases}
$$

Attractive equilibrium

Equilibrium $x_E$ is \textbf{attractive} if there exists a neighborhood $\mathcal{N}(x_E)$ of $x_E$ such that

$$
\lim_{t \to +\infty} x(t) = x_E, \quad \forall x_0 \in \mathcal{N}(x_E)
$$

The property that an equilibrium is attractive does not depend of the transitory phase $x(t_0), x(t_0 + 1), \ldots, x(T)$
The concept of asymptotic stability combines transitories and asymptotics properties

Asymptotically stable equilibrium

The equilibrium $x_E$ is said to be **asymptotically stable** when any trajectory $x(t)$, $t = t_0, t_0 + 1, \ldots$, starting close enough to $x_E$ ($x(t_0) \approx x_E$):

- remains in the vicinity of $x_E$ ($x(t) \approx x_E$, $t = t_0, t_0 + 1, \ldots$): **stability concerns transitories**
- and converges towards $x_E$ ($x(t) \rightarrow x_E$): **attractivity only focuses on asymptotics**
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As a starter, we highlight stability of the zero equilibrium for scalar linear systems

- Scalar $x(t) \in \mathbb{R}$
- A dynamical linear scalar system has the form
  \[ x(t + 1) = Mx(t) \quad \text{where} \quad M \in \mathbb{R} \]

- The solution has the expression $x(t) = M^{t-t_0} x(t_0)$
  - $|M| > 1$
  - $|M| < 1$
  - $|M| = 1$

Proposition

Equilibrium $x_E = 0$ is asymptotically stable if and only if $|M| < 1$
P. H. Leslie introduced mortality-natality matrix models in forestry

Leslie, P.H. (1945)
"The use of matrices in certain population mathematics"
Biometrika, 33(3), 183–212
The stability of the nul equilibrium of a linear system is related to the location of eigenvalues with respect to the unit disk.

- Vector $x(t) \in \mathbb{R}^n$ and dynamical linear system
  $$x(t + 1) = M x(t)$$
  square matrix

- Eigenvalues $\lambda_i(M)$ of the square matrix $M$

Theorem

Equilibrium $x_E = 0$ is asymptotically stable if and only if all eigenvalues have modulus strictly less than unity, that is,

$$| \lambda_i(M) | < 1, \quad \forall i$$

eigenvalue
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The dynamics of the increments is (almost) linear

\[
x(t + 1) = \text{Dyn}(x(t), u_E)
\]

\[
x_E = \text{Dyn}(x_E, u_E)
\]

\[
\begin{align*}
x(t + 1) - x_E &= \text{Dyn}(x_E + (t) - x_E, u_E) - \text{Dyn}(x_E, u_E) \\
\text{increment} &= \frac{\partial \text{Dyn}}{\partial x}(x_E, u_E)(x(t) - x_E) + \cdots
\end{align*}
\]
The Jacobian matrix provides the linear part of a nonlinear dynamics

\[
\frac{\partial \text{Dyn}}{\partial x}(x_E, u_E) = \begin{pmatrix}
\frac{\partial \text{Dyn}^1}{\partial x_1}(x_E, u_E) & \frac{\partial \text{Dyn}^1}{\partial x_2}(x_E, u_E) & \cdots & \frac{\partial \text{Dyn}^1}{\partial x_n}(x_E, u_E) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \text{Dyn}^n}{\partial x_1}(x_E, u_E) & \frac{\partial \text{Dyn}^n}{\partial x_2}(x_E, u_E) & \cdots & \frac{\partial \text{Dyn}^n}{\partial x_n}(x_E, u_E)
\end{pmatrix}
\]

where \( \text{Dyn}(x, u) = \begin{pmatrix}
\text{Dyn}^1(x_1, x_2, \ldots, x_n, u) \\
\vdots \\
\text{Dyn}^n(x_1, x_2, \ldots, x_n, u)
\end{pmatrix} \)
Stability of an equilibrium can often be deduced from linearization of the dynamics at the equilibrium.

**Theorem**

Assume that the dynamics $\text{Dyn}$ is a continuously differentiable mapping on a neighborhood of the equilibrium $(x_E, u_E)$. Then,

$$\forall i, \quad |\lambda_i(\frac{\partial \text{Dyn}}{\partial x}(x_E, u_E))| < 1 \implies x_E \text{ asymptotically stable}$$

for the nonlinear dynamical system

$$x(t + 1) = \text{Dyn}(x(t), u_E)$$
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Exercise: study stability when the biological dynamics is a Beverton-Holt function

We consider the Schaefer model

\[ B(t + 1) = \frac{R(B(t) - h_E)}{1 + b(B(t) - h_E)} = \text{Dyn}(B(t), h_E) \]

built upon the Beverton-Holt biological dynamics

- Equilibrium: \( h_E = B_E - \frac{B_E}{R - bB_E} \)
- Exercise: compute \( \frac{\partial \text{Dyn}(B,h_E)}{\partial B} \bigg|_{B=B_E} = \frac{\partial}{\partial B} \bigg|_{B=B_E} \frac{R(B-h_E)}{1 + b(B-h_E)} = \left( \frac{R - bB_E}{\sqrt{R}} \right)^2 \)
- Asymptotic stability if

\[ B_{\text{MSE}} < B_E \leq K \]

