Introduction to Optimization

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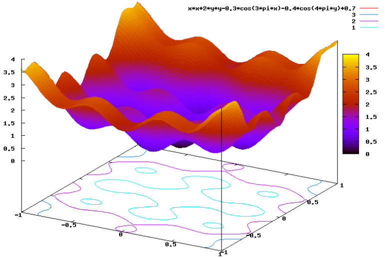

Mathematics of Bio-Economics (MABIES) - IHP

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Contents

Optimization problems

In order to formulate an optimization problem, the following concepts must be very clear:

- decision variables
- restrictions
- objective function

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Decision variables

- One or more variables on which we can decide (harvesting rate or effort, level of investment, distribution of tasks, parameters)
- Objective: to find the *best* value for the decision variable
- We denote by x the decision variable
- x can be a number, a vector, a sequence, a function, ...
- In a general framework, we denote by X the set where the decision variable is ($x \in X$)

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This tutorial is on *Continuous Optimization*

That means that decision variables are continuous variables

What that means?

Firstly, the set X where the decision variable belongs, is a (linear) vector space

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Restrictions

- Constraints that the decision variable has to satisfy
- If for a certain value of the decision variable the restrictions are satisfied, we say that it is a feasible solution
- In a general framework, we denote by $C \subseteq X$ the set of all feasible solutions
- If we have two decision variables, x_1 and x_2 and they have to satisfy the following constraints: $x_1 \geq 0,\ x_2 \geq 0,$ $2\,x_1 + 3\,x_2 \leq 5$, we denote

$$C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ 2\,x_1 + 3x_2 \leq 5\}$$

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It is the mathematical representation for measuring the *goodness* of values for the decision variable

This representation is done through a function

$$f : X \longrightarrow \mathbb{R}$$

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Optimization problems

To find the *best* value for the decision variable from all feasible solutions

To find $\bar{x} \in C \subseteq X$ such that

$$f(\bar{x}) \leq f(x) \quad (\text{or } f(\bar{x}) \geq f(x)) \quad \text{for all } x \in C$$

$$\begin{cases} \min\limits_{x \in X} f(x) & \left(\text{or } \max\limits_{x \in X} f(x)\right) \\ \text{subject to} & \\ x \in C & \end{cases}$$

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Optimization problems

Example: maximizing a rectangular surface

With a given fence, to enclose the largest rectangular area

- Objective: maximize the rectangular surface
- Decision variables: lengths of the rectangular figure
- Constraints:
  - positive lengths
  - length of the fence

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Example: maximizing a rectangular surface

- Decision variables: a (height) and b (width)
- Objective: maximize $s(a, b) = a\,b$
- Constraints:
  - positive lengths: $a > 0, \; b > 0$
  - length of fence ($L > 0$): $2(a + b) = L$
- $b = \frac{L}{2} - a$
- New objective function $f(a) = a \; \left(\frac{L}{2} - a\right)$
- Rewrite restrictions: $a > 0, \; a < \frac{L}{2}$

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Example: maximizing a rectangular surface

$$\left\{ \begin{array}{l} \max\limits_{a \in \mathbb{R}} f(a) = a \ \left(\frac{L}{2} - a\right) \\ \text{subject to} \\ a > 0 \\ a < \frac{L}{2} \end{array} \right.$$

Optimization problems

Example: Optimizing a portafolio

A firm wishes to maximize the utilities of its portafolio

- Objective: maximize profits
- Decision variables: amount to invest in each fund
- Constraints:
  - nonnegative investments
  - budget restrictions
  - minimal or maximal bounds for investments

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Example: Optimizing a portafolio

- Decision variables: $x_1, x_2, \ldots, x_n$, where x_j the quantity to invest in the fund j
- Objective: maximize

$$(1 + r_1)x_1 + (1 + r_2)x_2 + \ldots + (1 + r_n)x_n$$

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\right.
$$

Given points on the plane, to find the *line of best fit* through these points

- Objective: To minimize the (squared) error between a line and the points
- Decision variables: the parameters of a line

Optimization problems

Example: Least square problem

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Example: Least square problem

Given points

$$(x_1, y_1); (x_2, y_2); \ldots ; (x_n, y_n)$$

to find the *line of best fit* through these points

- Decision variables (the parameters of a line): m and n, where the expression of a line in the plane is

$$y = mx + n$$

- Objective: To minimize

$$\sum_{k=1}^{n} (mx_k + n - y_k)^2$$

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Example: Least square problem

$$\max_{m,\, n \in \mathbb{R}} \sum_{k=1}^{n} (mx_k + n - y_k)^2$$

Optimization problems

Example: Harvesting of a renewable resource

To maximize the benefit from the harvesting of a natural resource

- Objective: maximize present utilities
- Decision variables: harvesting levels (or effort) at each period
- Constraints:
  - nonnegative harvesting
  - biology
  - relation between harvesting and the amount of the resource
  - environmental constraints

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Example: Harvesting of a renewable resource

$$x(t+1) = F(x(t)) - h(t) \qquad t = t_0, t_0+1, \ldots, T-1$$

$$x(t_0) = x_0 \text{ given (e.g. the current state of the resource)}$$

where

- $x(t) \geq 0$ is the level of the resource at period t
- $h(t) \geq 0$ is the harvesting at period t
- $F : \mathbb{R} \longrightarrow \mathbb{R}$ is the biological growth function of the resource

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Example: Harvesting of a renewable resource

