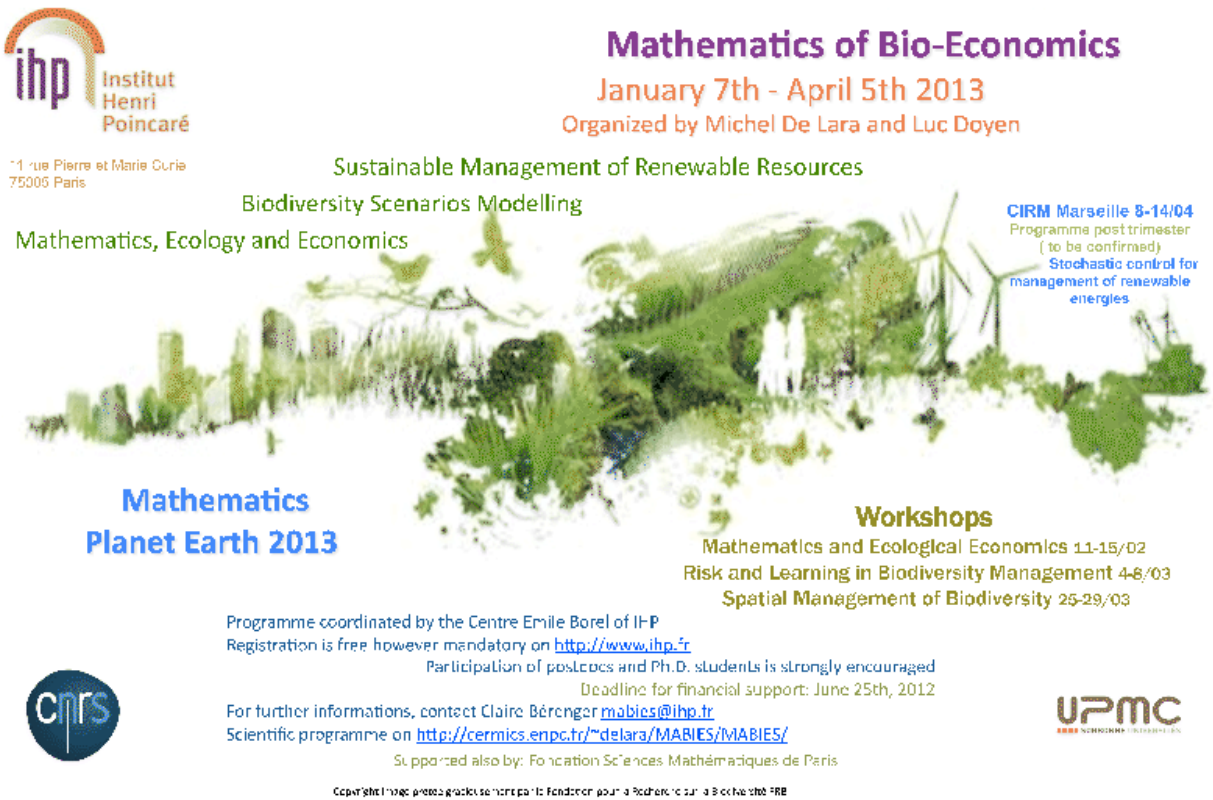
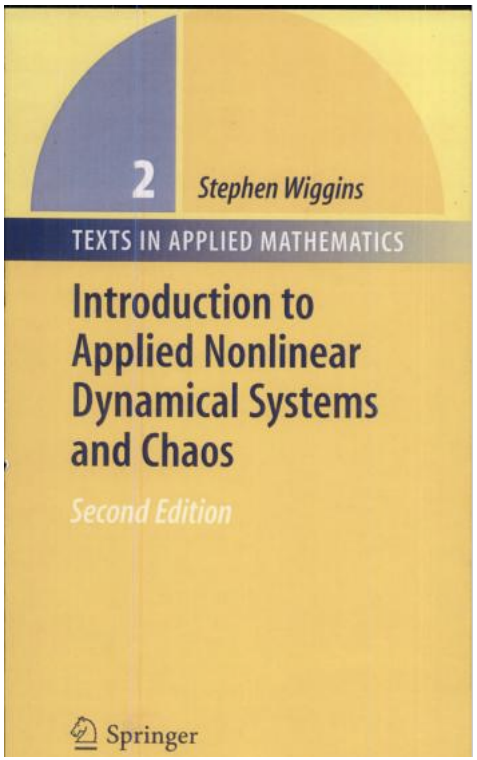
MABIES Program at IHP

Institut Henri Poincaré
quarterly thematic program MABIES
Mathematics of Bio-Economics
2013, January 7 - April 5

Michel De Lara, Luc Doyen

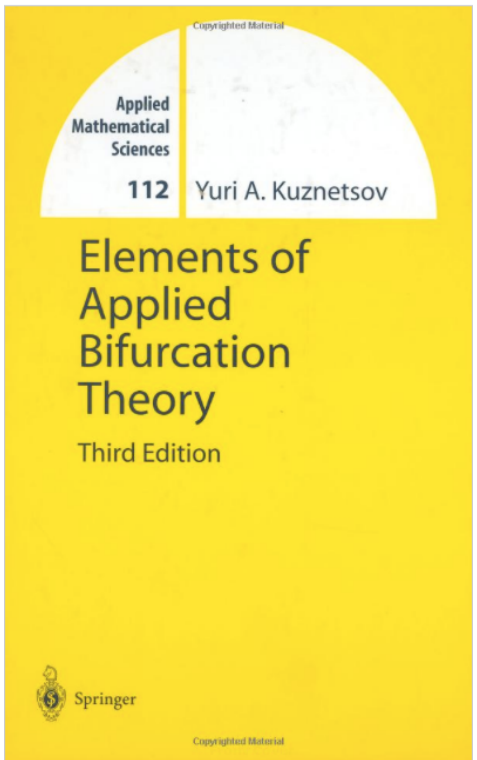


Book 1 (to read for the topic)



