Stochastic Programming, Risk and Statistical Analysis.

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Two and multistage stochastic programming

The concept of two-stage (linear) stochastic programming problem with recourse

$$\operatorname*{Min}_{x \in \mathcal{X}} c^{\mathsf{T}} x + \mathbb{E}[Q(x, \boldsymbol{\xi})], \tag{1}$$

where $\mathcal{X} = \{x : Ax = b,\ x \geq 0\}$ and $Q(x, \xi)$ is the optimal value of the second stage problem

$$\operatorname*{Min}_{y} q^{\mathsf{T}} y \text{ s.t. } Tx + Wy = h,\ y \geq 0, \tag{2}$$

with $\xi = (q, T, W, h)$. In general, the feasible set $\mathcal{X}$ can be finite, i.e., integer first stage problem. Both stages can be integer (mixed integer) problems.

Suppose that the probability distribution P of $\boldsymbol{\xi}$ has a finite support, i.e., $\boldsymbol{\xi}$ can take values $\xi_1, ..., \xi_K$ (called *scenarios*) with respective probabilities $p_1, ..., p_K$. In that case

$$\mathbb{E}_P[Q(x, \boldsymbol{\xi})] = \sum_{k=1}^{K} p_k Q(x, \xi_k),$$

where

$$Q(x, \xi_k) = \inf \left\{ q_k^{\mathsf{T}} y_k : T_k x + W_k y_k = h_k,\ y_k \geq 0 \right\}.$$

It follows that we can write problem (1)-(2) as one large linear program:

$$\begin{array}{cl} \underset{x, y_1, \ldots, y_K}{\text{Min}} & c^{\mathsf{T}} x + \sum_{k=1}^{K} p_k q_k^{\mathsf{T}} y_k \\ \text{subject to} & Ax = b, \\ & T_k x + W_k y_k = h_k,\ k = 1, ..., K, \\ & x \geq 0,\ y_k \geq 0,\ k = 1, ..., K. \end{array} \tag{3}$$

Even crude discretization of the distribution of the data vector $\boldsymbol{\xi}$ leads to an exponential growth of the number of scenarios with increase of its dimension d.

Could stochastic programming problems be solved numerically?

What does it mean to solve a stochastic program?

How do we know the probability distribution of the random data vector?

Why do we optimize the expected value of the objective (cost) function?

General formulation of two-stage stochastic programming problems

$$\underset{x \in \mathcal{X}}{\text{Min}} \Big\{ f(x) := \mathbb{E}[F(x,\omega)] \Big\}, \tag{4}$$

where $F(x,\omega)$ is the optimal value of the second stage problem

$$\underset{y \in \mathfrak{X}(x,\omega)}{\text{Min}} \; g(x,y,\omega). \tag{5}$$

Here $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathcal{X} \subset \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}$ and $\mathfrak{X} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ is a multifunction. In particular, the linear two-stage problem can be formulated in the above form with $g(x,y,\omega) := c^{\mathsf{T}} x + q(\omega)^{\mathsf{T}} y$ and

$$\mathfrak{X}(x,\omega) := \{ y : T(\omega)x + W(\omega)y = h(\omega), \; y \geq 0 \}. \tag{6}$$

The second stage problem (5) can be also written in the following equivalent form

$$\operatorname*{Min}_{y\in\mathbb{R}^m} \bar{g}(x,y,\omega), \tag{7}$$

where

$$\bar{g}(x,y,\omega) := \begin{cases} g(x,y,\omega), & \text{if } y \in \mathfrak{X}(x,\omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

By the interchangeability principle we have

$$\underbrace{\mathbb{E}\left[\inf_{y\in\mathbb{R}^m} \bar{g}(x,y,\omega)\right]}_{F(x,\omega)} = \inf_{\boldsymbol{y}\in\mathfrak{Y}} \mathbb{E}\Big[\bar{g}(x,\boldsymbol{y}(\omega),\omega)\Big], \tag{8}$$

where $\mathfrak{Y}$ is a functional space, e.g., $\mathfrak{Y} := \mathcal{L}_p(\Omega,\mathcal{F},P;\mathbb{R}^m)$ with $p \in [1,+\infty]$.

