Introduction to Optimization

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Contents

Introduction

Optimality conditions
Optimization problems

In order to formulate an optimization problem, the following concepts must be very clear:

- decision variables
- restrictions
- objective function
Optimization problems

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Decision variables

- One or more variables on which we can decide (harvesting rate or effort, level of investment, distribution of tasks, parameters)
- Objective: to find the best value for the decision variable
- We denote by $x$ the decision variable
- $x$ can be a number, a vector, a sequence, a function, ........
- In a general framework, we denote by $X$ the set where the decision variable is ($x \in X$)
Optimization problems

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Warning

This tutorial is on *Continuous Optimization*

That means that decision variables are continuous variables

What that means?

Firstly, the set $X$ where the decision variable belongs, is a (linear) vector space
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Optimization problems

Restrictions

- **Constraints that the decision variable has to satisfy**
  - If for a certain value of the decision variable the restrictions are satisfied, we say that it is a feasible solution.
  - In a general framework, we denote by $C \subseteq X$ the set of all feasible solutions.
  - If we have two decision variables, $x_1$ and $x_2$ and they have to satisfy the following constraints: $x_1 \geq 0$, $x_2 \geq 0$, $2x_1 + 3x_2 \leq 5$, we denote $C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, 2x_1 + 3x_2 \leq 5\}$.
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Objective function

It is the mathematical representation for measuring the *goodness* of values for the decision variable

This representation is done through a function

\[ f : X \rightarrow \mathbb{R} \]
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Optimization problems

To find the *best* value for the decision variable from all feasible solutions

To find $\bar{x} \in C \subseteq X$ such that

$$f(\bar{x}) \leq f(x) \quad (\text{or} \quad f(\bar{x}) \geq f(x))$$

for all $x \in C$

\[
\begin{align*}
\min_{x \in X} f(x) \quad \text{or} \quad \max_{x \in X} f(x)
\end{align*}
\]

subject to

$\begin{align*}
x \in C
\end{align*}$
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Optimization problems
Example: maximizing a rectangular surface

With a given fence, to enclose the largest rectangular area

- **Objective**: maximize the rectangular surface
- Decision variables: lengths of the rectangular figure
- Constraints:
  - positive lengths
  - length of the fence
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Example: maximizing a rectangular surface

- **Decision variables**: $a$ (height) and $b$ (width)

- Objective: maximize $s(a, b) = ab$

- Constraints:
  - Positive lengths: $a > 0, b > 0$
  - Length of fence ($L > 0$): $2(a + b) = L$

- $b = \frac{L}{2} - a$

- New objective function $f(a) = a \left( \frac{L}{2} - a \right)$

- Rewrite restrictions: $a > 0, a < \frac{L}{2}$
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\[
\begin{align*}
\max_{a \in \mathbb{R}} f(a) &= a \left( \frac{L}{2} - a \right) \\
\text{subject to} & \\
& \quad a > 0 \\
& \quad a < \frac{L}{2}
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Optimization problems

Example: Optimizing a portfolio

A firm wishes to maximize the utilities of its portfolio

- **Objective**: maximize profits
- Decision variables: amount to invest in each fund
- Constraints:
  - nonnegative investments
  - budget restrictions
  - minimal or maximal bounds for investments
Optimization problems

Example: Optimizing a portfolio

A firm wishes to maximize the utilities of its portfolio

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Example: Optimizing a portfolio

- **Decision variables**: $x_1, x_2, \ldots, x_n$, where $x_j$ is the quantity to invest in the fund $j$

- **Objective**: maximize

\[
(1 + r_1)x_1 + (1 + r_2)x_2 + \ldots + (1 + r_n)x_n
\]

where $r_j$ is the rentability of the fund $j$

- **Restrictions**:
  - Non-negative investments: $x_j \geq 0$, $j = 1, 2, \ldots, n$
  - Budget restriction:

\[
x_1 + x_2 + \ldots + x_n \leq B
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- Minimal/maximal bounds for investments:

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a_j \leq x_j \leq b_j \quad \text{for} \quad j = 1, 2, \ldots, n
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Optimization problems
Example: Optimizing a portafolio

- Decision variables: $x_1, x_2, \ldots, x_n$, where $x_j$ the quantity to invest in the fund $j$
- Objective: maximize
  \[(1 + r_1)x_1 + (1 + r_2)x_2 + \ldots + (1 + r_n)x_n\]
  where $r_j$ is the rentability of the fund $j$
- Restrictions:
  - Non-negative investments: $x_j \geq 0 \quad j = 1, 2, \ldots, n$
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    \[a_j \leq x_j \leq b_j \quad \text{for } j = 1, 2, \ldots, n\]
Optimization problems

Example: Optimizing a portfolio

\[
\begin{align*}
\max_{x_1, x_2, \ldots, x_n \in \mathbb{R}} & \quad (1 + r_1)x_1 + (1 + r_2)x_2 + \ldots + (1 + r_n)x_n \\
\text{subject to} & \\
& x_j \geq 0 \quad j = 1, 2, \ldots, n \\
& x_1 + x_2 + \ldots + x_n \leq B \\
& a_j \leq x_j \leq b_j \quad j = 1, 2, \ldots, n
\end{align*}
\]
Optimization problems
Example: Least square problem

