Controlled spatio-temporal dynamics of resources

Resource management in infinite dimensional spaces

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These lectures are based on results obtained through joint research with William Brock and Athanasios Yannacopoulos, on spatiotemporal dynamics of economic/ ecological systems.

Economic and ecological systems evolve in time and space. Interactions take place among units occupying distinct spatial points. Thus geographical patterns of production activities, urban concentrations, or species concentrations occur. My purpose is:

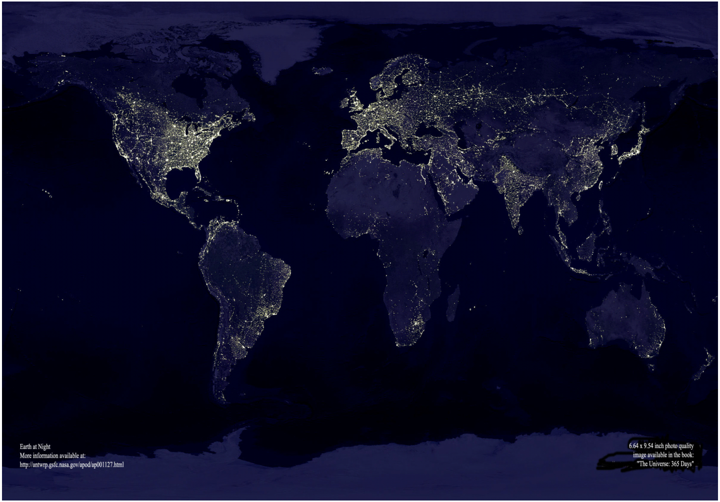
- to discuss approaches for modeling, in a meaningful way, economic and ecological processes evolving in space time.
- to examine mechanisms under which a spatially homogenous state – a flat landscape – acquires a spatial pattern.
- to examine how this pattern evolves in space-time.

The spatial dimension has been brought into the picture through:

- New economic geography/growth models (e.g., Krugman; Boucekkine; Brito; Camacho and Zou; Desmet and Rossi-Hansberg; Brock, Xepapadeas and Yannacopoulos)
- Models of resource management (e.g., Sanchirico; Wilen; Smith; Brock and Xepapadeas)
- In fields like biology or automatic control systems, spatially distributed parameter aspects in the dynamics have been used to study pattern formation
  - on biological agents (e.g., Murray; Levin)
  - of infinite platoons of vehicles over time (e.g., Bamieh, Paganini, and Dahleh; Curtain)
  - in groundwater management (e.g., Leizarowitz).

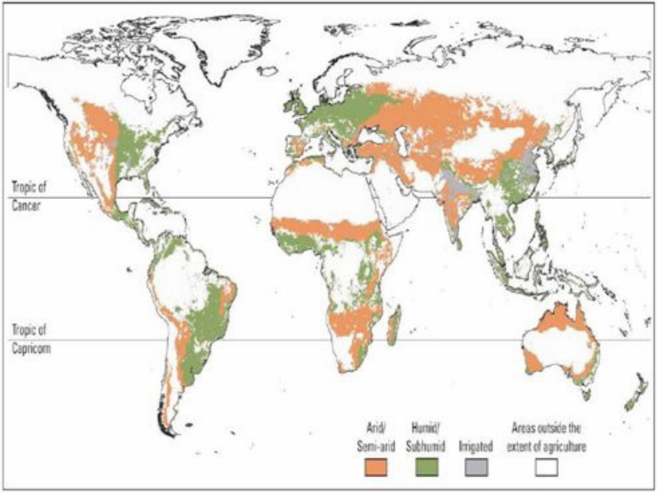
How the leopard got its spots . . .

http://gecon.yale.edu/large-pixeled-contour-globe



School of fish

Arid and Semi arid areas



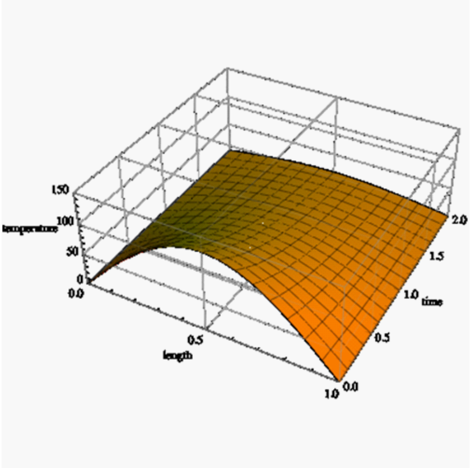
Agricultural landscape

- Modelling short-range (local) and long-range (nonlocal) spatial movements
- Modelling coupled economic - environmental models using systems of reaction-diffusion equations or integrodifferential equations
- Emergence of pattern formation and agglomerations through the classic Turing mechanism

- Optimal control of spatiotemporal economic/ecological models modelled as distributed parameter systems
- Emergence of pattern formation in the optimal control of these systems through optimal diffusion induced or spillover induced instabilities
- Global analysis and persistence of optimal spatial patterns and agglomerations in long run
- Robust control of spatiotemporal models

- Let $x(t,z)$ be a scalar quantity that denotes the concentration of a biological or economic variable which evolves in time and depends on the particular point z of the spatial domain $\mathcal{O}$. Thus $x(t,z)$ is described as a function of time t and space z, i.e. $x : I \times \mathcal{O} \to \mathbb{R}$ where $I = (0, T)$ is the time interval over which the temporal evolution of the phenomenon takes place. For an infinite horizon model, i.e. $I = \mathbb{R}_+$.
- The spatial behavior of x is modelled by assuming that the functions $x(t,\cdot)$ belong for all t to an appropriately chosen function space $\mathbb{H}$ that describes the spatial properties of the function x. Different choices for $\mathbb{H}$ are possible. A convenient choice is to let $\mathbb{H}$ be a Hilbert space, e.g., $\mathbb{H} = L^2(\mathcal{O})$, the space of square integrable functions on $\mathcal{O}$, or an appropriately chosen subspace, e.g. $L^2_{per}(\mathcal{O})$, the space of square integrable functions on $\mathcal{O} = [-L, L]$ satisfying periodic boundary conditions (this would model a circular economy).

- Thus spatial concentration or size is defined such that $(x(t))(z) := x(t,z)$. Therefore by $x(t)$ we denote an element of $\mathbb{H}$, which is in fact a function $x(t) : \mathcal{O} \to \mathbb{R}$ which describes the spatial structure of x at time t



- Let $\phi(t,z)$denote the flow of 'material' such as animals, commodities, or capital, past z at time t. The flux is proportional to the gradient of the concentration

$$\phi(t,z) = -D\frac{\partial}{\partial z}x(t,z) \tag{1}$$

- D is the diffusion coefficient and the minus sign indicates that material moves from high levels of concentration to low levels of concentration. In a small interval Δz:

$$\frac{d}{dt}\int_{z}^{z+\Delta z} x(t,s)\,ds = \phi(t,z) - \phi(t,z+\Delta z) \tag{2}$$

$$+\int_{z}^{z+\Delta z} Fx((t,s))\,ds \tag{3}$$

where $F(t,s)$ is a source or growth function.

- Dividing by Δz and taking limits as $\Delta z \rightarrow 0$ we obtain

$$\frac{\partial x(t,z)}{\partial t} = -\frac{\partial \phi(t,z)}{\partial z} + F(t,z) \quad (4)$$

$$\frac{\partial x(t,z)}{\partial t} = F(x(t,z)) + D\frac{\partial x^2(t,z)}{\partial z^2} \quad (5)$$

- For $F(x(t,z))$ a logistic population growth

$$\frac{\partial x(t,z)}{\partial t} = sx(t,z)\left[1 - \frac{rx(t,z)}{s}\right] + D\frac{\partial x^2(t,z)}{\partial z^2} \quad (6)$$

$$r/s \text{ carrying capacity} \quad (7)$$

Two interacting species $x(t,z), y(t,z)$, no cross diffusion

$$\frac{\partial x}{\partial t} = F_1(x,y) + D_x \nabla^2 x \quad (8)$$

$$\frac{\partial y}{\partial t} = F_2(x,y) + D_y \nabla^2 y \quad (9)$$

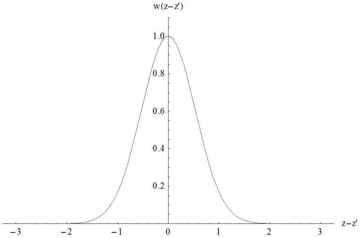
$$\nabla^2 = \frac{\partial^2}{\partial z^2} \quad (10)$$

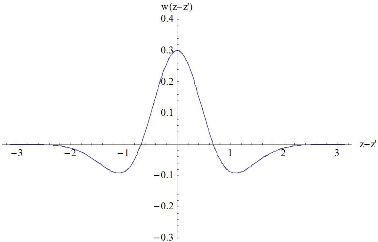
Long range effects can be modelled by integral equations. The evolution of $x(t,z)$ can be represented by:

$$\frac{\partial x(t,z)}{\partial t} = F(x(t,z)) + \int_{-L}^{L} w(z,z')\, x(t,z')\, dz' \qquad (11)$$

where $w : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ is an integrable kernel function modelling the effect that position s has on position z. This introduces nonlocal (spatial) effects, and may be understood as defining a mapping which takes an element $x(t,\cdot) \in \mathbb{H}$ and maps it to a new element $X(t,\cdot) \in \mathbb{H}$ such that (48) holds for every $z \in \mathcal{O}$. This mapping is understood as an operator $\mathrm{T} : \mathbb{H} \to \mathbb{H}$

Kernel	$w_1(\zeta) = b_1 \exp\left[-(\zeta/d_1)^2\right]$, $b_1, d_1 > 0,\ \zeta = z - z'$
Kernel	$w_2(\zeta) = b_1 \exp\left[-(\zeta/d_1)^2\right] - b_2 \exp\left[-(\zeta/d_2)^2\right]$ $b_1 > b_2,\ d_1 < d_2$





- Consider a discrete finite lattice $\mathfrak{L}$, e.g. $\mathfrak{L} = (\mathbb{Z}_N)^d$. The quantity that denotes the concentration of a biological or economic variable which evolves in time and depends on the particular point n of the spatial domain $\mathfrak{L}$ is described by a function $\check{x} : I \to \mathbb{R}^N$ such that $\check{x}(t) = \{x_n(t)\}$, $n \in \mathfrak{L}$, where $x_n(t)$ is the state of the system at site n at time t.
- We therefore consider the state variable x as taking values on a (finite dimensional) sequence space sequence space $\ell^2 := \ell^2(\mathbb{Z}_N) = \{\{x_n\}, \sum_{n\in\mathbb{Z}_N} x_n^2 < \infty\}$. This space is a Hilbert space with a norm derivable from the inner product $\langle x, y\rangle = \sum_{n\in\mathbb{Z}_N} x_n y_n$ and is in fact equivalent to $\mathbb{R}^N$.