Consequently, we can write two-stage problem (4)–(5) as one large problem:

$$\begin{array}{cl} \underset{x\in\mathbb{R}^n, \boldsymbol{y}\in\mathfrak{Y}}{\text{Min}} & \mathbb{E}\left[g(x, \boldsymbol{y}(\omega), \omega)\right] \\ \text{s.t.} & x \in \mathcal{X},\ \boldsymbol{y}(\omega) \in \mathfrak{X}(x, \omega) \text{ a.e. } \omega \in \Omega. \end{array} \tag{9}$$

In particular, if $\Omega = \{\omega_1, ..., \omega_K\}$ is finite, then problem (9) becomes

$$\begin{array}{cl} \underset{x, y_1, \ldots, y_k}{\text{Min}} & \sum_{k=1}^{K} g(x, y_k, \omega_k) \\ \text{s.t.} & x \in \mathcal{X},\ y_k \in \mathfrak{X}(x, \omega_k)\ k = 1, ..., K. \end{array} \tag{10}$$

Decision rules

Recall that the two stage problem (1) can be formulated as an optimization problem over $x \in \mathcal{X}$ and second stage decision $y(\cdot)$ considered as a function of the data. Suppose now that the recourse is fixed, i.e., only $T = T(\omega)$ and $h = h(\omega)$ of the second stage problem

$$\operatorname*{Min}_{y} q^{\mathsf{T}} y \text{ s.t. } Tx + Wy = h,\ y \geq 0,$$

are random. For a given $x \in \mathcal{X}$ the second stage problem attains its minimal value at an extreme point of the set $\{y : Wy = h - Tx,\ y \geq 0\}$, assuming that this set is nonempty and bounded. By the well known result of linear programming we have the following characterization of the extreme points

Multistage stochastic programming.

We can write the two-stage problem (1)–(2) using the following *nested formulation*:

$$\operatorname*{Min}_{\substack{Ax=b\\ x\geq 0}} c^{\mathsf{T}}x + \mathbb{E}\left[\operatorname*{Min}_{\substack{Tx+Wy=h\\ y\geq 0}} q^{\mathsf{T}}y\right]. \tag{11}$$

In the above, $y = y(\xi)$ is considered as a function of the random data $\xi = (q, T, W, h)$ and in that sense is random. If the number of scenarios is finite, we associate with every possible realization ξ_k of the data the corresponding second stage decision variable y_k.

This can be extended to the following nested formulation of a multistage stochastic programming problem:

$$\underset{x_1 \in \mathcal{X}}{\text{Min}}\, f_1(x_1) + \mathbb{E}\Bigg[\underset{x_2 \in \mathcal{X}_2(x_1,\xi_2)}{\text{Min}} f_2(x_2,\xi_2) + \cdots \\ + \mathbb{E}\Big[\underset{x_T \in \mathcal{X}_T(x_{T-1},\xi_T)}{\text{Min}} f_T(x_T,\xi_T) \Big] \Bigg], \tag{12}$$

where $\xi_1, ..., \xi_T$ is a random process (ξ_1 is deterministic), $x_t \in \mathbb{R}^{n_t}$ are decision variables, $f_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$ are objective functions and $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{d_t} \rightrightarrows \mathbb{R}^{n_t}$ are measurable closed valued multi-functions.

For example, in the linear case $f_t(x_t, \xi_t) := c_t^{\mathsf{T}} x_t$,

$$\mathcal{X}_t(x_{t-1}, \xi_t) := \{x_t : B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0\},$$

$\xi_t = (c_t, B_t, A_t, b_t)$, $t = 2, ..., T$, is considered as a random process, $\xi_1 = (c_1, A_1, b_1)$ is supposed to be known.

Hence in the linear case the nested formulation can be written as

$$\operatorname*{Min}_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^{\mathsf{T}} x_1 + \mathbb{E}\left[\operatorname*{Min}_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^{\mathsf{T}} x_2 + \cdots + \mathbb{E}\Big[\operatorname*{Min}_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^{\mathsf{T}} x_T \Big] \right]. \tag{13}$$

If the number of realizations (scenarios) of the process ξ_t is finite, then problem (13) can be written as one large linear programming problem. There are several possible formulations of the above multistage program.