- MSE at frontier between stability and unstability
- PPE is a stable equilibrium
- CPE can be unstable
Blue whale: CPE versus PPE

\[ K = 400,000 \text{ BWU} \]
\[ q = 0.0016 \text{ per whale catcher year} \]
\[ c = 600,000 \text{ $ per whale catcher year} \]

\[ R \approx 1.05 \]
\[ p = 7,000 \text{ $ per BWU} \]

CPE equilibrium: instability  
PPE equilibrium: stability

The red trajectory corresponds to the MSE
We show that, in a simple open access model, instability and extinction occur

\[
\begin{align*}
B(t + 1) &= \text{Biol}(B(t) - qE(t)B(t)) = \text{Dyn}^B(B(t), E(t)) \\
E(t + 1) &= E(t) + \alpha \left( pqE(t)B(t) - cE(t) \right) = \text{Dyn}^E(B(t), E(t))
\end{align*}
\]

The Jacobian matrix at equilibrium \((B_E, E_E) = \left( \frac{c}{\rho q}, E_E \right)\) is

\[
\begin{pmatrix}
(1 - qE_E)\text{Biol}'(B_E(1 - qE_E)) & -qB_E\text{Biol}'(B_E(1 - qE_E)) \\
\alpha pqE_E & 1
\end{pmatrix}
\]
In the linear biological dynamics case, the eigenvalues of the Jacobian matrix are outside the stability disk.

Assuming a linear biological dynamics $\text{Biol}(B) = RB$, the Jacobian matrix is

$$
\begin{pmatrix}
R(1 - qE_E) & -RqB_E \\
\alpha pqE_E & 1
\end{pmatrix}
= \begin{pmatrix}
1 & -\frac{Rc}{p} \\
\alpha p\frac{R-1}{R} & 1
\end{pmatrix}
$$

The two eigenvalues $(\lambda_1, \lambda_2)$ are

$$
\begin{cases}
\lambda_1 &= 1 - i\sqrt{\alpha c(R - 1)} \\
\lambda_2 &= 1 + i\sqrt{\alpha c(R - 1)}
\end{cases}
$$

Stability cannot be guaranteed since $|\lambda_i|^2 > 1$.
Biomass and effort trajectories display oscillations and divergence

(a) Biomass trajectory $B(t)$

(b) Effort trajectory $E(t)$
We consider a dynamical model of $n$ species in competition for the same resource

\[
N_i(t + 1) = N_i(t) + \Delta_t \left( N_i(t) (f_i R(t) - d_i) \right)
\]

\[
R(t + 1) = R(t) + \Delta_t \left( S - aR(t) - \sum_{i=1}^{n} w_i f_i R(t) N_i(t) \right)
\]

- $\Delta_t$ time unit
- $N_i(t)$ density of species $i$
- $R(t)$ resource for which all species compete
- $f_i R(t)$ growth, $d_i$ death rates of species $i$
- $S - a R(t)$ natural growth rate of the resource ($S$ a stationary input)
The competitive exclusion principle states that only one species survives

- If $\Delta_{t}$ is small enough, a positive stable equilibrium is (Tilman, 1988)

$$
\begin{align*}
R_{E} &= \min_{i=1,\ldots,n} \frac{d_{i}}{f_{i}} = \frac{d_{i_{E}}}{f_{i_{E}}} \\
N_{i,E} &= \begin{cases} 
\frac{S_{E}-R_{E}a}{R_{E}w_{i}f_{i}} & \text{if } i = i_{E} \\
0 & \text{if } i \neq i_{E}
\end{cases}
\end{align*}
$$

- Only species $i_{E}$ survives: this is the \textit{competitive exclusion principle}
Competitive exclusion principle

Densities trajectories

- N1
- N2
- N3
- N4
- N5
- R

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Summary

- Asymptotic stability tackles both transitories and asymptotics
- The stability of the nul equilibrium of a linear system is related to the location of eigenvalues with respect to the unit disk
- Local stability can often be deduced from linearization of the dynamics at the equilibrium
Outline of the presentation

1. Equilibrium
   - Definition of equilibrium (under constraint)
   - Examples of equilibria

2. Sustainable yield and related notions
   - Sustainable yield for surplus model
   - Maximum sustainable equilibrium
   - Private property equilibrium
   - Common property equilibrium
   - Examples

3. Stability
   - Definition of stability of an equilibrium state
   - Stability for dynamical linear systems
   - Linearized dynamics and stability
   - Examples

4. Summary
Equilibrium analysis is the basics of natural resource management relying upon biomass models: sustainability=equilibrium

Though being strongly criticized, the maximum sustainable yield (MSY) remains the reference till now

Three equilibria are worth being distinguished
- Maximum sustainable equilibrium (MSE)
- Private property equilibrium (PPE)
- Common property equilibrium (CPE)

Stability is important for conservation issues, to avoid biomass collapse

Stability analysis relies upon the study of the linearized dynamics at equilibrium