- Spatial effects such that the state of the system at point m has an effect on the state of the system at point n are quantified through an discretized version of an influence kernel which can be represented in terms of a matrix $\mathrm{A} = (a_{nm})$. The entry a_{nm} provides a measure of the influence of the state of the system at point m to the state of the system at point n. Network effects knowledge spillovers can be modelled for example through a proper choice of A.
$$\mathrm{A}x = \sum_m a_{nm} x_m \tag{12}$$
$\mathrm{A} : \mathbb{R}^N \to \mathbb{R}^N$ is a linear operator, representable by a finite matrix with elements a_{nm}
- With no spatial interactions at all then $\mathrm{A} = a_{nm} = \delta_{n,m}$ where $\delta_{n,m}$ is the Kronecker delta. If only next neighborhood effects are possible then a_{nm} is non-zero only if m is a neighbor of n. Such an example is the discrete Laplacian

$$D\frac{\partial^2 x(t,z)}{\partial z^2} \approx D\left[x_{n+1}(t) - 2x_n(t) + x_{n-1}(t)\right] \tag{13}$$

Matrix A in this case has a general form

$$\mathrm{A} = D\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & . \\ 0 & 1 & -2 & 1 & 0 & 0 & . \\ 0 & 0 & 1 & -2 & 1 & 0 & . \\ 0 & 0 & 0 & 1 & -2 & 1 & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

$$dx_n = (\sum_m a_{nm}x_m + \sum_m b_{nm}u_m)dt + \sum_m c_{nm}dw_m, \quad n \in \mathbb{Z}_N$$

$$dx = (\mathrm{A}x + \mathrm{B}u)\,dt + \mathrm{C}dw$$

where $\mathrm{A}, \mathrm{B}, \mathrm{C} : \mathbb{R}^N \to \mathbb{R}^N$ are linear operators, representable by finite matrices with elements a_{nm},b_{nm}, c_{nm}, respectively. The state equation is an Ornstein-Uhlenbeck equation on the finite dimensional Hilbert space $\ell^2(\mathbb{Z}_N) = \mathbb{R}^N$.

- A semi-arid system can be described in terms of spatiotemporal dynamics of three state variables: surface water, soil water, and plant biomass. Space is a circle and surface water is fixed by rainfall and uniformly distributed along the circle.
- Plant biomass is consumed in the process of producing cattle products. Cattle products are produced by a conventional production function with two inputs, plant biomass and grazing effort.

$$P_t(t,z) = G(W(t,z), P(t,z)) - b(P(t,z)) - \tag{14}$$
$$TH(t,z) + D_P P_{zz}(t,z) \tag{15}$$
$$W_t(t,z) = F(P(t,z), R) - V(W(t,z), P(t,z)) -$$
$$r_W W(t,z) + D_W W_{zz}(t,z) \tag{16}$$
$$P(0,z),\ W(0,z) \text{ given}$$
$$P(t,0) = P(t,L) = \bar{P}(t),$$
$$W(t,0) = W(t,L) = \bar{W}(t)\ \forall t, \tag{17}$$

$P(t,z)$: plant density (biomass); $W(t,z)$: soil water at time $t \in [0,\infty)$ and site $z \in [0,L]$; R : fixed rainfall; $TH(t,z)$: harvesting of the plant biomass through grazing; $G(W,P)$: plant growth, increasing in soil water and plant density; $b(P)$: plant senescence, $F(P,R)$: water infiltration ; $V(W,P)$: water uptake by plants ; r_W : specific rate of water loss due to evaporation and percolation; D_P and D_W : diffusion coefficients for plant biomass (plant dispersal) and soil water.

$$
\begin{aligned}
\frac{\partial x_1(t,z)}{\partial t} &= f_1\left(x_1(t,z), x_2(t,z), \mathbf{u}(t,z)\right) + D_{x_1}\frac{\partial^2 x_1(t,z)}{\partial z^2} \\
\frac{\partial x_2(t,z)}{\partial t} &= f_2\left(x_1(t,z), x_2(t,z), \mathbf{u}(t,z)\right) + D_{x_2}\frac{\partial^2 x_2(t,z)}{\partial z^2} \\
&\mathbf{x}(0,z) \text{ given, } \mathbf{x}(t,0) = \mathbf{x}(t,L) = \bar{x}(t), \ \forall t.
\end{aligned}
$$

Economic agents maximize utility or profits myopically in each z

$$
\begin{aligned}
u_j^0(t,z) &= \arg\max_{u_j} U\left(\mathbf{x}(t,z), \mathbf{u}(t,z)\right) \ , j = 1, ..., m \\
u_j^0(z,t) &= h_j^0\left(\mathbf{x}(t,z)\right) \ , j = 1, ..., m
\end{aligned}
$$

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$$
\hat{\mathbf{u}}(t,z) \quad : \quad U\left(\mathbf{x}(t,z), \hat{\mathbf{u}}(t,z)\right) = 0 \tag{18}
$$

$$
\hat{u}_j(t,z) = \hat{h}_j\left(\mathbf{x}(t,z)\right) \ , j = 1, ..., m \tag{19}
$$

Reaction-diffusion system with optimizing agents

$$\frac{\partial x_1(t,z)}{\partial t} = f_1\left(x_1(t,z), x_2(t,z), \mathbf{h}^0(\mathbf{x}(t,z))\right) +$$
$$D_{x_1}\frac{\partial^2 x_1(t,z)}{\partial z^2}$$
$$\frac{\partial x_2(t,z)}{\partial t} = f_2\left(x_1(t,z), x_2(t,z), \mathbf{h}^0(\mathbf{x}(t,z))\right) +$$
$$D_{x_2}\frac{\partial^2 x_2(t,z)}{\partial z^2}$$

Define a spatially homogeneous or "flat steady state" for $D_{x_1} = D_{x_2} = 0$, as:

$$\mathbf{x}^0 : f_1\left(x_1^0, x_2^0, \mathbf{h}^0\left(\mathbf{x}^0\right)\right) = 0, \quad f_2\left(x_1^0, x_2^0, \mathbf{h}^0\left(\mathbf{x}^0\right)\right) = 0.$$

Let $\bar{x}(t) = \left(x_1(t) - x_1^0, x_2(t) - x_2^0\right)' = \left(\bar{x}_1(t), \bar{x}_2(t)\right)'$ denote deviations around $\mathbf{x}^0$ and define the linearization

$$\bar{x}_t(t) = J^P \bar{x}(t) \ , \ \bar{x}_t(t) = \begin{pmatrix} \frac{d\bar{x}_1(t)}{dt} \\ \frac{d\bar{x}_2(t)}{dt} \end{pmatrix}, \ J^P = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{20}$$

Assume that $\mathrm{tr} J^P = b_{11} + b_{22} < 0$ and $\det J^P = b_{11}b_{22} - b_{12}b_{21} > 0$ which means that both eigenvalues of J^P have negative real parts. This implies that the FSS $\mathbf{x}^0$ is locally stable to spatially homogeneous perturbations.

Theorem

Private optimizing behavior, as implied by choosing controls according to myopic optimization in the management of a reaction-diffusion system, generates spatial patterns around a flat steady state if

$$\frac{b_{22}D_{x_1} + b_{11}D_{x_2}}{2D_{x_1}D_{x_2}} > 0 \tag{21}$$

$$-\frac{\left(b_{22}D_{x_1} + b_{11}D_{x_2}\right)^2}{4D_{x_1}D_{x_2}} + \det J^P < 0. \tag{22}$$

Note that the elements of the Jacobian matrix, evaluated at $\mathbf{x}^0$, are defined as:

$$b_{11} = \frac{\partial f_1}{\partial x_1} + \sum_{j=1}^{m} \frac{\partial f_1}{\partial u_j}\frac{\partial u_j}{\partial x_1}, \quad b_{12} = \frac{\partial f_1}{\partial x_2} + \sum_{j=1}^{m} \frac{\partial f_1}{\partial u_j}\frac{\partial u_j}{\partial x_2} \quad (23)$$

$$b_{21} = \frac{\partial f_2}{\partial x_1} + \sum_{j=1}^{m} \frac{\partial f_2}{\partial u_j}\frac{\partial u_j}{\partial x_1}, \quad b_{22} = \frac{\partial f_2}{\partial x_2} + \sum_{j=1}^{m} \frac{\partial f_2}{\partial u_j}\frac{\partial u_j}{\partial x_2}. \quad (24)$$

Following Murray (2003), the linearization of the full reaction diffusion system is:

$$\bar{x}_t(t,z) = J^P \bar{x}(t,z) + D\bar{x}_{zz}(t,z), \quad \bar{x}_{zz}(t,z) = \begin{pmatrix} \frac{\partial^2 \bar{x}_1(t,z)}{\partial z^2} \\ \frac{\partial^2 \bar{x}_2(t,z)}{\partial z^2} \end{pmatrix}$$

$$D = \begin{pmatrix} D_{x_1} & 0 \\ 0 & D_{x_2} \end{pmatrix}.$$

Spatial patterns emerge if the FSS is unstable to spatially *heterogeneous* perturbations which take the form of spatially varying solutions defined as:

$$\bar{x}_i\left(t,z\right)=\sum_k c_{ik}e^{\sigma t}\cos\left(kz\right),\ i=1,2\,,k=\frac{2n\pi}{L},\ n=\pm1,\pm2,\ldots \tag{25}$$

where k is called the *wavenumber* and $1/k$, which is a measure of the wave-like pattern, is proportional to the wavelength $\omega:\omega=2\pi/k=L/n$ at *mode* n. σ is the eigenvalue which determines temporal growth and c_{ik}, $i=1,2$ are constants determined by initial conditions and the eigenspace of σ.

Substituting (25) and noting that they satisfy circle boundary conditions at $z = 0$ and $z = L$, we obtain our result because the linearization becomes

$$\bar{x}_t(t,z) = J^L \bar{x}(t,z) \ , J^L = \begin{pmatrix} b_{11} - D_{x_1} k^2 & b_{12} \\ b_{21} & b_{22} - D_{x_2} k^2 \end{pmatrix} \tag{26}$$

Since $\mathrm{tr} J^L = b_{11} + b_{22} - D_{x_1} k^2 - D_{x_2} k^2 < 0$, destabilization of the FSS under spatially heterogenous perturbations requires that

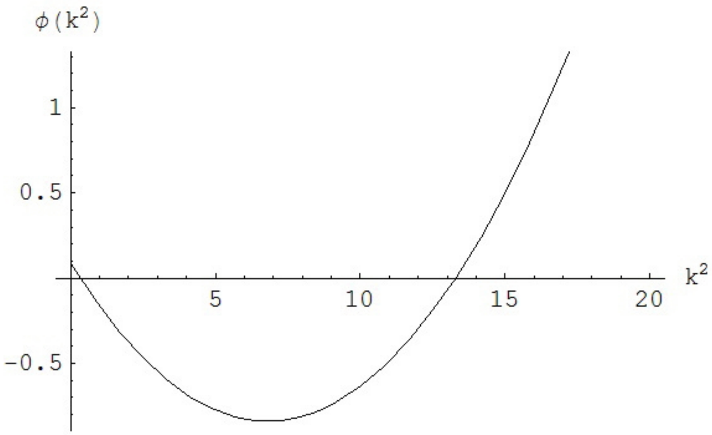
$$\det J^L = \phi\left(k^2\right) = D_{x_1} D_{x_2} k^4 - \left(b_{11} D_{x_2} + b_{22} D_{x_1}\right) k^2 + \det J^P < 0 \tag{27}$$

where $\det J^P > 0$ by the stability assumption about the FSS.