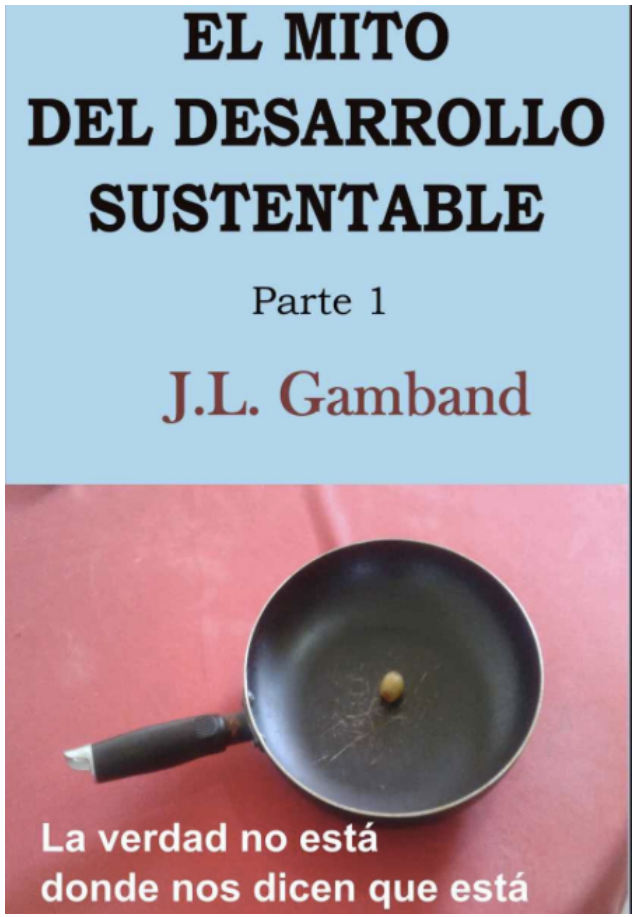
Stephen Wiggins book for Nonlinear Dynamics

Book 2 (to read for the topic)



Yuri Kuznetsov book for Bifurcations

Book 3 (to read for the trimester)



J.L. Gambland book (in Spanish) for breaking myths?

Why do we need mathematical models?

- uncontrolled-model prediction (dynamics)
- control through some optimality criterion (or not, just control)
- controlled-model prediction

Prey-Predator systems

$$\dot{S} = \gamma S\left(1 - \frac{S}{K}\right) - \frac{S^\alpha}{a_1 + S^\beta} m_1 x_1 - \frac{S^\alpha}{a_2 + S^\beta} m_2 x_2$$

$$\dot{x_1} = \frac{S^\alpha}{a_1 + S^\beta} m_1 x_1 - d_1 x_1$$

$$\dot{x_2} = \frac{S^\alpha}{a_2 + S^\beta} m_2 x_2 - d_2 x_2$$

Dengue disease (Aedes Aegypti)

$$
\begin{aligned}
\dot{y}_1 &= \mu N - \beta y_1 \frac{y_5}{y_4 + y_5} - \mu y_1, \\
\dot{y}_2 &= \beta y_1 \frac{y_5}{y_4 + y_5} - (\mu + \theta)\, y_2, \\
\dot{y}_3 &= \theta y_2 - \mu y_3, \\
\dot{y}_4 &= \omega y_6 - \lambda \frac{y_2}{N} y_4 - \delta y_4, \\
\dot{y}_5 &= \lambda \frac{y_2}{N} y_4 - \delta y_5, \\
\dot{y}_6 &= \phi\,(y_4 + y_5) \left(1 - \frac{y_6}{\gamma y_7}\right) - (\omega + \epsilon)\, y_6, \\
\dot{y}_7 &= v y_7 \left(1 - \frac{y_7}{\kappa}\right)
\end{aligned}
$$

Sustainable development

Resource dynamics:

$$\frac{dS}{dt} = \rho\left(\frac{S}{k} - 1\right)\left(1 - \frac{S}{K}\right)S - \alpha\beta LS$$

- Natural Growth
- Harvesting rate

Population dynamics:

$$\frac{dL}{dt} = \left(\gamma\lambda(1-\beta)^{\delta}L^{\delta-1} + \phi\alpha\beta S - \sigma\right)L$$

- Per-capita income
- Poverty line

Dynamical Systems
Introduction & Bifurcations

Gerard Olivar

ABC Dynamics - PCI

http://www.manizales.unal.edu.co/gta/abcdynamics
http://www.manizales.unal.edu.co/gta/PCI