Dynamic programming equations

Consider the last stage problem

$$\underset{x_T \in \mathcal{X}_T(x_{T-1},\xi_T)}{\text{Min}} f_T(x_T, \xi_T). \tag{14}$$

The optimal value of this problem, denoted $Q_T(x_{T-1}, \xi_T)$, depends on the decision vector x_{T-1} and data ξ_T. At stage $t = 2, ..., T-1$, we write the problem:

$$\begin{array}{ll} \underset{x_t}{\text{Min}} & f_t(x_t, \xi_t) + \mathbb{E}\left\{Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \Big| \xi_{[t]}\right\} \\ \text{s.t.} & x_t \in \mathcal{X}_t(x_{t-1}, \xi_t). \end{array} \tag{15}$$

Its optimal value depends on the decision x_{t-1} at the previous stage and realization of the data process $\xi_{[t]} = (\xi_1, ..., \xi_t)$, and denoted $Q_t\left(x_{t-1}, \xi_{[t]}\right)$. The idea is to calculate the (so-called *cost-to-go* or *value*) functions $Q_t\left(x_{t-1}, \xi_{[t])}\right)$, recursively, going backward in time.

At the first stage we finally need to solve the problem:

$$\operatorname*{Min}_{x_1 \in \mathcal{X}} f_1(x_1) + \mathbb{E}\left[Q_2\left(x_t, \xi_2\right)\right]. \tag{16}$$

The dynamic programming equations:

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{ f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) \right\}, \tag{17}$$

where

$$\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) := \mathbb{E}\left\{ Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \middle| \xi_{[t]} \right\}.$$

If the random process is *Markovian* (i.e., the conditional distribution of ξ_{t+1} given $\xi_{[t]} = (\xi_1, ..., \xi_t)$ is the same as the conditional distribution of ξ_{t+1} given ξ_t), then $Q_t\left(x_{t-1}, \xi_t\right)$ is a function of x_{t-1} and ξ_t, and if it is *stagewise independent* (i.e., ξ_{t+1} is independent of $\xi_{[t]}$), then $\mathbb{E}\left[Q_{t+1}\left(x_t, \xi_{t+1}\right) \middle| \xi_t\right] = \mathcal{Q}_{t+1}(x_t)$ does not depend on ξ_t.

A sequence of (measurable) mappings $x_t(\xi_{[t]})$, $t = 1, ..., T$, is called a *policy* (recall that ξ_1 is deterministic). A policy is said to be feasible if it satisfies the feasibility constraints, i.e.,

$$x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t),\ t = 2, ..., T, \text{ w.p.1.} \tag{18}$$

We can formulate the multistage problem (12) in the form

$$\begin{array}{cl} \underset{x_1, x_2(\cdot), ..., x_T(\cdot)}{\text{Min}} & \mathbb{E}\Big[f_1(x_1) + f_2(x_2(\xi_2), \xi_2) + ... + f_T\left(x_T(\xi_{[T]}), \xi_T\right)\Big] \\ \text{s.t.} & x_1 \in \mathcal{X},\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t),\ t = 2, ..., T. \end{array}$$

Note that the above optimization is performed over feasible policies. A policy $\bar{x}_t(\xi_{[t]})$ is *optimal* if it satisfies the dynamic programming equations, i.e.,

$$\bar{x}_t(\xi_{[t]}) \in \arg \min_{x_t \in \mathcal{X}_t\left(\bar{x}_{t-1}(\xi_{[t-1]}), \xi_t\right)} \left\{ f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) \right\}, \text{ w.p.1.}$$

Monte Carlo sampling methods

Consider (two-stage) stochastic programming problem:

$$\operatorname*{Min}_{x \in \mathcal{X}} \Big\{ f(x) := \mathbb{E}[F(x, \xi)] \Big\}. \tag{19}$$

Even a crude discretization of the distribution of the random data vector $\xi \in \mathbb{R}^d$ typically results in an exponential growth of the number of scenarios with increase of the number d of random variables. For example, if components of random vector ξ are independently distributed and distribution of each component is discretized by r points, then the total number of scenarios is r^d. That is, although the input data grows linearly with increase of the dimension d, the number of scenarios grows exponentially. The standard approach to dealing with this issue is to generate a manageable number of scenarios in some “representative” way.