Relationship (27) is a dispersion relationship. The instability condition will be satisfied if there exist wavenumbers k_1 and k_2 such that $\phi\left(k^2\right) < 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$, which implies that matrix (26) has a positive eigenvalue $\sigma\left(k^2\right)$ for $k^2 \in \left(k_1^2, k_2^2\right)$. This in turn requires that: (i) $k_{\min}^2$ which corresponds to the wavenumber which minimizes $\phi\left(k^2\right)$ be positive and, (ii) $\phi\left(k_{\min}^2\right) < 0$ or

$$
\begin{aligned}
\frac{b_{22} D_{x_1} + b_{11} D_{x_2}}{2 D_{x_1} D_{x_2}} &> 0 \\
-\frac{\left(b_{22} D_{x_1} + b_{11} D_{x_2}\right)^2}{4 D_{x_1} D_{x_2}} + \det J^P &< 0.
\end{aligned}
$$

The dispersion relationship

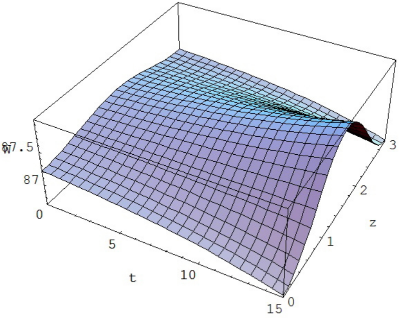


- Linear instability is local
- It is hypothesized that the nonlinear kinetics of the system bound the solution $\mathbf{x}(t, z)$ which eventually settles to a spatial pattern.
- A spatially heterogeneous steady state is obtained by:

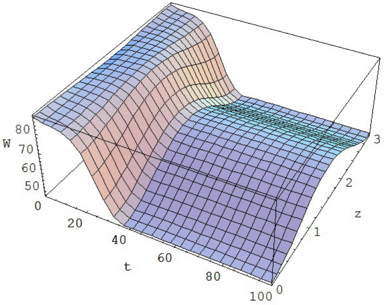
$$0 = f_i\left(\mathbf{x}(z), \mathbf{h}^0(\mathbf{x}(z))\right) + D_{x_i}\frac{\partial^2 x_i(z)}{\partial z^2}, i = 1, 2 \quad (28)$$

- If persistence patterns emerge in this set-up, their creation is a result of the Turing mechanism

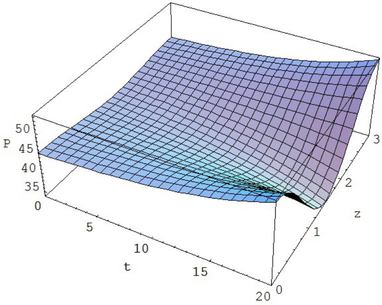
Local instability



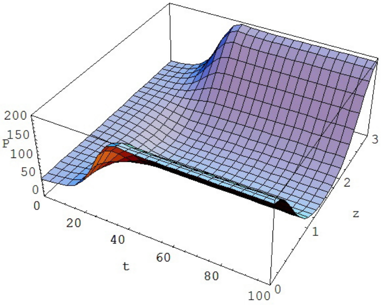
Long run behavior



Local instability



Long run behavior



- There is a large literature in mathematical biology (e.g., Murray, 2003) that studies spatial agglomeration problems.
- To my knowledge, none of this literature deals with optimization problems as we do here. There are many differences between the "backward-looking" dynamics in mathematical biology problems and other natural science problems, and the "forward-looking" dynamics of economic problems.
- These suggest the possibility of a potential agglomeration at the social optimum or at a rational expectations equilibrium related to the incomplete internalization of the spatial externality by optimizing agents

$$\max_{\{\mathbf{u}(t,z)\}} \int_0^\infty \int_0^L e^{-\rho t} \left[U \left(\mathbf{x}(t,z), \mathbf{u}(t,z) \right) \right] dzdt \quad (29)$$

subject to

$$\frac{\partial x_1(t,z)}{\partial t} = f_1\left(x_1(t,z), x_2(t,z), \mathbf{u}(t,z)\right) + D_{x_1} \frac{\partial^2 x_1(t,z)}{\partial z^2}$$

$$\frac{\partial x_2(t,z)}{\partial t} = f_2\left(x_1(t,z), x_2(t,z), \mathbf{u}(t,z)\right) + D_{x_2} \frac{\partial^2 x_2(t,z)}{\partial z^2}$$

$$\mathbf{x}(0,z) \text{ given, } \mathbf{x}(t,0) = \mathbf{x}(t,L) = \bar{x}(t), \ \forall t.$$

$$
\begin{aligned}
u_j^*(t,z) &= \arg\max_{u_j} \mathcal{H}\left(\mathbf{x}(t,z), \mathbf{u}(t,z), \mathbf{p}(t,z)\right) \text{ or} \quad (30)\\
u_j^*(t,z) &= g_j^*\left(\mathbf{x}(t,z), \mathbf{p}(t,z)\right), j = 1, \ldots m.
\end{aligned}
$$

where $\mathcal{H}$ is the current value Hamiltonian function

$$
\begin{aligned}
\mathcal{H} &= U\left(\mathbf{x}(t,z), \mathbf{u}(t,z)\right) + \\
&\sum_{i=1,2} p_i(t,z)\left[f_i\left(\mathbf{x}(t,z), \mathbf{u}(t,z)\right) + D_{x_i}\frac{\partial^2 x_i}{\partial z^2}\right] \quad (31)
\end{aligned}
$$

which is a generalization of the “flat” Hamiltonian function

$$
H = U(\mathbf{x}, \mathbf{u}) + \sum_{i=1,2} p_i f_i(\mathbf{x}, \mathbf{u}). \quad (32)
$$

The vector of the costate variables is $\mathbf{p}(t,z) = (p_1(t,z), p_2(t,z))$ and satisfies for $i = 1, 2$:

$$\frac{\partial p_i(t,z)}{\partial t} = \rho p_i - H_{x_i}(\mathbf{x}(t,z), \mathbf{p}(t,z), \mathbf{g}^*(\mathbf{x},\mathbf{p})) - \quad (33)$$

$$D_{x_i} \frac{\partial^2 p_i(t,z)}{\partial z^2} \quad (34)$$

where $\mathbf{g}^*(\mathbf{x}(t,z), \mathbf{p}(t,z))$ is the vector of the optimal control functions defined by (30).[1]

Temporal and spatial transversality conditions:

$$\lim_{T\to\infty} e^{-\rho T} \int_0^L p_i(T,z)\, x_i(T,z)\, dz = 0 \;, i = 1,2 \quad (35)$$

$$p_i(t,0) = p_i(t,L) \;, i = 1,2. \quad (36)$$

[1] To ease notation we sometimes use subscripts to denote partial derivatives. Thus $H_{\mathbf{v}}$ denotes a vector of partial derivatives, while $H_{\mathbf{vv}}$ denotes a matrix of second order partial derivatives with respect to variables $\mathbf{v}$.

The reaction diffusion system of $(x_1(t,z), x_2(t,z))$, with $\mathbf{u}$ replaced by the optimal controls $\mathbf{g}^*(\mathbf{x}(t,z), \mathbf{p}(t,z))$, and the system of (33) constitute the Hamiltonian system of four partial differential equations:

$$\begin{aligned}
\frac{\partial x_i(t,z)}{\partial t} &= H_{p_i}(\mathbf{x}(t,z), \mathbf{p}(t,z), \mathbf{g}^*(\mathbf{x},\mathbf{p})) + D_{x_i}\frac{\partial^2 x_i}{\partial z^2} \\
\frac{\partial p_i(t,z)}{\partial t} &= \rho p_i - H_{x_i}(\mathbf{x}(t,z), \mathbf{p}(t,z), \mathbf{g}^*(\mathbf{x},\mathbf{p})) - D_{x_i}\frac{\partial^2 p_i}{\partial z^2} \\
i &= 1,2
\end{aligned}$$

Theorem

Assume that for problem (29) with $D_{x_1} = D_{x_2} = 0$, the FOSS $(x_1^*, x_2^*, p_1^*, p_2^*)$ has the local saddle point property with either two positive and two negative real roots, or with complex roots with two of them having negative real parts. Then there is a $(D_{x_1}, D_{x_2}) > 0$ and wave numbers $k \in (k_1, k_2) > 0$ such that if: **(a)**

$$\frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]}{2\left(D_{x_1}^2 + D_{x_2}^2\right)} > 0 \qquad (37)$$

$$\frac{\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right]^2}{4\left(D_{x_1}^2 + D_{x_2}^2\right)} + K^0 > 0 \qquad (38)$$

$$0 < \det J^S\left(k^2\right) \leq (K/2)^2$$

then all the eigenvalues of the Jacobian matrix $J^S\left(k^2\right)$ are real and positive;

Theorem (Continued)

(b)

$$\det J^S\left(k^2\right) < 0 \tag{39}$$

then $J^S\left(k^2\right)$ *has one negative real eigenvalue, while all the other eigenvalues have positive real parts;* **(c)**

$$K^2 - 4\det J^S\left(k^2\right) < 0 \tag{40}$$

$$\det J^S\left(k^2\right) \quad < \quad \left(K/2\right)^2 + \rho^2\left(K/2\right),$$

then all the eigenvalues of $J^S\left(k^2\right)$ *are complex with positive real parts. In all cases above the optimal dynamics associated with the reaction-diffusion system are unstable in the neighborhood of the FOSS in the time-space domain and optimal diffusion induced instability emerges.*

Let $\bar{x}(t,z), \bar{p}(t,z)$ denote deviations from the FOSS, and define the linearization of the Hamiltonian system at the FOSS as:

$$\begin{pmatrix} \bar{x}_t(t,z) \\ \bar{p}_t(t,z) \end{pmatrix} = J^0 \begin{pmatrix} \bar{x}(t,z) \\ \bar{p}(t,z) \end{pmatrix} + D \begin{pmatrix} \bar{x}_{zz}(t,z) \\ \bar{p}_{zz}(t,z) \end{pmatrix} \quad (41)$$

$$J^0 = \begin{pmatrix} H_{\mathbf{px}} & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & \rho I_2 - H_{\mathbf{xp}} \end{pmatrix}, \quad D = \begin{pmatrix} D_{\mathbf{x}} I_2 & \mathbf{0} \\ \mathbf{0} & -D_{\mathbf{x}} I_2 \end{pmatrix},$$

$$D_{\mathbf{x}} = \begin{pmatrix} D_{x_1} \\ D_{x_2} \end{pmatrix}$$

where $H_{\mathbf{pp}}, H_{\mathbf{xx}}, H_{\mathbf{px}} = H_{\mathbf{xp}}$ are (2×2) matrices of second derivatives of the Hamiltonian with $\mathbf{u}^* = \mathbf{g}^*(\mathbf{x}, \mathbf{p})$, I_2 is the (2×2) identity matrix, $\mathbf{0}$ is a (2×2) zero matrix, and J^0 is the Jacobian of the flat Hamiltonian system $(D_{x_1} = D_{x_2} = 0)$.