CeiBA Complejidad

Universidad Nacional de Colombia, Sede Manizales

Mathematics of Bio-Economics at IHP
January 2013

Contents

- Today: very basics
- Tomorrow: bifurcations

Outline for today

1. Introductory motivation
2. Basics
3. Nonlinear tools
4. Poincaré maps
5. Examples

A word about models

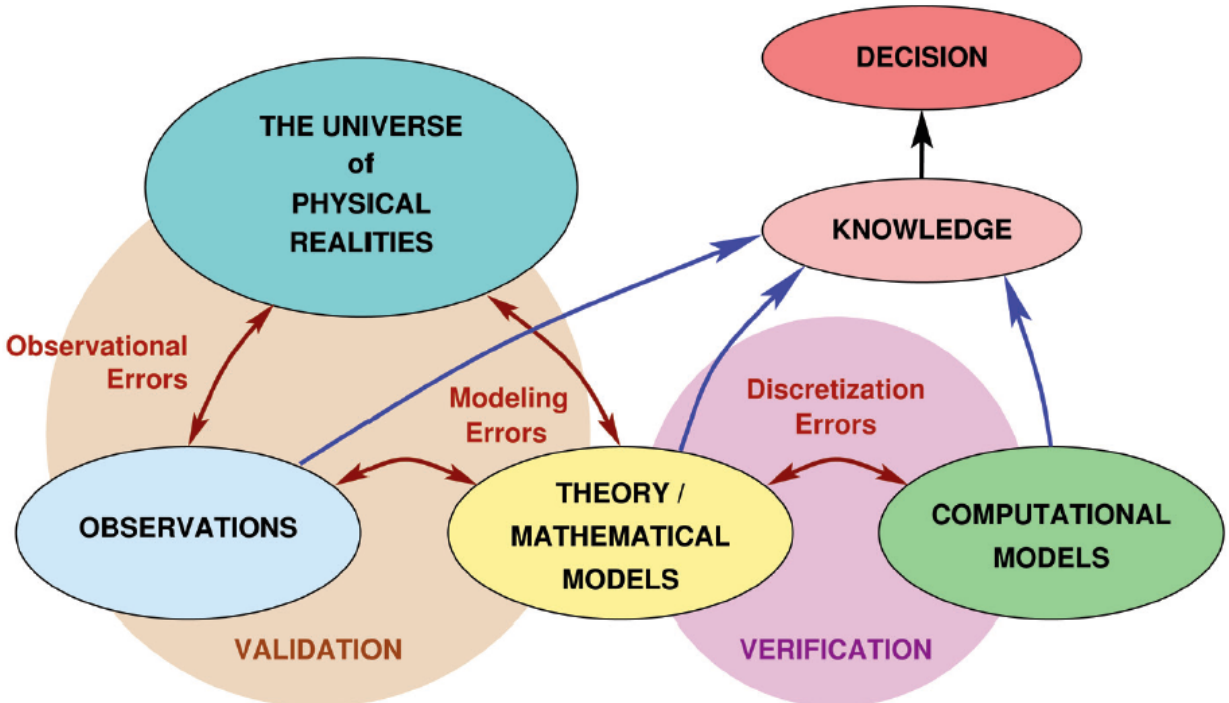


Figure 1. *Imperfect computational modeling: Imperfections in the mathematical models, incomplete observational data, observations delivered by imperfect instruments, and corruption of the model itself in the discretization needed for computation all lead to imperfect paths to knowledge. Reproduced from J.T. Oden, "A Brief View of V & V & UQ," a presentation to the Board on Mathematical Sciences and Their Applications, National Research Council, October 2009.*

How close can we get?

Dynamical systems: ODEs

As far as this first part of the tutorial, we will only consider sytems of Ordinary Differential Equations (ODEs)

$$\dot{X}(t) = F(t, X(t))$$

where X is the state vector and F is smooth (say differentiable)

- NOT F non-smooth (piecewise-smooth)
- NOT PDEs
- NOT maps (discrete-time systems), since they will be studied in detail next week
- NOT adding stochasticity (stochastic ODEs or PDEs)
- NOT dynamics on networks (graphs)

although all these other cases appear quite usually in practice

Dynamical systems: F non-smooth

They usually appear in some control systems where the control is also non-smooth (bang-bang, for example)

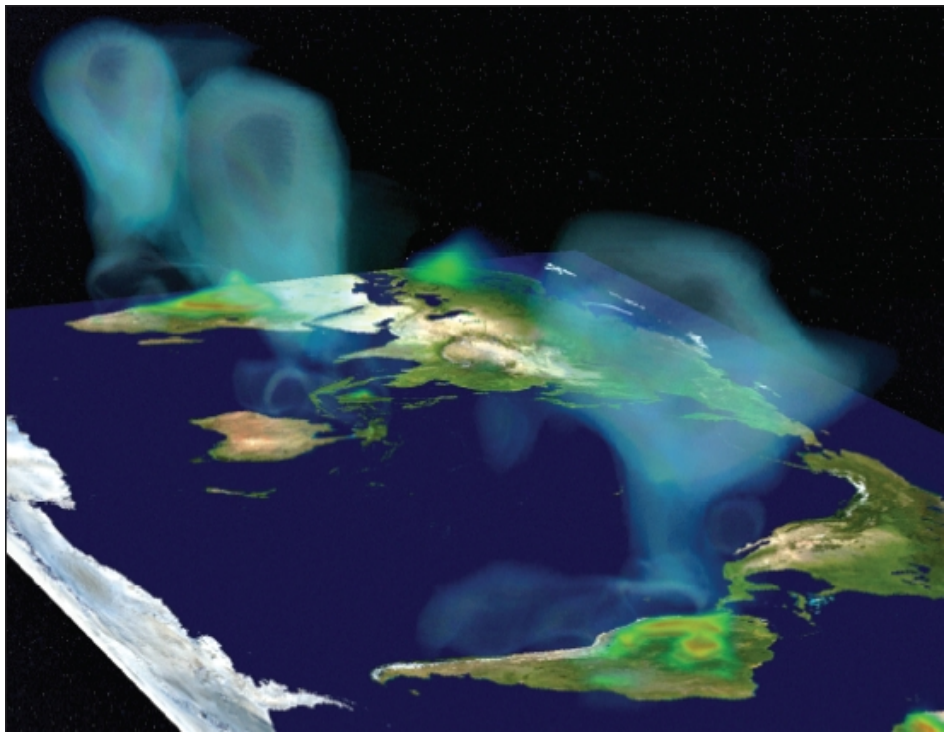
Depending on the non-smoothness degree, they can be classified into

- Impact systems (heaviest non-smoothness, like rigid walls)
- Filippov systems (sliding systems from control)
- Piecewise-continuous systems (vector field is continuous at the border)

Dynamical systems: PDEs I

Partial differential equations (PDEs)

- when spatial dimension matters



Dynamical systems: PDEs II

Wave equation ($x = x(t, l_1(t), l_2(t))$)

$$\frac{\partial^2 x}{\partial t^2} = c^2 \left(\frac{\partial^2 x}{\partial {l_1}^2} + \frac{\partial^2 x}{\partial {l_2}^2} \right)$$

http://en.wikipedia.org/wiki/Wave_equation

Heat equation ($x = x(t, l_1(t), l_2(t))$)

$$\frac{\partial x}{\partial t} = b^2 \left(\frac{\partial^2 x}{\partial {l_1}^2} + \frac{\partial^2 x}{\partial {l_2}^2} \right)$$

http://en.wikipedia.org/wiki/Heat_equation

Dynamical systems: Maps

The discrete TRAP-TRICK

Maps are discrete-time systems. My view is:

- We would be really interested in continuous-time systems but they are more difficult to study and simulate than discrete-time systems
- Also, in practice, is almost impossible to obtain continuous-time DATA
- And, even if we had the DATA, probably we could not process it adequately

Thus, we are happy enough with a discretization in time of a really continuous-time system

Dynamical systems: Stochastics and Networks

Adding stochasticity poses a quite difficult problem (at least for me!). Something will be seen in the last part of this trimester.

On networks, the geometry of the state space becomes important (usually is no more isotropic nor homogeneous).

Check `http://acn2013.dei.polimi.it`

(Milano, 20-22 February 2013)

Dynamical systems: ODEs

Thus we will study

$$\dot{X}(t) = F(t, X(t))$$

where

$$X(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

is the state vector (for short, we will write X or $(x_1, x_2, \ldots, x_n)$), which lives in $\mathbb{R}^n$,

$$\dot{X} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$$

are the time-derivatives of the state and F (so-called vector field) is smooth (say $\mathbb{C}^1$)

From control systems to dynamical systems

Usually, control theorists use the state-space framework

$$\begin{aligned} \dot{X} &= F(X, u) \\ Y &= H(X) \end{aligned} \tag{1}$$

where Y stands for the observable variables and $u(Y)$ is the control action.