For example, we can generate a random sample $\xi^1, ..., \xi^N$ of N realizations of the random vector ξ by using Monte Carlo sampling techniques. Then the expected value function $f(x) := \mathbb{E}[F(x, \xi)]$ can be approximated by the sample average function

$$\hat{f}_N(x) := N^{-1} \sum_{j=1}^{N} F(x, \xi^j).$$

Consequently the true (expected value) problem is approximated by the so-called *sample average approximation* (SAA) problem:

$$\operatorname*{Min}_{x \in \mathcal{X}} \hat{f}_N(x). \tag{20}$$

Note that once the sample is generated, the above SAA problem can be viewed as a (two-stage) problem with the corresponding set of scenarios $\left\{\xi^1, ..., \xi^N\right\}$ each scenario with equal probability $1/N$.

A (naive) justification of the SAA method is that for a given $x \in \mathcal{X}$, by the Law of Large Numbers (LLN), $\hat{f}_N(x)$ converges to $f(x)$ w.p.1 as N tends to infinity. It is possible to show that, under mild regularity conditions, this convergence is uniform on any compact subset of $\mathcal{X}$ (uniform LLN). It follows that the optimal value $\hat{v}_N$ and an optimal solution $\hat{x}_N$ of the SAA problem (20) converge w.p.1 to their counterparts of the true problem.

Central Limit Theorem type results. Notoriously slow convergence of order $O_p(N^{-1/2})$. By the CLT, for a given $x \in \mathcal{X}$,

$$N^{1/2}\left[\hat{f}_N(x) - f(x)\right] \Rightarrow N(0, \sigma^2(x)),$$

where $\sigma^2(x) := \mathsf{Var}[F(x,\xi)]$ and " $\Rightarrow$ " denotes convergence in distribution.

Delta method

Let $Y_N \in \mathbb{R}^d$ be a sequence of random vectors, converging in probability to a vector $\mu \in \mathbb{R}^d$. Suppose that there exists a sequence $\tau_N \to +\infty$ such that $\tau_N(Y_N - \mu) \Rightarrow Y$. Let $G : \mathbb{R}^d \to \mathbb{R}^m$ be a vector valued function, differentiable at μ. That is $G(y) - G(\mu) = M(y-\mu)+r(y)$, where $M := \nabla G(\mu)$ is the $m \times d$ Jacobian matrix of G at μ, and the remainder $r(y)$ is of order $o(\|y - \mu\|)$. It follows that $\tau_N\,[G(Y_N) - G(\mu)] \Rightarrow MY$. In particular, suppose that $N^{1/2}(Y_N - \mu)$ converges in distribution to a (multivariate) normal distribution with zero mean vector and covariance matrix Σ. Then it follows that

$$N^{1/2}\,[G(Y_N) - G(\mu)] \Rightarrow N(0, M\Sigma M^{\mathsf{T}}).$$

Infinite dimensional Delta Theorem. Let B_1 and B_2 be two Banach spaces, and $G : B_1 \to B_2$ be a mapping. It is said that G is directionally differentiable at a point $\mu \in B_1$ if the limit

$$G'_\mu(d) := \lim_{t \downarrow 0} \frac{G(\mu + td) - G(\mu)}{t} \tag{21}$$

exists for all $d \in B_1$. If, in addition, the directional derivative $G'_\mu : B_1 \to B_2$ is linear and continuous, then it is said that G is Gâteaux differentiable at μ. It is said that G is directionally differentiable at μ in the sense of Hadamard if the directional derivative $G'_\mu(d)$ exists for all $d \in B_1$ and, moreover,

$$G'_\mu(d) = \lim_{\substack{t \downarrow 0 \\ d' \to d}} \frac{G(\mu + td') - G(\mu)}{t}. \tag{22}$$