Given points on the plane, to find the line of best fit through these points

- **Objective**: To minimize the (squared) error between a line and the points

- Decision variables: the parameters of a line
Optimization problems

Example: Least square problem

Given points on the plane, to find the \textit{line of best fit} through these points

- Objective: To minimize the (squared) error between a line and the points

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Optimization problems

Example: Least square problem

Given points

\[(x_1, y_1); (x_2, y_2); \ldots; (x_n, y_n)\]

to find the line of best fit through these points

- **Decision variables (the parameters of a line):** \(m\) and \(n\), where the expression of a line in the plane is

\[y = mx + n\]

- **Objective:** To minimize

\[
\sum_{k=1}^{n} (mx_k + n - y_k)^2
\]
Introduction

Optimality conditions

Optimization problems

Example: Least square problem

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Optimization problems

Example: Least square problem

\[
\max_{m, n \in \mathbb{R}} \sum_{k=1}^{n} (mx_k + n - y_k)^2
\]
Optimization problems

Example: Harvesting of a renewable resource

To maximize the benefit from the harvesting of a natural resource

- **Objective**: maximize present utilities

- Decision variables: harvesting levels (or effort) at each period

- Constraints:
  - nonnegative harvesting
  - biology
  - relation between harvesting and the amount of the resource
  - environmental constraints
Optimization problems

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Example: Harvesting of a renewable resource

\[
x(t + 1) = F(x(t)) - h(t) \quad t = t_0, t_0 + 1, \ldots, T - 1
\]

\[
x(t_0) = x_0 \text{ given (e.g. the current state of the resource)}
\]

where

- \( x(t) \geq 0 \) is the level of the resource at period \( t \)
- \( h(t) \geq 0 \) is the harvesting at period \( t \)
- \( F : \mathbb{R} \rightarrow \mathbb{R} \) is the biological growth function of the resource
Optimization problems

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Example: Harvesting of a renewable resource

The total benefits can be represented by

\[
B = \sum_{t=t_0}^{T-1} \rho^{t-t_0} U(x(t), h(t))
\]

where

- \( U(x, h) \) is the instantaneous profit if we have \( x \) and we harvest \( h \)
- \( 0 \leq \rho \leq 1 \) is a discount factor
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Example: Harvesting of a renewable resource

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$$B = \sum_{t=t_0}^{T-1} \rho^{t-t_0} U(x(t), h(t))$$

where

- $U(x, h)$ is the instantaneous profit if we have $x$ and we harvest $h$

- $0 \leq \rho \leq 1$ is a discount factor
Optimization problems

Example: Harvesting of a renewable resource

\[
\begin{align*}
\max_{h(t_0), h(t_0+1), \ldots, h(T-1)} & \sum_{t=t_0}^{T-1} \rho^{(t-t_0)} U(x(t), h(t)) \\
\text{subject to} & \\
x(t + 1) &= F(x(t)) - h(t) \quad t = t_0, t_0 + 1, \ldots, T - 1 \\
x(t_0) &= x_0 \\
x(t) &\geq 0 \quad t = t_0 + 1, t_0 + 2, \ldots, T \\
0 &\leq h(t) \leq F(x(t)) \quad t = t_0, t_0 + 1, \ldots, T - 1
\end{align*}
\]
Optimization problems

Important remark: min is equivalent to max

We can restrict our study only to minimization problems

To find \( \bar{x} \in C \subseteq X \) such that

\[
f(\bar{x}) \leq f(x) \quad \text{for all } x \in C
\]

\[
\min_{x \in X} f(x) \quad \text{subject to} \quad x \in C
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Optimization problems

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\begin{align*}
\max_{x \in X} & \quad f(x) \\
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\end{align*}
\]

is equivalent to problem

\[
\begin{align*}
\min_{x \in X} & \quad -f(x) \\
\text{subject to} & \quad x \in C
\end{align*}
\]

In the following sense:

- \(x\) is solution of \((P_{\text{max}})\) if and only if it is solution of \((P_{\text{min}})\)

- The optimal value of \((P_{\text{max}})\) is the negative of the optimal value of \((P_{\text{min}})\)
Optimization problems

Important remark: min is equivalent to max

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(P_{\text{max}}) \begin{cases}
    \max_{x \in X} f(x) \\
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Optimization problems

Existence of solutions

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\begin{align*}
\min_{x \in X} f(x) \\
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\end{align*}
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When \((P)\) has solution?

- If \([a, b] = C \subseteq X = \mathbb{R}\)
- When \(X = \mathbb{R}^n\), there exists at least one solution if \(C\) is closed and bounded
Optimization problems
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Contents

Introduction

Optimality conditions
Optimality conditions

\[
(P) \begin{cases} 
\min_{x \in X} f(x) \\
\text{subject to} \\
x \in C 
\end{cases}
\]

Optimality conditions are mathematical expressions: equations or system of equations, system of inequalities, ordinary differential equations, partial differential equations.