This is the so-called control problem, since $u(Y)$ is still not defined.

But if $u(Y)$ is already defined, then we have

$$\dot{X} = F(X, u) = F(X, u(Y)) = F(X, u(H(x))) = G(x)$$

and we recover just a dynamical system.

From optimal control systems to dynamical systems

Finally, in this trimester we want to deal with

Dynamics - Constraints - Optimal Control

Thus...

Our framework is...

Dynamical system

$$\dot{X} = F(t, X, u)$$

Constraints

$$X \in \mathbb{A} \qquad u \in \mathbb{B}$$

Control optimization criteria

$$Max \quad L(t, X, u) \qquad\qquad Min \quad L(t, X, u)$$

Back again to dynamical systems

Once we have

$$u^* = u^*(t, X)$$

which optimizes L, then solve

$$\dot{X} = F(t, X, u^*(t, X)) = G(t, X)$$

which is just a dynamical system again

(taking the constraints into account)

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2 Basics

3 Nonlinear tools

4 Poincaré maps

5 Examples

Systems of ODEs

$$\dot{X}(t) = F(t, X(t), p)$$

where

$$X(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

is the state vector (for short, we will write X or $(x_1, x_2, \ldots, x_n)$), which lives in $\mathbb{R}^n$,

$$\dot{X} = (\dot{x_1}, \dot{x_2}, \ldots, \dot{x_n})$$

are the time-derivatives of the state,

$$F = (F_1(x_1, x_2, \ldots, x_n), F_2(x_1, x_2, \ldots, x_n), \ldots, F_n(x_1, x_2, \ldots, x_n))$$

is the smooth vector field (say $\mathbb{C}^1$), and

$$p = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m$$

is the parameters (parameters are unknown constants) vector.

Study cases

It may be that:

- F does not depend explicitly on t (autonomous system)
- F depends T-periodically on t (periodically-forced system)
- F depends non-periodically on t (non-autonomous system)

Also, it may be that

- F is linear
- F is nonlinear

(comment on linear is to systems as dogs is to animals)

Equilibrium points

For the rest of the day, assume that the system is autonomous and parameters p are fixed to p^*. Thus we have

$$\dot{X} = F(X)$$

An equilibrium is a state X^* such that

$$F(X^*) = 0$$

Moreover, X^* is

- stable, if all orbits which start with initial conditions in a neighbourhood Ω close enough to X^*, remain in a neighbourhood of X^*.
- assymptotically stable, if it is stable, and all the orbits starting at Ω tend to X^* as time tends to infinity.
- unstable, in other case.

Linear systems I

Assume we have a linear system

$$\dot{X} = F(X) = AX + b$$

- If $det(A) \neq 0$ then the unique equilibrium point is $X^* = -A^{-1}b$
- If $det(A) = 0$ then there is an infinite number of non-isolated equilibrium points (a linear manifold: a line, a plane,...)

Linear systems II

Assume we have a linear system

$$\dot{X} = F(X) = AX + b$$

with equilibrium point $X^* = -A^{-1}b$.

- If all eigenvalues of A have strictly negative real part then X^* is assymptotically stable
- If there is an eigenvalue of A with strictly positive real part then X^* is unstable
- If there are no eigenvalues of A with strictly positive real part, and all eigenvalues with zero real part (thus they are non-zero pure imaginary eigenvalues) have multiplicity one, then X^* is stable (but not assymptotically). In this case, the equilibrium point is called a center, and it is surrounded by an infinite family of isochronic periodic orbits.

(other cases are more technical)

Different configurations in the linear case

See blackboard

Nonlinear systems I

Assume we have a nonlinear system

$$\dot{X} = F(X)$$

Now, we can have any number of isolated equilibrium points (even noone). Moreover there are other stationary sets:

- Limit cycles (isolated periodic orbits)
- Quasiperiodic orbits (if dimension of the state space is at least 3)
- Chaotic orbits (if dimension of the state space is at least 3)

which can be stable or unstable.

Nonlinear systems II

Assume we have a nonlinear system

$$\dot{X} = F(X)$$

and an isolated equilibrium point X^*.

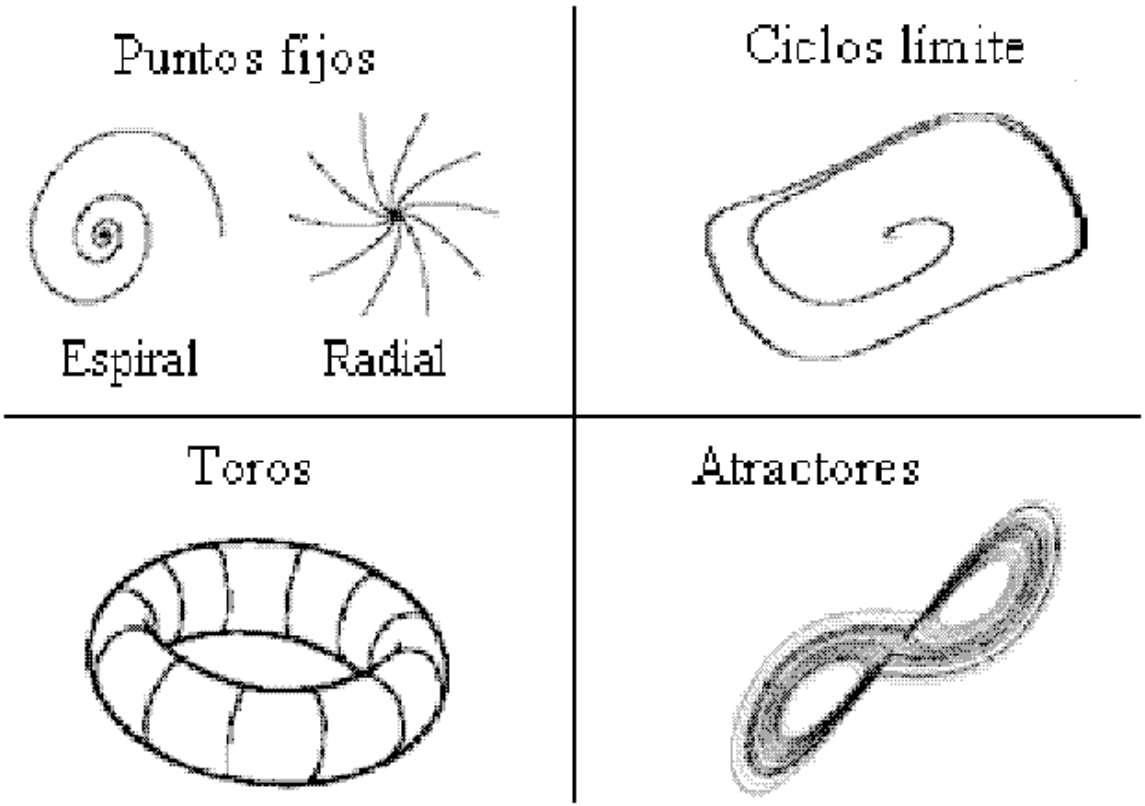
We compute the jacobian at the equilibrium point

$$A = Jac(F(X^*)) = DF(X^*)$$

- If all eigenvalues of A have strictly negative real part then X^* is assymptotically stable
- If there is an eigenvalue of A with strictly positive real part then X^* is unstable