The total benefits can be represented by

$$B = \sum_{t=t_0}^{T-1} \rho^{(t-t_0)} U(x(t), h(t))$$

where

- $U(x, h)$ is the instantaneous profit if we have x and we harvest h
- $0 \leq \rho \leq 1$ is a descount factor

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\begin{cases}
\displaystyle\max_{h(t_0),h(t_0+1),\dots,h(T-1)} \sum_{t=t_0}^{T-1} \rho^{(t-t_0)} U(x(t),h(t)) \\
\text{subject to} \\
x(t+1) = F(x(t)) - h(t) \qquad t = t_0, t_0+1, \dots, T-1 \\
x(t_0) = x_0 \\
x(t) \geq 0 \qquad t = t_0+1, t_0+2, \dots, T \\
0 \leq h(t) \leq F(x(t)) \qquad t = t_0, t_0+1, \dots, T-1
\end{cases}
$$

Optimization problems

Important remark: min is equivalent to max

We can restrict our study only to minimization problems

To find $\bar{x} \in C \subseteq X$ such that

$$f(\bar{x}) \leq f(x) \qquad \text{for all } x \in C$$

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Optimization problems

Important remark: min is equivalent to max

We can restrict our study only to minimization problems

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$$(P_{min}) \begin{cases} \min\limits_{x \in X} \ -f(x) \\ \text{subject to} \\ x \in C \end{cases}$$

In the following sense:

- $\bar{x}$ is solution of (P_{max}) if and only if it is solution of (P_{min})
- The optimal value of (P_{max}) is the negative of the optimal value of (P_{min})

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Optimization problems

Existence of solutions

$$(P)\begin{cases}\min\limits_{x\in X} f(x)\\ \text{subject to}\\ x\in C\end{cases}$$

When (P) has solution?

- If $[a,b]=C\subseteq X=\mathbb{R}$
- When $X=\mathbb{R}^n$, there exists at least one solution if C is closed and bounded

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Optimality conditions

$$(P)\begin{cases}\min\limits_{x\in X} \; f(x)\\ \text{subject to}\\ x\in C\end{cases}$$

Optimality conditions are mathematical expressions: equations or system of equations, system of inequalities, ordinary differential equations, partial differential equations

Solutions of the mathematical problems obtained from optimality conditions are related with the solutions of the optimization problem

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Necessary conditions

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- Suppose that optimality conditions of this problem are represented by a system of equations (S)
- We say that (S) is a necessary condition if a solution of (P) is a solution of (S)
- The idea is to find solutions of (S) in order to have a list of candidates for the solutions of (P)
- If (S) is solved only for one point, then it is solution of (P) or (P) does not have solution
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Optimality conditions

Necessary condition: unrestricted case

$$(P)\left\{\ \min_{x\in\mathbb{R}} f(x)\right.$$

where $f:\mathbb{R}\longrightarrow\mathbb{R}$

- A necessary condition is $f'(x)=0$ where

$$f'(x)=\lim_{t\to 0}\frac{f(x+t)-f(x)}{t}$$

- That means: If $\bar{x}$ is a solution of (P) then $f(\bar{x})=0$
- If we know the expression of f' we can solve the equation $f'(x)=0$ in order to have candidates of solutions
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Necessary conditions: restrictions

$$(P)\begin{cases}\displaystyle\min_{x\in\mathbb{R}} f(x)\\ a\le x\le b\end{cases}$$

where $f:\mathbb{R}\longrightarrow\mathbb{R}$

- At a point x observe that f increase in the sense of (the sign of) $f'(x)$
- There are three posibilities:
  - an interior point $a<\bar{x}<b$ is a solution. Then $f'(\bar{x})=0$
  - $\bar{x}=a$ is a solution. Then $f'(\bar{x})=f'(a)\ge 0$
  - $\bar{x}=b$ is a solution. Then $f'(\bar{x})=f'(b)\le 0$
- Observe that if $f'(a)>0$ and $f'(b)<0$, then there exists $a<\bar{x}<b$ such that $f'(\bar{x})=0$

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A necessary condition of the problem

$$(P) \begin{cases} \min_{x \in \mathbb{R}} f(x) \\ a \leq x \leq b \end{cases}$$

is

$$(S) \begin{cases} \text{Find } x, \ \alpha, \ \beta \text{ such that} \\ f'(x) - \alpha + \beta = 0 \\ x \in [a, b] \\ \alpha, \ \beta \geq 0 \\ \alpha(x - a) = 0 \\ \beta(x - b) = 0 \end{cases}$$

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Optimality conditions

Global vs local minimum

$$(P)\begin{cases} \min\limits_{x\in X} \; f(x) \\ \text{subject to} \\ x \in C \end{cases}$$

- We say that $\bar{x} \in C$ is a global minimum, if it is a solution of (P)
- We say that $\bar{x} \in C$ is a local minimum, if for every $x \in C$ close enough, one has $f(\bar{x}) \leq f(x)$
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- In the general case, this is very difficult
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- A (local) sufficient condition to (P) is $f'(x) = 0$ and $f''(x) > 0$

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$$(P)\begin{cases}\min\limits_{x\in X}\ f(x)\\ \text{subject to}\\ x\in C\end{cases}$$

- To formulate an optimization problem, the following concepts must be clear:
  - Decision variable(s): $x \in X$
  - Objective function: $f : X \longrightarrow \mathbb{R}$
  - Restrictions: $C \subset X$
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$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n = X$$

- Necessary and sufficient conditions
- Why the convexity (or concavity) is relevant?

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