Solutions of the mathematical problems obtained from optimality conditions are related with the solutions of the optimization problem.
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The idea is to obtain optimality conditions and then to solve the associated mathematical problems (analytically or numerically)
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Necessary conditions

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\begin{align*}
\min_{x \in X} f(x) \\
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\end{align*}
\]

(P)

- Suppose that optimality conditions of this problem are represented by a system of equations (S).

- We say that (S) is a necessary condition if a solution of (P) is a solution of (S).

- The idea is to find solutions of (S) in order to have a list of candidates for the solutions of (P).

- If (S) is solved only for one point, then it is solution of (P) or (P) does not have solution.

- Warning: A solution of (S) may be not a solution of (P).
Optimality conditions

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Optimality conditions

Necessary condition: unrestricted case

\[
(P) \left\{ \min_{x \in \mathbb{R}} f(x) \right\}
\]

where \( f : \mathbb{R} \rightarrow \mathbb{R} \)

- A necessary condition is \( f'(x) = 0 \)

\[
f'(x) = \lim_{t \to 0} \frac{f(x + t) - f(x)}{t}
\]

That means: If \( \bar{x} \) is a solution of \( (P) \) then \( f(\bar{x}) = 0 \)

- If we know the expression of \( f' \) we can solve the equation \( f'(x) = 0 \) in order to have candidates of solutions

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Optimality conditions

Necessary conditions: restrictions

\[(P) \begin{cases} \min_{x \in \mathbb{R}} f(x) \\ a \leq x \leq b \end{cases}\]

where \( f : \mathbb{R} \rightarrow \mathbb{R} \)

- At a point \( x \) observe that \( f \) increase in the sense of (the sign of) \( f'(x) \)

- There are three possibilities:
  - an interior point \( a < \bar{x} < b \) is a solution. Then \( f'(\bar{x}) = 0 \)
  - \( \bar{x} = a \) is a solution. Then \( f'(\bar{x}) = f'(a) \geq 0 \)
  - \( \bar{x} = b \) is a solution. Then \( f'(\bar{x}) = f'(b) \leq 0 \)

- Observe that if \( f'(a) > 0 \) and \( f'(b) < 0 \), then there exists \( a < \bar{x} < b \) such that \( f'(\bar{x}) = 0 \)
Optimality conditions

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\[ \min_{x \in \mathbb{R}} f(x) \]
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Optimality conditions

Necessary conditions: restrictions

A necessary condition of the problem

\[(P) \quad \left\{ \begin{array}{l}
\min_{x \in \mathbb{R}} f(x) \\
\quad a \leq x \leq b
\end{array} \right.\]

is

\[(S') \quad \left\{ \begin{array}{l}
\text{Find } x, \alpha, \beta \text{ such that } \\
f'(x) - \alpha + \beta = 0 \\
x \in [a, b] \\
\alpha, \beta \geq 0 \\
\alpha(x - a) = 0 \\
\beta(x - b) = 0
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Optimality conditions

Global vs local minimum

\[
(P) \begin{cases}
\min_{x \in X} f(x) \\
\text{subject to}
\end{cases}
\]

- We say that \( \bar{x} \in C \) is a global minimum, if it is a solution of \((P)\)

- We say that \( \bar{x} \in C \) is a local minimum, if for every \( x \in C \) close enough, one has \( f(\bar{x}) \leq f(x) \)

- A global optimum is a local optimum
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Sufficient condition: unrestricted case

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- The idea is to find solutions of \((S)\) in order to have solutions of \((P)\)

- In the general case, this is very difficult

- It is much realistic to have that solutions of \((S)\) are local solutions of \((P)\)

- A (local) sufficient condition to \((P)\) is \(f''(x) = 0\) and \(f'''(x) > 0\)
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Sufficient condition

If we find $\bar{x} \in \mathbb{R}$ such that

\[
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\end{align*}$$

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To formulate an optimization problem, the following concepts must be clear:

- Decision variable(s): $x \in X$
- Objective function: $f : X \rightarrow \mathbb{R}$
- Restrictions: $C \subset X$

A maximization problem is equivalent to a minimization problem (deal with $-f(x)$ instead of $f(x)$)
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Optimality conditions are mathematical systems

The idea is to solve these systems in order to:

- have candidates of solutions (if the system is a necessary condition)
- obtain a local solution (if the system is a sufficient condition)
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Conclusions

Optimality conditions are mathematical systems

The idea is to solve these systems in order to:

- have candidates of solutions (if the system is a necessary condition)
- obtain a local solution (if the system is a sufficient condition)
Next part

\[
\begin{aligned}
\min_{x \in X} f(x) \\
\text{subject to} \\
x \in C
\end{aligned}
\]  

- Several decision variables \( x_1, x_2, \ldots, x_n \). That is

\[
x = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} \in \mathbb{R}^n = X
\]

- Necessary and sufficient conditions

- Why the convexity (or concavity) is relevant?
Next part

\[
(P) \begin{cases}
\min_{x \in X} f(x) \\
\text{subject to} \\
x \in C
\end{cases}
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- Several decision variables \(x_1, x_2, \ldots, x_n\). That is

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