(the case with zero real-part eigenvalues is not solved)

Stationary states



Transient versus Stationary states

The role of stationary states: take into account that

- If the system has a slow time-constant, sometimes the study of the stationary behavior is nonsense, and what is really important are the possible transient states (sustainable development, sludges, bioreactors,...)
- If the system has a fast time-constant, although the transient may be important, the study of the satationary state is worthwhile (circuits, power electronics, powers systems, ...)

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Invariant manifolds (linear case)

In the linear case, we can consider

- n^- = number of strictly negative real-part eigenvalues
- n^+ = number of strictly positive real-part eigenvalues
- n^0 = number of zero real-part eigenvalues.

Then we can decompose

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^0$$

where

- E^s is the linear stable manifold with dimension n^-
- E^u is the linear unstable manifold with dimension n^+ and,
- E^0 is the linear neutral manifold with dimension n^0.

They are expanded by the corresponding eigenvalues (or generalized eigenvalues).

Note that, since they are linear, they can only meet at the equilibrium point X^*

Invariant manifolds (nonlinear case)

In the nonlinear case, we can do something similar with the Jacobian $A = DF(X^*)$. We also consider

- n^- = number of strictly negative real-part eigenvalues
- n^+ = number of strictly positive real-part eigenvalues
- n^0 = number of zero real-part eigenvalues.

Then we can decompose (at least, locally)

$$\mathbb{R}^n = W^s \oplus W^u \oplus W^0$$

where

- W^s is the stable manifold with dimension n^-
- W^u is the unstable manifold with dimension n^+ and,
- W^0 is the neutral manifold with dimension n^0.

Invariant manifolds (nonlinear case)

They are expanded locally close to the equilibrium point X^* also by the corresponding eigenvalues (or generalized eigenvalues). This means that the linear manifolds are tangent to the invariant manifolds at the equilibrium point.

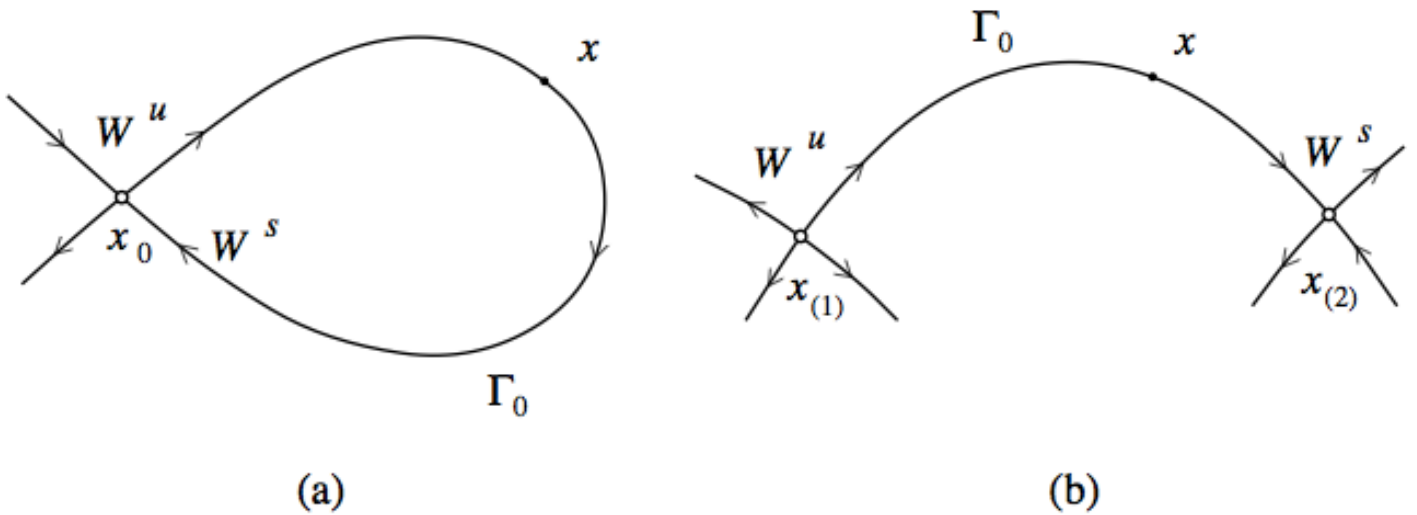
Note that now, since they are nonlinear surfaces, or curves, a priori, they may meet at points other than the equilibrium point X^*.

But note also that the existence and uniqueness theorem for ODEs avoids this, and then

- the invariant manifolds coincide, or
- we have another equilibrium point, or in general, another stationary set