Consider spatially heterogeneous perturbations of the FOSS of the form

$$\begin{aligned} \bar{x}_i(t,z) &= \sum_k c_{ik}^x e^{\sigma t}\cos(kz),\ \ \bar{p}_i(t,z) = \sum_k c_{ik}^p e^{\sigma t}\cos(kz) \\ k &= \frac{2n\pi}{L},\ \ n = \pm 1, \pm 2, \ldots \end{aligned}$$

and define the following:

$$K_i = \begin{vmatrix} H_{p_i x_i} - D_{x_i}k^2 & H_{p_i p_i} \\ -H_{x_i x_i} & \rho - H_{x_i p_i} + D_{x_i}k^2 \end{vmatrix},\ \ i = 1,2$$

$$K_3 = \begin{vmatrix} H_{p_1 x_2} & H_{p_1 p_2} \\ -H_{x_1 x_2} & -H_{x_1 p_2} \end{vmatrix}, \tag{42}$$

$$K\left(k^2\right) = K_1 + K_2 + 2K_3 \tag{43}$$

Substituting the spatially heterogenous perturbations into the linearized Hamiltonian system we obtain:

$$\begin{pmatrix} \bar{x}_t\left(t,z\right) \\ \bar{p}_t\left(t,z\right) \end{pmatrix} = J^S \begin{pmatrix} \bar{x}\left(t,z\right) \\ \bar{p}\left(t,z\right) \end{pmatrix},$$

$$J^S = \begin{pmatrix} H_{\mathbf{px}} - D_{\mathbf{x}} k^2 I_2 & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & \rho I_2 - H_{\mathbf{xp}} + D_{\mathbf{x}} k^2 I_2 \end{pmatrix}.$$

Define the matrix

$$Z\left(\frac{\rho}{2}\right) = \begin{pmatrix} H_{\mathbf{px}} - D_{\mathbf{x}} k^2 I_2 - \frac{\rho}{2} I_2 & H_{\mathbf{pp}} \\ -H_{\mathbf{xx}} & -H_{\mathbf{xp}} + D_{\mathbf{x}} k^2 I_2 + \frac{\rho}{2} I_2 \end{pmatrix}.$$

Following Kurz (1968, Theorem 2) we obtain that if σ_1, σ_2 are eigenvalues of J^S, then they satisfy $\sigma_{1,2} = \frac{\rho}{2} \pm \psi$, where ψ is a pair of eigenvalues for Z. The eigenvalues of matrix Z are determined by the solution of the characteristic equation:

$$\psi^4 - M_3\psi^3 + M_2\psi^2 - M_1\psi + \det Z = 0 \qquad (44)$$

where $M_3 = \mathrm{tr}(Z) = 0$. By rather tedious calculation we can obtain $M_2 = \left(K - \frac{\rho^2}{2}\right)$, with K defined in (42), and with M_2 being the sum of six principal minors of Z of second order; $M_3 = 0$, with M_3 being the sum of four principal minors of Z of third order; and $\det Z = \left(\frac{\rho}{2}\right)^4 - \left(\frac{\rho}{2}\right)^2 K + \det J^S$.

Substituting in (44) and using the Kurz Theorem we obtain the eigenvalues of J^S as:

$$ {}_1^3\sigma_2^4 = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 - \frac{K}{2} \pm \sqrt{\left(\frac{K}{2}\right)^2 - \det J^S}} \qquad (45) $$

which is an extension of Dockner's (1985) formula for the eigenvalues of the Hamiltonian system for optimal control problems with two state variables, for the case where the state variables diffuse in space.

- The FOSS will have the saddle point property (two positive and two negative eigenvalues) under spatially heterogenous perturbations if (i) $K < 0$ and (ii) $0 < \det J^S < \left(\frac{K}{2}\right)^2$.
- If $K > 0$ while (ii) is still satisfied, the two negative eigenvalues will become positive.
- $$K\left(k^2\right) = -\left(D_{x_1}^2 + D_{x_2}^2\right)k^4 + \quad (46)$$
  $$\left[\sum_{i=1,2} D_{x_i}(2H_{x_i p_i} - \rho)\right] k^2 + K^0 \ , K^0 < 0 \quad (47)$$

  where $K^0 < 0$ because of the saddle point assumption for the FOSS. For instability we want $K\left(k^2\right) > 0$ for some wavenumber k, thus (46) is a dispersion relationship.

Let $\left(\sigma_3\left(k^2\right), \sigma_4\left(k^2\right)\right) > 0, k^2 \in \left(k_1^2, k_2^2\right)$ the eigenvalues that turn positive under spatial perturbation, then the patterned state and costate paths in the neighborhood of the FOSS can be approximated as:

$$\begin{pmatrix} \bar{x}(t,z) \\ \bar{p}(t,z) \end{pmatrix} \sim$$

$$\sum_{n_1}^{n_2} \mathbf{c}_{3n} \exp\left[\sigma_3\left(k^2\right) t\right] \cos(kz) +$$

$$\sum_{n_1}^{n_2} \mathbf{c}_{4n} \exp\left[\sigma_4\left(k^2\right) t\right] \cos(kz), \;\; k = \frac{2n\pi}{L}, n = 1, 2, ..$$

- Note that the two constants which correspond to eigenvalues σ_1, σ_2 with positive real parts should still be set equal to zero, so that the use of the temporal transversality condition at infinity will allow us to choose initial costates $\mathbf{p}$ for any initial state $\mathbf{x}$, which will set the system on the *spatially heterogeneous - spatiotemporally unstable* "optimal" manifold.
- The length L of space should be adequate to allow the existence of these unstable modes.

- Fourier methods are used to reduce the original problem to a countable number of "ordinary" finite dimensional optimal control problems in which the dynamics are described by ordinary differential equations (Brock and Xepapadeas 2008). These mode-n control problems correspond to each mode $k = 2n\pi/L$, $n = 0, 1, 2, \ldots$.
- Then the unstable nodes are identified through the dispersion relationship
- Optimal pre-patterns occur along the *spatially heterogeneous - spatiotemporally unstable* "optimal" manifold

- This steady state will satisfy the system of second-order differential equations in the space variable z, defined by:

$$
\begin{aligned}
0 &= H_{p_i}\left(\mathbf{x}\left(t,z\right),\mathbf{p}\left(t,z\right),\mathbf{g}^*\left(\mathbf{x},\mathbf{p}\right)\right)+D_{x_i}\frac{\partial^2 x_i}{\partial z^2} \\
0 &= \rho p_i - H_{x_i}\left(\mathbf{x}\left(t,z\right),\mathbf{p}\left(t,z\right),\mathbf{g}^*\left(\mathbf{x},\mathbf{p}\right)\right)-D_{x_i}\frac{\partial^2 p_i}{\partial z^2}
\end{aligned}
$$

- *However no convergence results.*

- Use global analysis based on monotone operator theory, combined with local analysis based on spectral theory, to obtain insights regarding the endogenous emergence (or not) of optimal agglomerations at a rational expectations equilibrium and the social optimum of dynamic economic systems.
- The possibility of a potential agglomeration at a rational expectations equilibrium is related to the incomplete internalization of the spatial externality by optimizing agents.
- A "no agglomerations" theorem at the social optimum stems from the full internalization of the spatial externality by a social planner and the strict concavity of the production function.

- Production at time t and site z is given by the strictly concave production function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in terms of $f(x(t,z), X(t,z))$.
- $$X(t,z) = \int_{\mathcal{O}} w(z,s)x(t,s)ds \tag{48}$$
- $x(t,z)$: denotes the capital stock at point $z \in \mathcal{O}$ at time $t \in [0, +\infty)$
- $w : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ is an integrable kernel function modeling the effect that position s has on position z. There is a mapping which takes an element $x(t,\cdot) \in \mathbb{H}$ and maps it to a new element $X(t,\cdot) \in \mathbb{H}$ such that (48) holds for every $z \in \mathcal{O}$.
- $\mathbb{H} = L^2(\mathcal{O})$, the space of square integrable functions on $\mathcal{O}$

$$c\left(t,z\right)+\frac{\partial x\left(t,z\right)}{\partial t}=f\left(x\left(t,z\right),\check{X}\left(t,z\right)\right)-$$
$$\eta x\left(t,z\right)-\frac{\alpha}{2}\left[\frac{\partial x\left(t,z\right)}{\partial t}\right]^{2},\frac{\partial x\left(t,z\right)}{\partial t}=u\left(t,z\right)$$

$$0=\mathcal{C}(z):=\int_0^\infty e^{-rt}\left[x_0+f\left(x(t,z),\check{X}(t,z)\right)\right. \quad (49)$$
$$\left.-\lambda x(t,z)-c(t,z)-\frac{\alpha}{2}u^2(t,z)\right]dt$$

$$0=\mathcal{C}^{\lozenge}:=\int_{\mathcal{O}}\int_0^\infty e^{-rt}\left[x_0+f\left(x(t,z),\check{X}(t,z)\right)\right. \quad (50)$$
$$\left.-\lambda x(t,z)-c(t,z)-\frac{\alpha}{2}u^2(t,z)\right]dtdz$$

$$
\begin{aligned}
& \max_c (J_{RE}(c))(z) := \int_0^\infty e^{-\rho t} U(c(t,z))dt, \\
0 \quad = \quad & \mathcal{C}(z) := \int_0^\infty e^{-rt} \left[x_0 + f\left(x(t,z), \check{X}(t,z)\right) \right. \\
& \left. -\lambda x(t,z) - c(t,z) - \frac{\alpha}{2} u^2(t,z) \right] dt \\
& \check{X}(t,z) = X^e \text{ exogenous}
\end{aligned}
$$

$$
\begin{aligned}
J_{SO}(c) \quad : \quad & = \int_{\mathcal{O}} \psi(z)(J_s c)(z) dz = \\
& \int_{\mathcal{O}} \int_0^{\infty} e^{-\rho t} \psi(z) U(c(t,z)) dt dz. \\
& 0 = \mathcal{C}^{\lozenge} := \int_{\mathcal{O}} \int_0^{\infty} e^{-rt} \left[x_0 + f\left(x(t,z), \check{X}(t,z)\right)\right. \\
& \left. -\lambda x(t,z) - c(t,z) - \frac{\alpha}{2} u^2(t,z) \right] dt dz \\
& \check{X}(t,z) = X(t,z) = \int_{\mathcal{O}} w(z,s) x(t,s) ds
\end{aligned}
$$

- The influence kernel function $w : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ is continuous and symmetric, i.e. $w(z, s) = w(s, z) = w(z - s)$. Then following integral operator is defined

$$X(t, z) = (\mathrm{K}x)(t, z) := \int_{\mathcal{O}} w(z - s)x(t, s)ds. \quad (51)$$

- The production function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a strictly increasing, strictly concave function of the (real) variables (x, X).
- The second order derivatives f_{xX}, f_{XX} are uniformly bounded below in $\mathcal{O}$,

$$-\mu := \inf_{(x,X)\in\mathbb{R}^2} f_{XX}, \quad \xi := \inf_{(x,X)\in\mathbb{R}^2} f_{xX}, \quad \mu, \xi \in \mathbb{R}_+.$$

- The utility function $U : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and strictly concave C^2 function in consumption c and satisfies the Inada conditions

$$\lim_{c \to 0} \partial_c U(c) = +\infty, \quad \lim_{c \to +\infty} \partial_c U(c) = 0.$$

- The operator $\mathrm{K} : \mathbb{H} \to \mathbb{H}$ is strictly positive
- It holds that $\mu/\xi < \mu_1$ where μ_1 is the largest (positive) eigenvalue of operator K.

$x(t,z) = x_0(z) + \int_0^t u(s,z)ds, \; u = k'$

Rational expectations equilibrium

$$\max_{x'} \int_0^\infty e^{-rt} \left(f(x(t,z), (\mathrm{K}x)(t,z)) - \lambda x(t,z) - \frac{\alpha}{2}(x'(t,z))^2 \right) dt \tag{52}$$

Social Optimum

$$\max_{x'} \int_0^\infty \int_{\mathcal{O}} e^{-rt} \left[f(x(t,z), (\mathrm{K}x)(t,z)) - \lambda x(t,z) - \frac{\alpha}{2}(x'(t,z))^2 \right] dz\, dt. \tag{53}$$

Define the nonlinear operators $\mathrm{A}_{\nu} : \mathbb{H} \rightarrow \mathbb{H}$, $\nu = RE, SO$, by

$$\mathrm{A}_{RE}x := -\alpha^{-1}(f_x(x, \check{X}) - \lambda), \;\; \check{X} = \mathrm{K}x,$$
$$\mathrm{A}_{SO}x := -\alpha^{-1}(f_x(x, \check{X}) + \mathrm{K}f_X(x, \check{X}) - \lambda), \;\; \check{X} = \mathrm{K}x.$$

Theorem

The first order necessary condition for problems (52) and (53) is of the form

$$x^{''} - rx^{'} - A_{\nu}x = 0, \;\; \nu = RE, SO \tag{54}$$

where A_{ν} are the nonlinear operators above. The first order necessary conditions have to be complemented with the transversality condition

$$\lim_{t\rightarrow\infty} e^{-rt}xx' = \lim_{t\rightarrow\infty} \frac{1}{2}e^{-rt}(x^2)' = 0. \tag{55}$$

Definition

A solution $x : I \longrightarrow \mathbb{H}$, if it exists, of the nonlinear integro-differential equation

$$x'' - rx' - \mathrm{A}_{\nu} x = 0 \tag{56}$$

is called an RE equilibrium if $\nu = RE$ and an SO equilibrium if $\nu = SO$.

Consider

$$J = \int_0^\infty \int_{\mathcal{O}} e^{-r\,t} \left[f(x(t,z),(\mathrm{K}x)(t,z)) - \right. \tag{57}$$

$$\left. \lambda x(t,z) - \frac{\alpha}{2} (x^{'}(t,z))^2 \right] dz\, dt. \tag{58}$$

as a functional of $u = x'$ and $x = x_0 + \int_0^t u(s)ds$. The FONC will be of the form $(\nabla J, \phi) = 0$ where ∇ denotes the Gâteaux derivative and ϕ is a test function in $\mathbb{H}$

The Gâteaux derivative: we fix any direction $v \in \mathbb{H}$, define $u_\epsilon = u + \epsilon v$, $V = \int_0^t v(s)ds$ and calculate

$$\left. \frac{d}{d\epsilon} J(u_\epsilon) \right|_{\epsilon=0} = \int_0^\infty \int_{\mathcal{O}} e^{-rt} \left[\partial_x f(x,\mathrm{K}x) + \mathrm{K}^* \partial_X f(x,\mathrm{K}x) V - \right. \\ \left. \lambda V - \alpha u v \right] dz\, dt \tag{59}$$

Since $v = V'$, by integration by parts over t and using the transversality condition, the first order condition becomes

$$\int_0^\infty \int_{\mathcal{O}} e^{-rt} \left[\partial_x f(x, \mathrm{K}x) + \mathrm{K}^* \partial_X f(x, \mathrm{K}x) - \lambda + \alpha u' - r\alpha u\right] V dz\, dt = 0.$$

This must be true for all v therefore for all V which implies that the first order condition becomes

$$\partial_x f(x, \mathrm{K}x) + \mathrm{K}^* \partial_X f(x, \mathrm{K}x) - \lambda + \alpha u' - r\alpha u = 0,$$

(a.e.) and keeping in mind that $u = x'$, we obtain:

$$x'' - rx' - \mathrm{A}_{SO} x = 0 \qquad (60)$$

The following theorem provides important information on the long-run dynamics

Theorem (Convergence)

(a) The operator equations $\mathrm{A}_\nu x = 0$, $\nu = RE, SO$, *have unique solutions. (b) All bounded solutions of* $x'' - rx' - \mathrm{A}_\nu x = 0$ *have as weak limit the solution of* $\mathrm{A}_\nu x = 0$, $\nu = RE, SO$.

Assumption P

The operator $\mathrm{K} : \mathbb{H} \to \mathbb{H}$ is strictly positive.[2]

It holds that $\mu/\xi < \mu_1$ where μ_1 is the largest (positive) eigenvalue of operator K.

$-\mu := \inf_{(x,X)\in\mathbb{R}^2} f_{XX}, \quad \xi := \inf_{(x,X)\in\mathbb{R}^2} f_{xX}, \quad \mu, \xi \in \mathbb{R}_+$

[2] K is a positive operator if $(\mathrm{K}h, h) \geq 0$ for all $h \in \mathbb{H}$, and strictly positive if furthermore $(\mathrm{K}h, h) = 0$ implies $h = 0$.

- The operators $\mathrm{A}_\nu : \mathbb{H} \rightarrow \mathbb{H}$, $\nu = RE, SO$ are maximal monotone.[3]
- We now use Theorem 3.3. of Rouhani and Khatibzadeh (2009) to obtain convergence results. According to a special case of this theorem a bounded solution of

$$x'' - rx' = \mathrm{A}_\nu x$$

for any initial condition x_0, converges weakly as $t \rightarrow \infty$ to an element of $\mathrm{A}_\nu^{-1}(0)$, if A_ν is a maximally monotone.

[3] A possibly nonlinear operator $\mathrm{A} : \mathbb{H} \rightarrow \mathbb{H}$ is called monotone if $(\mathrm{A}x - \mathrm{A}y, x - y) \geq 0$ for all $x, y \in \mathbb{H}$ and maximal monotone if its graph is not properly contained in the graph of any other monotone operator. Observe that monotonicity is related to positivity if the operator is linear.

- The results of the Theorem about convergence hold for SO without Assumption P. Assumption P is a sufficient condition for the theorem to hold in the RE case.
- Therefore, convergence to the RE steady state depends on the strength of diminishing returns with respect to spatial spillovers (f_{XX}), the strength of the complementarity between the capital stock and spatial spillovers in the production function (f_{xX}), and the structure of the spatial domain as reflected in the largest eigenvalue of K.
- Furthermore, relaxing the monotonicity assumption on operator K, there may exist multiple solutions for the RE steady-state equation for appropriate values of λ.

- For strictly concave production functions f, if the steady state equation $\mathrm{A}_{SO}x = 0$ admits a flat solution then all bounded solutions of the time dependent system will finally tend weakly to that flat solution as $t \to \infty$. Thus agglomeration is not a socially optimal outcome in this case.
- The uniqueness of the solution of $\mathrm{A}_{SO}x = 0$ precludes the existence of any steady state other than the flat steady state as long as total spillover effects are the same across all sites of the spatial domain. Then the socially optimal spatial distribution of economic activity is the uniform distribution in space. This is always true in the case of periodic boundary conditions, when α is independent of z. This result is a generalization of classical turnpike theory to infinite dimensional spatial models

- Convergence to the RE steady state is not guaranteed by the strict concavity of the production function, as in the SO case, but depends, according to part (a) of this theorem, on the relation between diminishing returns, complementarities, and the spatial geometry;
- If a unique globally stable RE steady state exists it will be flat. Hence for both the RE and SO the unique steady state is the flat steady state.[4]
- If the conditions of this theorem leading to (a) are not satisfied, a more complex behavior is expected in the RE equilibrium. In this case, multiple RE steady states cannot be eliminated, and a potential agglomeration at the RE equilibrium takes the form of instability of the flat steady state.

[4] The RE and SO steady states will in general be different from each other, which calls for spatially dependent economic policy if the SO steady state is to be attained.

- For a flat steady state $\bar{x}$, let

$$\begin{aligned} s_{11} &: = \alpha^{-1}\partial^2_{xx} f(\bar{x}, \mathrm{K}\bar{x}), \quad s_{22} := \alpha^{-1}\partial^2_{XX} f(\bar{x}, \mathrm{K}\bar{x}), \\ s_{12} &: = \alpha^{-1}\partial^2_{xX} f(\bar{x}, \mathrm{K}\bar{x}) > 0, \end{aligned} \tag{61}$$

and define the linear bounded operators $\mathrm{L}_\nu : \mathbb{H} \to \mathbb{H}$ by

$$\begin{aligned} \mathrm{L}_{RE}\hat{x} &:= s_{11}\hat{x} + s_{12}\mathrm{K}\hat{x} \\ \mathrm{L}_{SO}\hat{x} &:= s_{11}\hat{x} + 2s_{12}\mathrm{K}\hat{x} + s_{22}\mathrm{K}^2\hat{x}. \end{aligned}$$

- These operators govern the behavior of spatiotemporal perturbations, $\hat{x}$, from the flat steady state $\bar{x}$: Inserting the ansatz $x = \bar{x} + \epsilon\hat{x}$ into the equilibrium condition

$$x'' - rx' - \mathrm{A}_\nu x = 0 \tag{62}$$

and expanding in ϵ, we obtain the linearized equation for the evolution of the perturbation $\hat{x}(t, z)$ as follows:

$$\hat{x}'' - r\hat{x}' + \mathrm{L}_\nu\hat{x} = 0, \quad \nu = RE, SO. \tag{63}$$

Let $\{\mu_j\}$ be the eigenvalues of operator K and $\{\phi_j\}$ the corresponding eigenfunctions. Then,

Proposition

An arbitrary initial perturbation of the flat steady state of the form

$$\hat{x}(0,z) = \sum_j a_j \phi_j(z), \quad \hat{x}'(0,z) = \sum_j b_j \phi_j(z),$$

evolves under the linearized system (63) to

$$\hat{x}_\nu(t,z) = \sum_j c_{\nu,j}(t)\phi_j(z)$$

where $\{c_{\nu,j}(t)\}$ is the solution of the countably infinite system of ordinary differential equations

$$c''_{\nu,j} - r c'_{\nu,j} + \Lambda_{\nu,j} c_{\nu,j} = 0, \quad \nu = RE, SO, \quad j \in \mathbb{N} \qquad (64)$$
$$c_{\nu,j}(0) = a_j, \quad c'_{\nu,j}(0) = b_j$$

Proposition

-
  1. *If* $\Lambda_{\nu,j} < 0$, *then* $c_{\nu,j}(t) = \bar{A}_j e^{\sigma_1 t} + \bar{B}_j e^{\sigma_2 t}$ *where* $\sigma_1 < 0 < \frac{r}{2} < \sigma_2$ *(saddle path behaviour).*
  2. *If* $0 < \Lambda_{\nu,j} < \left(\frac{r}{2}\right)^2$, *then* $c_{\nu,j}(t) = \bar{A}_j e^{\sigma_1 t} + \bar{B}_j e^{\sigma_2 t}$ *where* $0 < \sigma_1 < \frac{r}{2} < \sigma_2$ *(unstable solutions).*
  3. *If* $\left(\frac{r}{2}\right)^2 < \Lambda_{\nu,j}$, *then* $c_j(t) = e^{\frac{r}{2}t}\left(\bar{A}_j \cos(\sigma t) + \bar{B}_j \sin(\sigma t)\right)$, $\sigma \in \mathbb{R}$

and $\bar{A}_j$, $\bar{B}_j$ *are constants related to the initial conditions.*

- The perturbations from the flat steady state which contain modes ϕ_j such that $\Lambda_{\nu,j} < 0$ will die out and the system will converge to the flat steady state – no possible agglomeration is expected.
- The perturbations from the flat steady state which contain modes ϕ_j such that $\Lambda_{\nu,j} > 0$ will turn unstable and lead to possible potential agglomeration spatial patterns, either monotone in time or oscillatory in time.

This instability can be contrasted with the celebrated Turing instability mechanism (Turing, 1952), which leads to pattern formation in biological and chemical systems. The important differences here are that:

(a) in our model the instability is driven not by the action of the diffusion operator (which is a differential operator) but rather by a compact integral operator that models geographical spillovers, and

(b) contrary to the spirit of the Turing model, here the instability is driven by optimizing behavior, so it is the outcome of forward-looking optimizing behavior by economic agents and not the result of reaction diffusion in chemical or biological agents. It is the optimizing nature of our model which dictates precisely the type of unstable modes which are "accepted" by the system, in the sense that they are compatible with the long-term behavior imposed on the system by the policy maker.