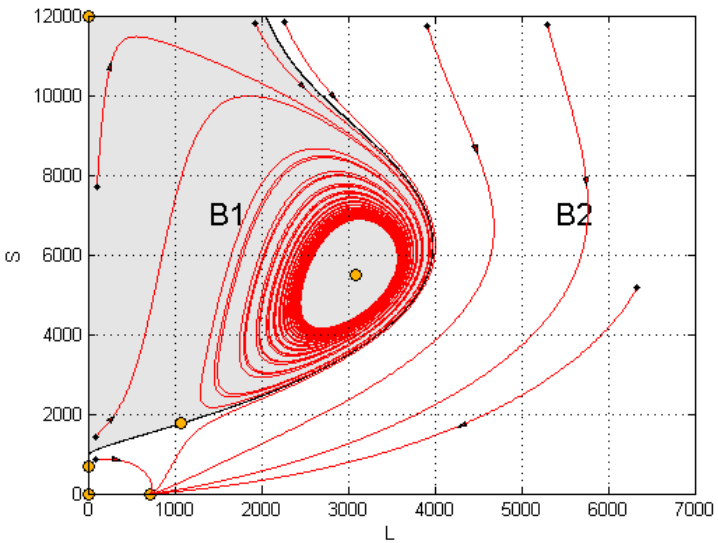
Homoclinic and heteroclinic orbits

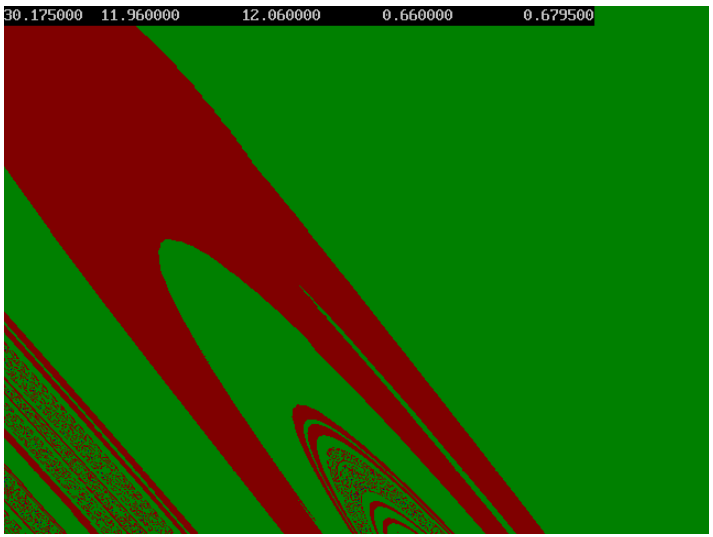
In the first case we have a so-called homoclinic connection and, in the second case we have an heteroclinic connection.



Basins of attraction

Also, in the nonlinear case, different attractors have their own basin of attraction, which includes all initial conditions in the state space whose orbits in positive time reach the attractor.





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Poincaré map I

Poincaré maps are often used for discretizing continuous-time systems, keeping most of the properties of the original one.

Consider

$$\dot{X} = F(t, X)$$

and the flow ϕ_t corresponding to the orbits of the system.

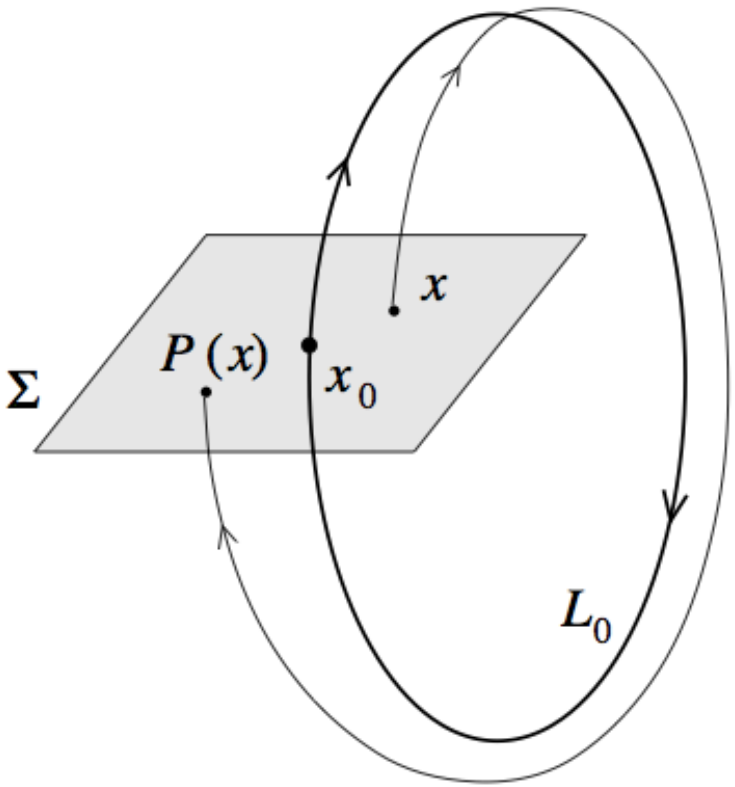
Consider (locally) a smooth surface Π of dimension $n - 1$.

Choose an initial condition X_0 on Π, and the orbit through X_0. If we assume that this orbit hits again Π for first time at time t_1 at a point $X_1 \in \Pi$, we can define this point as the image of X_0 through a so-called Poincaré map P,

$$P : \Pi \to \Pi \qquad \Pi(X_0) = X_1$$

Poincaré map I

We can do this for a convenient neighbourhood of X_0 as this (local) map is well-defined (some orbits can never go back again to Π)



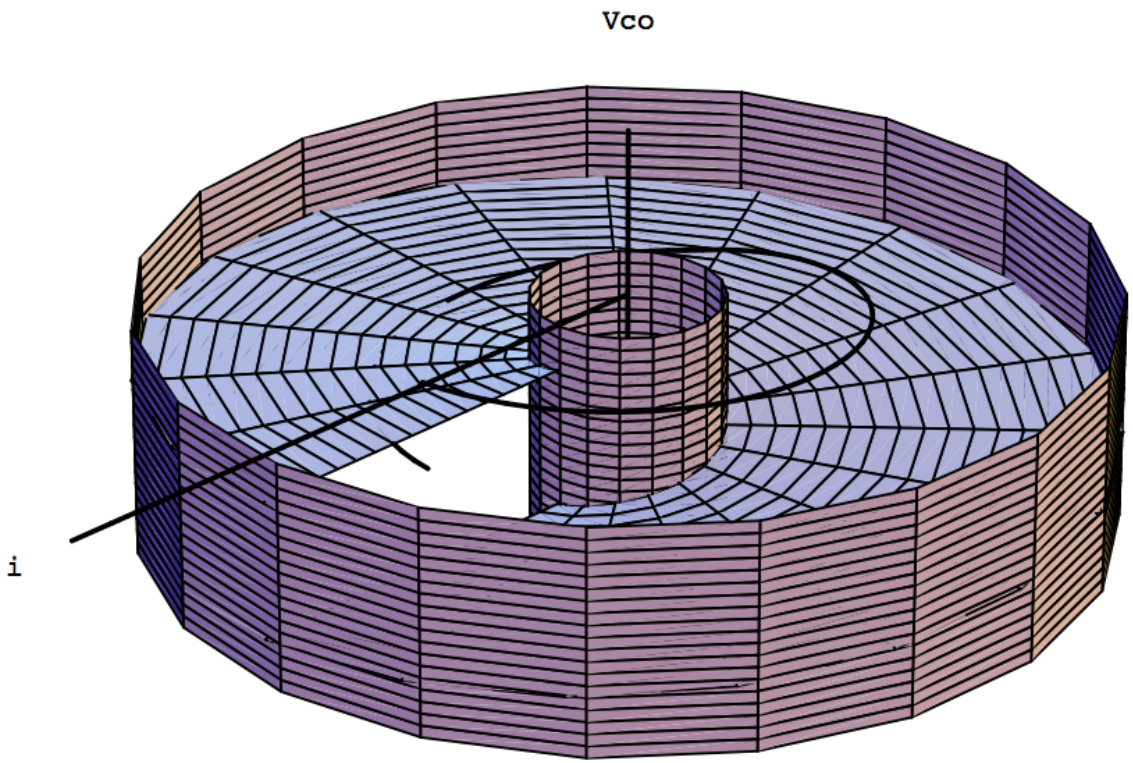
Poincaré map II

This procedure is specially useful in two situations:

- When the system is T-periodically forced
- Close to a periodic orbit

A T-periodically forced system

In this case, we can rewrite the problem in a cylindrical space $\mathbb{R}^n \times S^1$, and choose our Poincaré section as the set $\{t = 0 = T\}$. Then the Poincaré map is globally defined (and it is known as a stroboscopic map).



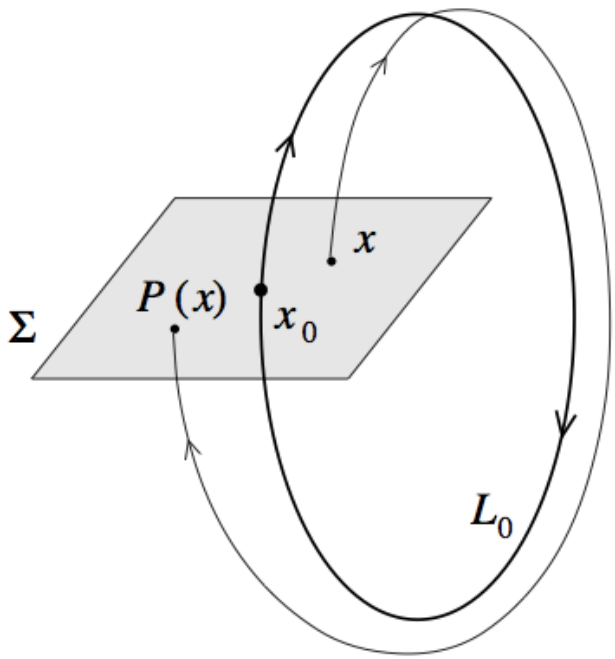
Close to a periodic orbit

In this case, if L is the periodic orbit, we choose a (local) Poincaré section which is normal to L.

If we choose point q^* as the intersection of the periodic orbit and the section, we will have that

$$P(q^*) = q^*$$

and thus q^* is a fixed point of map P.



Close to a periodic orbit

Consider now a neighbourhood of q^* in Π such that this Poincaré map is well-defined.

Then all stability properties of L (in the continuous-time system) are equivalent to those of q^* (as a fixed point of the Poincaré map).
The same applies if we consider a quasiperiodic orbit or a chaotic orbit

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Simple model in Sustainable development

Resource dynamics (forest):

$$\frac{dS}{dt} = \rho\left(\frac{S}{k} - 1\right)\left(1 - \frac{S}{K}\right)S - \alpha\beta LS$$

- Natural growth
- Resource profit

Population dynamics:

$$\frac{dL}{dt} = \left(\gamma\lambda\left(1-\beta\right)^{\delta} L^{\delta-1} + \phi\alpha\beta S - \sigma\right)L$$

- Population growth
- Resource profit

Simple model in Sustainable development

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- Resource profit

Population dynamics:

$$\frac{dL}{dt} = (\gamma\lambda(1-\beta)^{\delta}L^{\delta-1} + \phi\alpha\beta S - \sigma)L$$

- Rents
- Population increase

Simple model in Sustainable development

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- Natural growth
- Resource profit

Population dynamics:

$$\frac{dL}{dt} = (\gamma\lambda(1-\beta)^{\delta}L^{\delta-1} + \phi\alpha\beta S - \sigma)L$$

- Rents
- Poverty threshold

Simple model in Sustainable development

Resource dynamics (forest):

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- Natural growth
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- Rents
- Poverty threshold

Simple model in Sustainable development

Resource dynamics (forest):

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Population dynamics:

$$\frac{dL}{dt} = \left(\gamma\lambda\left(1-\beta\right)^{\delta} L^{\delta-1} + \phi\alpha\beta S - \sigma\right)L$$