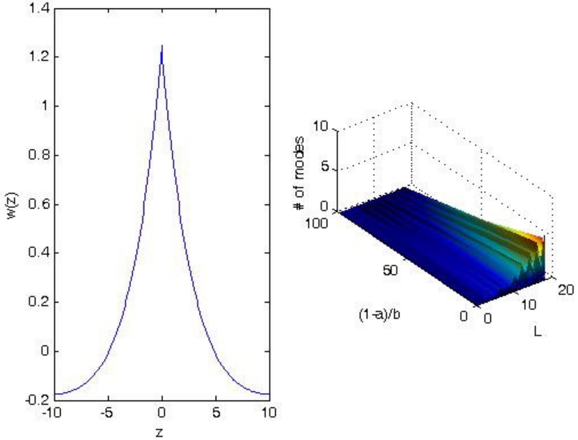
Assume periodic boundary conditions, i.e., $\mathbb{H} = L_{per}(\mathcal{O})$, $\mathcal{O} = [-L, L]$. Then,

(a) The eigenfunctions of operator K are the Fourier modes $\phi_n(z) = \cos(n\pi z/L)$, $n \in \mathbb{N}$ with corresponding eigenvalues $W_n = \int_{-L}^{L} w(z)\,\phi_n(z)dz$.

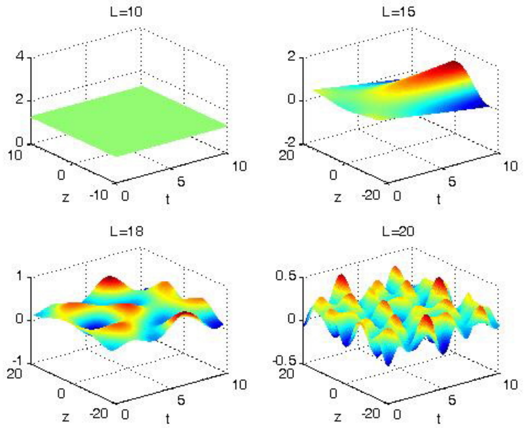
(b) The action of operator K on a flat state returns a flat state, $\mathrm{K}\bar{x} = \bar{x}\int_{-L}^{L} w(z)dz$

Cobb-Douglas and composite exponential kernels

$$
\begin{aligned}
f(x,X) &= C_0 x^a X^b, \;\; a+b<1, \\
w(z) &= \sum_{i=1}^{N} C_i \exp(-\gamma_i |z|), \;\; \gamma_i \geq 0, \;\; C_i \in \mathbb{R}. \\
\bar{x}_{RE} &= \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{a+b-1}} W^{-\frac{b}{a+b-1}} \\
\bar{x}_{SO} &= \left(\frac{\bar{\lambda}}{a+b}\right)^{\frac{1}{a+b-1}} W^{-\frac{b}{a+b-1}}
\end{aligned}
$$



The shape of the composite kernel for $\gamma_1 = 0.3$, $C_1 = 2$, $\gamma_2 = 0.1$, $C_2 = -0.75$ (left panel). The number of unstable modes for this choice of kernel function, as a function of the parameter $\frac{1-a}{b}$ and L (right panel).



Emerging patterns from the instability for different domain sizes.

- The need for **robustness** emerges when a decision-making agent has concerns about possible deviations of the actual model underlying the decision-making process from the model specified
- or when the decision maker has concerns about possible misspecifications of the reference model and wants to incorporate these concerns into the decision-making rules.
- A rule is robust if it continues to behave well even if the actual model deviates from a specified or a benchmark model .

- Robust control problems have been traditionally analyzed in the context of:
  - risk sensitive linear quadratic Gaussian (LEQG) models and
  - the H^{∞} models. The H^{∞} criterion implies decision making for protection against the 'worst case' and is related to a minimax approach.
- More recently Hansen and Sargent interpreted concerns about model misspecification in economics as a situation where a decision maker or a regulator distrusts her model and wants good decisions over a cloud of models that surrounds the regulator's approximating or benchmark model, which are difficult to distinguish with finite data sets.

- There is a fictitious 'adversarial agent' - Nature.
- Nature promotes robust decision rules by forcing the regulator, who seeks to maximize (minimize) an objective, to explore the fragility of decision rules to departures from the benchmark model.
- A robust decision rule means that lower bounds to the rule's performance are determined by Nature – the adversarial agent – which acts as a minimizing (maximizing) agent when constructing these lower bounds.
- Hansen and Sargent show that robust control theory can be interpreted as a recursive version of max-min expected utility theory.
- Robust control methods have not been extended, as far as we know, to models that evolve both in time and space.

Robust Control and Spatial Models

- Concerns about model misspecification refer now to the benchmark or reference model that describes the spatiotemporal dynamics of each specific site.
- If potential deviations from the specified model differ from site to site, then concerns for one site might affect the robust rules for other sites.
- Thus robust rules should account not only for the spatial characteristics of the problem in a specific location, but also for the degree to which the regulator distrusts her model across locations.
- If concerns about the benchmark model in a given site differ from concerns in other sites, a spatially dependent robust rule should capture these differences.

Hot Spots (Hot Spots)

- We formally identify, for the first time to our knowledge in economics, spatial hot spots – which are sites where:
  - robust control breaks down
  - robust control is very costly as a function of the degree of the regulator's concerns about model misspecification across all sites
  - the need to apply robust control induces spatial agglomerations and breaks down spatial symmetry
    - This is, as far as we know, a new source for generating spatial patterns as compared to the classic Turing diffusion induced instability.
- Thus hot spots are specific sites where uncertainties in these sites are such that when concerns about local misspecifications are incorporated into the decision rules for the entire spatial domain, the global rule **could break down**, **could be very costly** or **could induce spatial clustering**.

- Our economy is located on a discrete lattice $\mathfrak{L}$. The "economy" is a collection of state variables $x = \{x_n\}$, $n \in \mathfrak{L}$.
- We consider an optimal linear regulator problem: optimization of a quadratic objective defined over the whole lattice by exerting on each lattice site a control $u_n \in \mathbb{R}$.
- The economy evolves according to an infinite dimensional stochastic differential equation

$$dx_n = \left(\sum_m a_{nm} x_m + \sum_m b_{nm} u_m\right) dt + \sum_m c_{nm} dw_m, \; n \in \mathbb{Z}$$

where the last term, describes the fluctuations of the state due to the stochasticity.
- In compact form this can be expressed as

$$dx = (\mathrm{A}x + \mathrm{B}u)\, dt + \mathrm{C}dw$$

where $\mathrm{A}, \mathrm{B}, \mathrm{C} : \ell^2 \rightarrow \ell^2$ are linear operators, related to the doubly infinite matrices with elements a_{nm}, b_{nm}, c_{nm}, respectively.

- The economy at point m has an effect on the state of the economy at point n. This effect is quantified through an influence "kernel" which assumes the form of a double sequence $\mathrm{A} = (a_{nm})$. The entry a_{nm} provides a measure of the influence of the state of the system at point m on the state of the system at point n.
  - If the economies do not interact at all, then $\mathrm{A} = a_{nm} = \delta_{n,m}$ where $\delta_{n,m}$ is the Kronecker delta.
  - If only next neighbor effects are possible, then a_{nm} is non-zero only if m is a neighbor of n.
- The controls at different point of the lattice u_m are assumed to have an effect on the state of the system at site n, through the term $\sum_m b_{nm} u_m$.
  - Fishing effort at a given site may affect harvesting costs at other sites knowledge or productivity spillovers.
- The term $\sum_m c_{nm} dw_m$ tells us how the uncertainty at site m is affecting the uncertainty concerning the state of the system at site n.

- Assume now that there is some uncertainty concerning the "true" statistical distribution of the state of the system. This corresponds to a family of probability measures $\mathcal{Q}$ such that each $Q \in \mathcal{Q}$ corresponds to an alternative stochastic model (scenario) concerning the state of the system.
- By Girsanov's theorem, $\bar{w}_n(t) = w_n(t) - \int_0^t v_n(s)ds$ is a Q-Brownian motion for all $n \in \mathbb{N}$, where the drift term v_n may be considered as a measure of the model misspecification at lattice site n.
- The adoption of the family $\mathcal{Q}$ of alternative measures concerning the state of the system leads to a family of different equations for the state variable

$$dx_n = \Big(\sum_m a_{nm} x_m + \sum_m b_{nm} u_m + \sum_m c_{nm} v_m\Big)dt + \sum_m c_{nm} d\bar{w}_m, \;\; n \in \mathbb{Z}.$$

$$\min_u \mathbb{E}_Q \left[\int_0^\infty e^{-rt} \sum_{n,m} \left(p_{nm} x_n(t) x_m(t) + q_{nm} u_n(t) u_m(t) \right) dt \right]$$

or in compact form

$$\min_u \mathbb{E}_Q \left[\int_0^\infty e^{-rt} (\langle \mathrm{P} x(t), x(t) \rangle + \langle \mathrm{Q} u(t), u(t) \rangle) dt \right]$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the Hilbert space ℓ^2, and $\mathrm{P}, \mathrm{Q} : \ell^2 \to \ell^2$ are symmetric positive operators with infinite matrix representation $\mathrm{P} = \{p_{nm}\}$, $\mathrm{Q} = \{q_{nm}\}$.
If $p_{nm} = p\,\delta_{nm}$, $q_{nm} = q\,\delta_{nm}$, then

$$\min_u \mathbb{E}_Q \left[\int_0^\infty e^{-rt} \sum_n p(x_n(t))^2 + q(u_n(t))^2) dt \right]$$

First sum: Total deviation of the states of the system at each site from the desired state 0. Second sum: The total control cost to drive it to 0.

Being uncertain about the true model, the decision maker will choose the strategy that will work even in the worst case scenario.

$$\min_u \max_v \mathbb{E}_Q \left[\int_0^\infty e^{-rt} \sum_n \sum_m (p_{nm} x_n(t) x_m(t) \right.$$
$$\left. + q_{nm} u_n(t) u_m(t) - \theta r_{nm} v_n(t) v_m(t)) dt \right]$$

or in compact form

$$\min_u \max_v \mathbb{E}_Q \left[\int_0^\infty e^{-rt} (\langle (\mathrm{P}x)(t), x(t) \rangle + \langle (\mathrm{Q}u)(t), u(t) \rangle \right.$$
$$\left. - \theta \langle (\mathrm{R}v)(t), v(t) \rangle) dt \right]$$

subject to the dynamic constraint. The third term corresponds to a quadratic loss function related to the "cost" of model misspecification.

The optimization problem for the choice $R = I$, is related to a robust control problem with an entropic constraint of the form

$$\inf_{u} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[\int_0^\infty e^{-rt} (\langle \mathrm{P} x(t), x(t) \rangle + \langle \mathrm{Q} u(t), u(t) \rangle) dt \right],$$

$$\text{subject to } \ \mathcal{H}(P \mid Q) < H_0$$

and the dynamic constraint, where by $\mathcal{H}(P \mid Q)$ we denote the Kullback-Leibler entropy of the probability measures P and Q.

$$\mathcal{H}(Q \mid P) := \mathbb{E}_Q \left[\ln \left(\frac{dQ}{dP} \right) \right] = \frac{1}{2} \int_0^T \sum_n v_n^2(t) dt$$

We now consider the robust optimization problem

$$\inf_{u} \sup_{Q \in \mathcal{Q}} J(x,u;v)$$

$$\text{subject to} \qquad \mathcal{H}(Q \mid P) \leq H_0$$

and the dynamic constraint where
$J(x,u;v) := \mathbb{E}_Q \left[\int_0^T e^{-rt} \left(\langle \mathrm{P}x(t), x(t) \rangle + \langle \mathrm{Q}u(t), u(t) \rangle \right) dt \right]$.
Using Lagrange multipliers we see that a solution of the relative entropy constraint problem is equivalent to the solution of

$$\inf_{u} \sup_{Q \in \mathcal{Q}} J(x,u;v) - \theta(\mathcal{H}(Q|P) - H_0),$$

The optimization problem is related to a robust control problem with an entropic constraint of the form

$$\inf_{u} \sup_{Q\in\mathcal{Q}} \mathbb{E}_Q \left[\int_0^\infty e^{-rt}(\langle \mathrm{P}x(t), x(t)\rangle + \langle \mathrm{Q}u(t), u(t)\rangle)dt\right],$$

$$\text{subject to } \ \mathcal{H}(P_n \mid Q_n) < H_n, \ \ n \in \mathbb{Z}$$

and the dynamic constraint, where by $\mathcal{H}(P_n \mid Q_n)$ is the Kullback-Leibler entropy of the marginal probability measures P_n and Q_n.

The localized relative entropy constraint problem is equivalent to the solution of

$$\sup_{Q\in\mathcal{Q}} J(x,u;v) - \sum_n \theta_n \left(\mathcal{H}(\bar{Q}_n \mid \bar{P}_n) - H_n\right)$$

- The introduction of the local entropic constraints means that the policy maker's concerns differ at various spatial points.
- The maximizing adversarial agent - Nature - chooses a $\{v_n(t)\}$ where $\theta_n \in (\underline{\theta}_n, +\infty]$, $\underline{\theta}_n > 0$, is a penalty parameter restraining the maximizing choice of Nature.
- θ_n is associated with the Lagrange multiplier of the entropy constraint at each site. In the entropy constraint H_n is the maximum misspecification error that the decision maker is willing to consider given the existing information about the system at site n.

- The lower bound $\underline{\theta}_n$ is a so-called breakdown point beyond which it is fruitless to seek more robustness because the adversarial agent is sufficiently unconstrained so that she/he can push the criterion function to $+\infty$ despite the best response of the minimizing agent.
- Thus when $\theta_n < \underline{\theta}_n$ for a specific site, robust control rules cannot be attained. In our terminology this site will be **a hot spot** since misspecification concerns for this site will break down robust control for the whole spatial domain.
- On the other hand when $\theta_m \to \infty$ or equivalently $H_m = 0$, there are no misspecification concerns for this site and the benchmark model can be used.
- The effects of spatial connectivity can be seen in this extreme example. The spatial relation of site m to site n breaks down regulation for both sites. If site m was spatially isolated from n, there would be no problem with regulation at m.

- Assume that $a_{nm} = a_{n-m}$, i.e. the effect that a site m has on site n depends only on the distance between n and m and not on the actual positions of the sites. Thus the operators A, B and C are translation invariant.
- Denote the discrete Fourier transform by $\mathfrak{F}$. The Fourier transform has the property of turning a convolution operator into a multiplication operator, i.e. $\mathfrak{F}(\mathrm{A}u) = \mathfrak{F}(\mathrm{A})\mathfrak{F}(u)$.
- We will use the convention $\hat{u}_k := \mathfrak{F}(u)(k)$ where now k takes values on the dual lattice, $k \in \mathfrak{L}$.
- Applying the Fourier transform $\mathfrak{F}$ and Plancherel theorem

$$\begin{aligned} d\hat{x}_k(t) &= (\hat{a}_k \hat{x}_k(t) + \hat{b}_k \hat{u}_k(t) + \hat{c}_k \hat{v}_k(t))dt + \hat{c}_k \hat{w}_k(t) \\ \sum_n u_n^2 &= \sum_k [\mathfrak{F}(u)(k)]^2 = \sum_k \hat{u}_k^2 \text{ , } \mathrm{P} = pI \text{ and } \mathrm{Q} = qI \end{aligned}$$

$$\min_{\hat{u}_k} \max_{\hat{v}_k} \mathbb{E}_Q \left[\int_0^\infty e^{-rt} p(\hat{x}_k(t))^2 + q(\hat{u}_k(t))^2 - \theta \sum_k (\hat{v}_k(t))^2)$$

subject to the state constraint.

Solution

$$d\hat{x}_k^* = R_k \hat{x}_k^* dt + \hat{c}_k d\hat{w}_k$$

$$R_k := \hat{a}_k - \frac{\hat{b}_k^2 M_{2,k}}{2q} + \frac{\hat{c}_k^2 M_{2,k}}{2\theta}.$$

and $M_{2,k}$ is the solution of

$$\left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q} \right) M_{2,k}^2 + (2\,\hat{a}_k - r)\, M_{2,k} + 2p = 0.$$

The optimal controls are given by the feedback laws

$$\hat{u}_k^* = -\frac{\hat{b}_k M_{2,k}}{2q} \hat{x}_k^*, \qquad \hat{v}_k^* = \frac{\hat{c}_k\, M_{2,k}}{2\theta} \hat{x}_k^*.$$

Let $\mathcal{L}_k : C^2(\mathbb{R}) \to C(\mathbb{R})$ be the generator operator of the diffusion process $\{\hat{x}_k(t)\}$, $t \in \mathbb{R}_+$ defined by

$$(\mathcal{L}_k \Phi)(\hat{x}_k) = (\hat{a}_k \hat{x}_k + \hat{b}_k \hat{u}_k + \hat{c}_k \hat{v}_k) \frac{\partial \Phi}{\partial \hat{x}_k} + \frac{1}{2} \hat{c}_k^2 \frac{\partial^2 \Phi}{\partial \hat{x}_k^2}.$$

The Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation becomes

$$rV_k = \bar{H}\left(\hat{x}_k, \frac{\partial V_k}{\partial \hat{x}_k}, \frac{\partial^2 V_k}{\partial \hat{x}_k^2}\right)$$

$$\bar{H}\left(\hat{x}_k, \frac{\partial \Phi}{\partial \hat{x}_k}, \frac{\partial^2 \Phi}{\partial \hat{x}_k^2}\right) := \inf_{\hat{u}_k} \sup_{\hat{v}_k} \left(p\hat{x}_k^2 + q\hat{u}_k^2 - \theta \hat{v}_k^2 + \mathcal{L}_k \Phi\right),$$

The solution to the primal problem $(\min_{\hat{u}_k} \max_{\hat{v}_k})$ is the same as the solution to the dual $(\max_{\hat{v}_k} \min_{\hat{u}_k})$. **There is no duality gap.**

We will call the qualitative changes in the behavior of the system **hot spots**.
Hot Spots

- **Hot spot of type I:** This is a breakdown of the solution procedure, i.e., a set of parameters where a solution to the above problem does not exist.

Proposition

A hot spot of type I occurs for low enough values of θ. In particular a mode k corresponds to a hot spot of type I if

$$\theta < \theta_{cr} := \frac{p\hat{c}_k^2}{\left(\hat{a}_k - \frac{r}{2}\right)^2 + \frac{p}{q}\hat{b}_k^2}$$

In terms of regulatory objectives this means that concerns about model misspecification make regulation impossible.

- **Hot spot of type II:** This corresponds to the case where the solution exists but may lead to spatial pattern formation, i.e., to spatial instability similar to the Turing instability.

Proposition (Pattern formation for the primal problem)

There exist pattern formation behaviour for the primal problem if there exist modes k such that $R_k > 0$, i.e., if

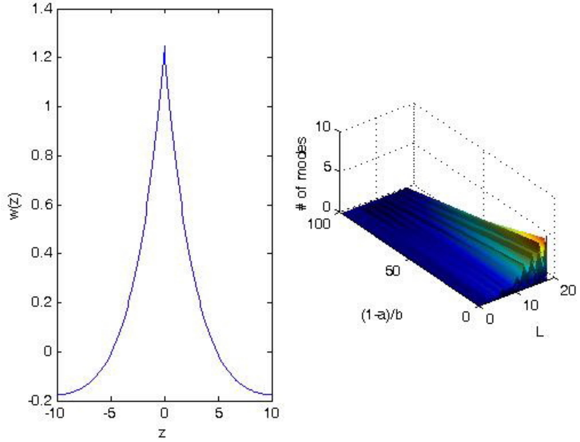
$$\hat{a}_k - \frac{\hat{b}_k^2 M_{2,k}}{2q} + \frac{\hat{c}_k^2 M_{2,k}}{2\theta} > 0.$$

θ may have a destabilizing effect on a mode, since it contributes a positive term to the expression for R_k. This effect is more pronounced the smaller θ is (but of course $\theta > \theta_{cr}$).

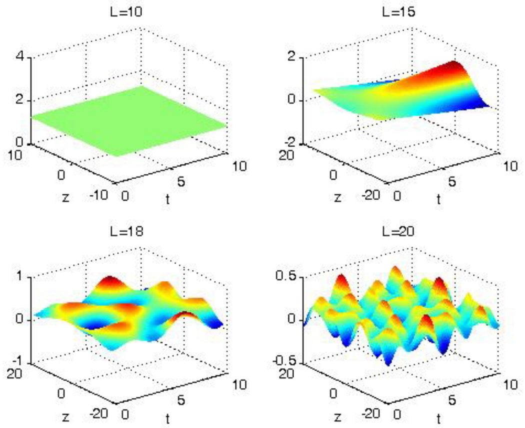
Optimal robustness-induced spatial instability

- Assume that when $|\theta| \to \infty$, then $R_k < 0$ for all modes k. This means that the control problem is stable when there are no concerns about model misspecification.
- If there exists a $\hat{\theta}$, $\theta_{cr} < \left|\hat{\theta}\right| < \infty$ such that $R_k\left(\hat{\theta}\right) > 0$ for some mode k, then concerns about model misspecification induce the emergence of a type II hot spot.
- That is, the regulator's desire for robustness causes the emergence of spatial patterns.
- This results connects uncertainty aversion and the robust control with the emergence of spatial clustering and agglomerations.

We will call this result optimal robustness-induced spatial instability.



The shape of the composite kernel for $\gamma_1 = 0.3$, $C_1 = 2$, $\gamma_2 = 0.1$, $C_2 = -0.75$ (left panel). The number of unstable modes for this choice of kernel function, as a function of a parameter and L (right panel).



Emerging patterns from the instability for different domain sizes.

- **Hot spot of type III:** This corresponds to the case where the cost of robustness becomes more than what it offers, i.e., where the relative cost of robustness may become very large.

$$\frac{1}{V}\frac{\partial V}{\partial \theta} = \frac{1}{M_{2,k}}\frac{\partial M_{2,k}}{\partial \theta}$$

Whenever $\frac{1}{M_{2,k}}\frac{\partial M_{2,k}}{\partial \theta} \to \infty$, then we say that the cost of robustness becomes more expensive than what it offers, and we will call that a hot spot of type **III**.

- We consider a commercial fishery occupying an area that consists of a ring of N cells or sites on a finite lattice.
- Let $x_n(t)$ denote biomass at time $t \geq 0$ and cell $n \in \mathbb{Z}_N$.
- Fish biomass moves from cell to cell. and the spatial movement can be modelled using the discrete Laplacian by a term $D\left[x_{n+1}(t) - 2x_n(t) + x_{n-1}(t)\right]$.
- Let $V_n(t)$ denote the number of identical vessels or firms operating at cell n of the ring, and $h_n(t)$ the harvest rate at cell n per unit time. Thus total harvesting at cell n is $h_n(t)\, V_n(t)$

Spatio-temporal dynamics

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$$dx_n(t) = \left[f(x_n(t)) + \sum_m \alpha_{nm} x_m(t) - h_n(t) V_n(t) \right] dt + \sum_m s_{nm} dw_m, \quad n, m \in \mathbb{Z}_N, \quad x(0, n) = x_0(n)$$

where $f(x)$, $x \geq 0$, is the recruitment rate or growth function for the fishery,with $f(\underline{x}) = f(\bar{x}) = 0$, $f'(x^0) = 0$, $f''(x) < 0$, $0 \leq \underline{x} < x^0 < \bar{x}$. When $f(x)$ is quadratic, growth is logistic

The cost per vessel operating at a cell n for harvesting rate h is determined by a cost function

$$c\left(h_n(t), x_n(t), C_n(t), P_n(t)\right)$$

.

$$C_n(t) := \sum_m \beta_{nm} V_m(t) = \mathrm{B}V,\ P_n(t) := \sum_m \gamma_{nm} h_m(t) = \grave{}\, h(t) \tag{65}$$

(i) $c_h > 0$, $c_{hh} \geq 0$; (ii) $c_x < 0$, which implies resource stock externalities; (iii) $c_C > 0$, which implies crowding externalities due to congestion effects; (iv) $c_P < 0$, which implies knowledge or productivity externalities because harvesting that takes place near cell n helps the development of harvesting knowledge in n and reduces operating costs.