- Rents
- Poverty threshold

Phase portrait

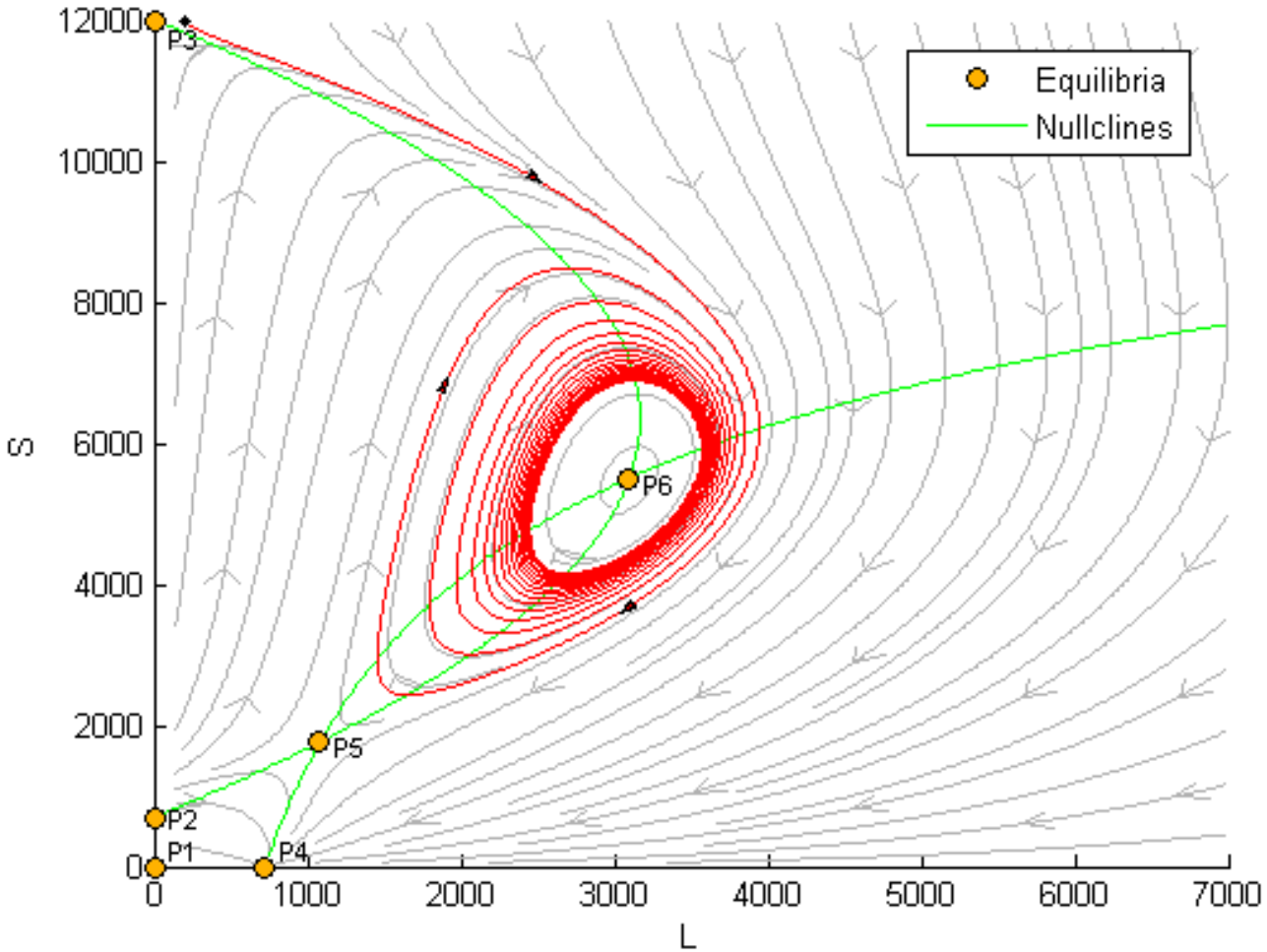

Figure : Equilibrium points and nullclines. *P*4 is assymptotically stable while *P*6 is unstable and *P*5 is a saddle

Basins of attraction

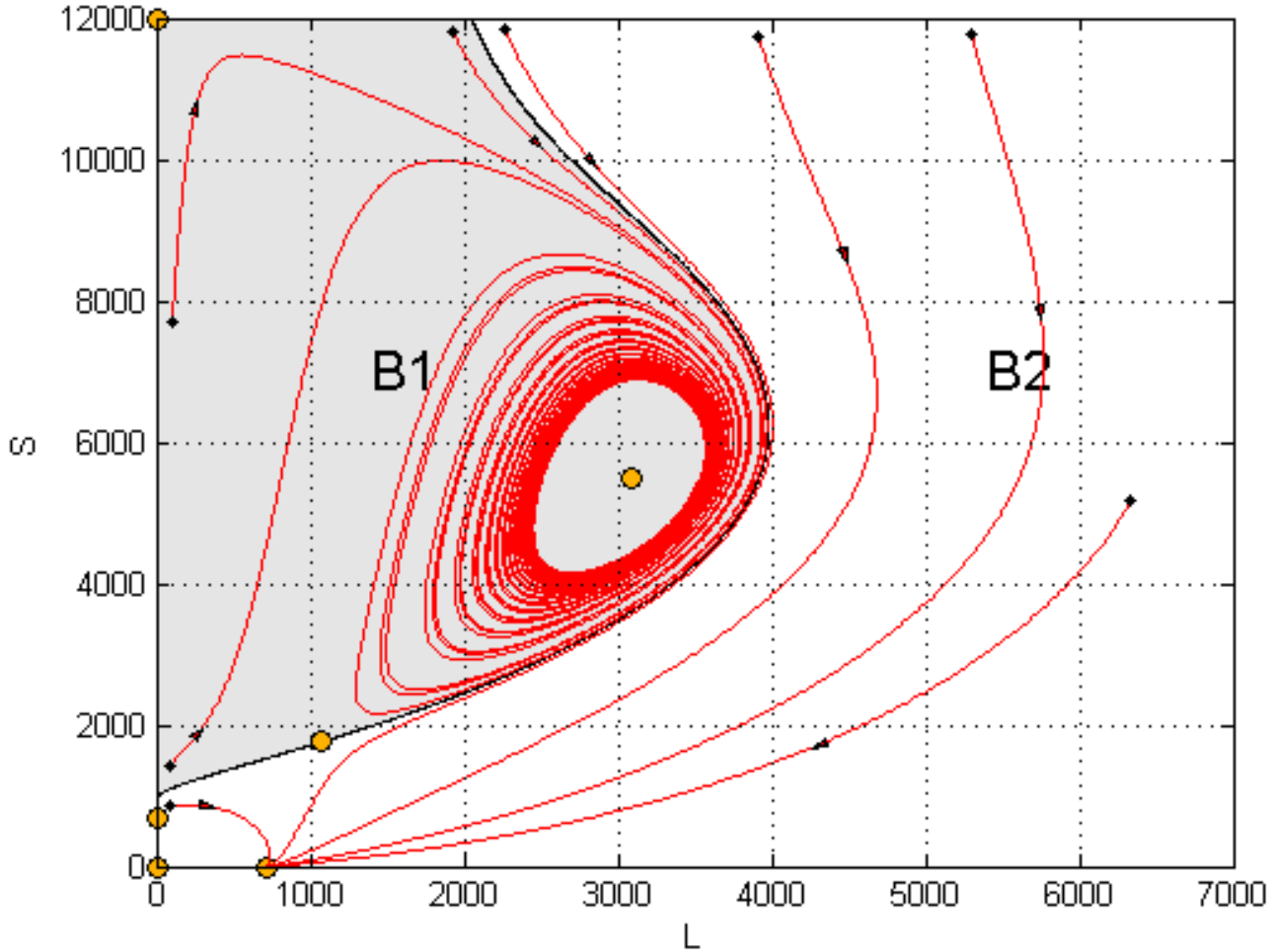

Figure : Basins

End of slides for today...

More examples on the blackboard