Harvested fish is sold at an exogenous world price p, Profit per vessel at n is defined as:

$$\pi_n(t) = ph_n(t) - c\left(h_n(t), x_n(t), \mathrm{B}V, \grave{}\,h\right) \tag{66}$$

Vessel movements

$$\frac{d}{dt}V_n(t) = \phi\left(\pi_n(t) - \frac{1}{N}\sum_m \pi_m(t)\right)V_n(t), \; \phi > 0 \tag{67}$$

$$V(0,n) = V_0(n) \tag{68}$$

$$\max_{\{h_n(t)\}} \sum_{n \in \mathbb{Z}} \left\{ \int_0^\infty e^{-\rho t} V_n(t) \left[p h_n(t) - c\left(h_n(t), x_n(t), \mathrm{B}V, \Gamma h\right)\right] dt \right. \tag{69}$$

subject to biomass evolution and vessel evolution
The robust control problem to be solved by the regulator is of the general form

$$J(h, v) = \tag{70}$$
$$\max_h \min_v \mathbb{E}_Q \int_0^\infty e^{-rt} \left[\sum_n V_n(t) \pi_Z(t) + \sum_n \theta_n (v_n(t))^2 \right] dt$$

$$\sup_h \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[\int_0^\infty e^{-rt} \sum_n V_n(t) \pi_Z(t) \, dt \right] \tag{71}$$

$$\text{subject to } \mathcal{H}(\mathcal{P}_n \mid Q_n) < H_n, \quad n \in \mathbb{Z} \tag{72}$$

Linearization around a deterministic optimal steady state We linearize the state equations around the state $s^{(0)} := \{x^{(0)}, V^{(0)}, h^{(0)}, v^{(0)} = 0\}$

$$\{x, V, h, v\} = \{x^0, V^0, h^0, 0\} + \epsilon\{x^1, V^1, h^1, v^1\}$$

$$dx^{(1)} = [A^{(1)}x^{(1)} + A^{(2)}V^{(1)} + B^{(1)}h^{(1)} + Sv^{(0)}]dt + Sd\bar{w}$$
$$dV^{(1)} = [A^{(2)} + A^{(2)}V^{(1)} + B^{(2)}h^{(1)}]dt$$

$$\min_u \max_v \mathbb{E}\left[\int_0^\infty e^{-rt} \sum_n \left(p(x_n^{(1)}(t))^2 + \bar{p}(V_n^{(1)}(t))^2 + q(h_n^{(1)}(t))^2 + r(v_n^{(1)}(t))\right.\right.$$

subject to biomass and vessel constraints.
This problem can be solved as a linear quadratic tracking problem

- **Hot spot of type I:** This is a breakdown of the solution procedure, Regulation is not possible with the existing misspecification concerns
- **Hot spot of type II:** Optimal regulation may lead to spatial pattern formation, i.e., to spatial instability similar to the Turing instability. Different levels of biomass, vessels and fishing quotas across locations
- **Hot spot of type III:** This corresponds to the case where the cost of robustness becomes more that what is offering us, i.e., where the relative cost of robustness may become very large.

- We now relax the simplifying (and restrictive) assumptions concerning the translation invariance property of the operators A, B, C as well as the overly restrictive assumption that $\mathrm{P} = pI$ and $\mathrm{Q} = qI$.
- The general form of the problem allows the study of a wider range of economic applications
- The relaxation of translation invariance leads to significant complications, and to the inability to derive solutions in closed form. However, as our subsequent analysis shows the qualitative properties of the solutions

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Theorem

The general LQ robust control problem has a solution for which the optimal controls are of the feedback control form

$$u = -\mathrm{Q}^{-1}\mathrm{B}^*\mathrm{H}^{sym}x, \quad v = \frac{1}{\theta}\mathrm{R}^{-1}\mathrm{C}^*\mathrm{H}^{sym}x,$$

and the optimal state satisfies the Ornstein-Uhlenbeck equation

$$dx = (\mathrm{A} - \mathrm{B}\mathrm{Q}^{-1}\mathrm{B}^*\mathrm{H}^{sym} + \frac{1}{\theta}\mathrm{C}\mathrm{R}^{-1}\mathrm{C}^*\mathrm{H}^{sym})x\,dt + \mathrm{C}dW$$

where H^{sym} *is the solution of the operator Riccati equation*

$$\mathrm{H}^{sym}\mathrm{A} + \mathrm{A}^*\mathrm{H}^{sym} - \mathrm{H}^{sym}\mathrm{E}^{sym}\mathrm{H}^{sym} - r\mathrm{H}^{sym} + \mathrm{P} = 0$$

and $\mathrm{E}^{sym} := \frac{1}{2}(\mathrm{E} + \mathrm{E}^*)$ *is the symmetric part of* $\mathrm{E} := \mathrm{B}\mathrm{Q}^{-1}\mathrm{B}^* - \frac{1}{\theta}\mathrm{C}\mathrm{R}^{-1}\mathrm{C}^*$.

- The operator Riccati equation is the generalization of the quadratic algebraic equation in the case where the operators A, B and C are not translation invariant, and thus amenable to analysis using the Fourier transform.
- When the state space is finite dimensional (i.e., in the case of finite lattices) the operator Riccati equation assumes the form of a matrix Riccati equation.

Proposition

Let $m = ||\mathrm{A}||$ defined as $m = \sup\langle \mathrm{A}x, x\rangle$ and assume that $m < r/2$. Then, for small enough values of $||\mathrm{E}||$ and $||\mathrm{P}||$ the operator Riccati equation admits a unique bounded solution.

- For the existence of a strong solution we need $||\mathrm{E}|| + ||\mathrm{P}|| < d$. This condition breaks down for small enough values of θ, which in fact is the analogue of the **hot spot of Type I** that was obtained before.
- **Hot spots of Type II** will correspond to these eigenfunctions of the operator $\mathcal{R} := \mathrm{A} - \mathrm{BQ}^{-1}\mathrm{B}^*\mathrm{H}^{sym} + \frac{1}{\theta}\mathrm{CR}^{-1}\mathrm{C}^*\mathrm{H}^{sym}$ that have positive eigenvalues.

- Assume for simplicity that C is diagonal and that the spatial domain is finite so that $\theta = (\theta_0, ..., \theta_{N-1})$ is the vector of local misspecification concerns.
- The low θ's will correspond to locations with the higher concerns.
- If one or more of these low θ's are such that the "smallness" condition on $||\mathrm{E}||$ and $||\mathrm{P}||$ is violated, then local concerns will cause global regulation to break down. **Hot spot of Type I.**
- If the low θ's are such that the operator $\mathcal{R}$ has positive eigenvalues, then local concerns may induce global spatial clustering. **Hot spot of Type II.**
- Even in the simple (2×2) case it is not possible to obtain the solution of the Riccati equation in closed form, as we did in the special case where the operators are translation invariant.

The mean field for the optimal state is

$$dx = \left[\mathrm{A} - \mathrm{B}\mathrm{Q}^{-1}\mathrm{B}^{*}\mathrm{H}^{sym} + \frac{1}{\theta}\mathrm{C}\mathrm{R}^{-1}\mathrm{C}^{*}\mathrm{H}^{sym} \right] x = \mathcal{R}x$$

Assume that matrix A is invertible but matrix $\mathcal{R}$ which embodies optimization and misspecification concerns is not invertible. In this case the steady state equation $0 = \mathcal{R}x$ will have more than one solution. This means that there will be vectors $x \neq \mathbf{0}$ that will satisfy $0 = \mathcal{R}x$. These vectors will be $\ker(\mathcal{R})$. If $\ker(\mathcal{R})$ consists of vectors with spatially non-uniformity, then pattern formation emerges.

- This pattern formation mechanism is however a non-Turing mechanism.

- Distance-dependent utility. Models of travel behavior where the impact of distance on trip preferences underlies the choice of an individual to consume at locations which are away from his/her current location.
- The distance-dependent utility relates to the concept of spatial discounting.
- A representative consumer is located at $n = 0, 1....N-1$. Each location is characterized by a stock of natural capital $x_n(t)$ which generates environmental services that can be consumed only in situ.

Consumption at location n is the sum of consumption of all individuals or $u_n(t) = \sum_{m=0}^{N-1} u_{nm}(t)$, where $u_{nm}(t)$ is the consumption of services at location n of an individual located at location $m = 0, 1, ...N-1$.

$$dx_n(t) = \sum_{m=0}^{N-1} \left[\alpha_{nm} x_m(t) - \gamma_{nm} u_n(t)\right] dt + \sum_{m=0}^{N-1} c_{nm}(t)\, dw_m,$$

$$\max_{\{u_{nm}\}} \; -\left[\sum_{m=0}^{n-1} \beta_{nm}\left(u_{nm}(t) - b_{nm}(t)\right)^2 + I_n(t) - \sum_{m=0}^{N-1} p_m(t)\, u_{nm}(t)\right] \text{ for all } n$$

$$\beta_{nm} \equiv \beta_{n-m} = \beta_{m-n} \equiv \beta_{mn}$$

Individual demand curves for consumption at each location

$$2\beta_{nm} b_{nm}(t) - 2\beta_{nm} u_{nm}(t) = p_m(t)$$

The aggregate demand at location m and time t, $u_n(t) := \sum_{n=0}^{N-1} u_{nm}(t)$

$$u_m(t) = \sum_{n=0}^{N-1} b_{nm}(t) - \left(\sum_{n=0}^{N-1} \frac{1}{2\beta_{nm}}\right) p_m(t) =: B_{0m}(t) - B_{1m}(t)$$

Site-dependent misspecification concerns

$$\min_{u}\max_{v}\mathbb{E}_P\left[\int_0^{\infty} e^{-rt}\sum_{n=0}^{N-1}\left[\langle(\mathrm{B}U)(t),U\rangle-\theta_n\langle(\mathrm{R}v),(v)\rangle\right]dt\right]$$

subject to $dx=(\mathrm{A}x+\mathrm{Z}u+\mathrm{C}v)\,dt+\mathrm{C}dw$.

Welfare: $\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\beta_{nm}\left(u_{nm}(t)-b_{nm}(t)\right)^2$

Optimal supply of services for an individual located at m and consuming at n

$$u^*_{nm}(t)=-\mathrm{B}^{-1`*}\mathrm{H}^{\mathrm{sym}}x_n(t)$$

Local equilibrium price at n will be

$$p^*_n(t)=B_{0n}(t)-\frac{1}{B_{1n}(t)}\left(\sum_{m=0}^{N-1}u^*_{nm}(t)\right).$$

- **Hot spot of type I:** Regulation breaks down for small θ. This means that because the regulator has very strong concerns about possible model misspecifications at specific site(s), the regulator can not set up markets for consumption of in situ services.
- **Hot spot of type II:** The regulator, due to misspecification concerns, allows a non-homogeneous spatial pattern of the stocks to emerge. There exists a system of local prices that supports the pattern.
- **Hot spot of type III:** The cost of controlling the in situ consumption at each location becomes very high in terms of deviations from the desired bliss points due to misspecification concerns.