Theorem 1 (Delta Theorem) *Let B_1 and B_2 be Banach spaces, equipped with their Borel σ-algebras, Y_N be a sequence of random elements of B_1, $G : B_1 \to B_2$ be a mapping, and τ_N be a sequence of positive numbers tending to infinity as $N \to \infty$. Suppose that the space B_1 is separable, the mapping G is Hadamard directionally differentiable at a point $\mu \in B_1$, and the sequence $X_N := \tau_N(Y_N - \mu)$ converges in distribution to a random element Y of B_1. Then*

$$\tau_N \left[G(Y_N) - G(\mu)\right] \Rightarrow G'_\mu(Y), \tag{23}$$

and

$$\tau_N \left[G(Y_N) - G(\mu)\right] = G'_\mu(X_N) + o_p(1). \tag{24}$$

Let $\mathcal{X}$ be a compact subset of $\mathbb{R}^n$ and consider the space $B = C(\mathcal{X})$ of continuous functions $\phi : \mathcal{X} \to \mathbb{R}$. Assume that:
(A1) For some point $x \in \mathcal{X}$ the expectation $\mathbb{E}[F(x,\xi)^2]$ is finite.
(A2) There exists a measurable function $C : \Xi \to \mathbb{R}_+$ such that $\mathbb{E}[C(\xi)^2]$ is finite and

$$\Big|F(x,\xi) - F(x',\xi)\Big| \le C(\xi)\|x - x'\|, \tag{25}$$

for all $x, x' \in \mathcal{X}$ and a.e. $\xi \in \Xi$.

We can view $Y_N := \hat{f}_N$ as a random element of $C(\mathcal{X})$. Consider the min-function $V : B \to \mathbb{R}$ defined as $V(Y) := \inf_{x\in\mathcal{X}} Y(x)$. Clearly $\hat{v}_N = V(Y_N)$. It is not difficult to show that for any $\mu \in C(\mathcal{X})$ and $\mathcal{X}^*(\mu) := \arg\min_{x\in\mathcal{X}} \mu(x)$,

$$V'_\mu(\delta) = \inf_{x\in\mathcal{X}^*(\mu)} \delta(x), \ \ \forall \delta \in C(\mathcal{X}),$$

and the above directional derivative holds in the Hadamard sense.

By a functional CLT, under assumptions (A1) and (A2), $N^{1/2}(\hat{f}_N - f)$ converges in distribution to a random element Y of $C(\mathcal{X})$. In particular, for any finite set $\{x_1, ..., x_m\} \subset \mathcal{X}$, the random vector $(Y(x_1), ..., Y(x_m))$ has a multivariate normal distribution with zero mean and the same covariance matrix as the covariance matrix of $(F(x_1, \xi), ..., F(x_m, \xi))$. In particular, for fixed $x \in \mathcal{X}$, $Y(x) \sim N(0, \sigma^2(x))$ with $\sigma^2(x) := \mathsf{Var}[F(x, \xi)]$.

Denote v^0 the optimal value and S^0 the set of optimal solutions of the true problem.

Theorem 2 *Suppose that the set $\mathcal{X}$ is compact, and assumptions (A1) and (A2) hold. Then*

$$\hat{v}_N = \min_{x \in S^0} \hat{f}_N(x) + o_p(N^{-1/2}),$$
$$N^{1/2}[\hat{v}_N - v^0] \Rightarrow \inf_{x \in S^0} Y(x).$$

In particular, if the optimal set (of the true problem) $S^0 = \{x^0\}$ is a singleton, then

$$N^{1/2}[\hat{v}_N - v^0] \Rightarrow N(0, \sigma^2(x^0)).$$

This result suggests that the optimal value of the SAA problem converges at a rate of $O_p(N^{-1/2})$. In particular, if $S^0 = \{x^0\}$, then $\hat{v}_N$ converges to v^0 at the same rate as $\hat{f}_N(x^0)$ converges to $f(x^0)$.

Validation analysis

How one can evaluate quality of a given (feasible) solution $\hat{x} \in \mathcal{X}$? The SAA approach – statistical test based on estimation of $f(\hat{x}) - v^0$, where v^0 is the optimal value of the true problem.
(i) Estimate $f(\hat{x})$ by the sample average $\hat{f}_{N'}(\hat{x})$, using sample of a large size N'.
(ii) Solve the SAA problem M times using M independent samples each of size N. Let $\hat{v}_N^{(1)}, ..., \hat{v}_N^{(M)}$ be the optimal values of the corresponding SAA problems. Estimate $\mathbb{E}[\hat{v}_N]$ by the average $M^{-1} \sum_{j=1}^M \hat{v}_N^{(j)}$. Note that

$$\mathbb{E}\left[\hat{f}_{N'}(\hat{x}) - M^{-1} \sum_{j=1}^M \hat{v}_N^{(j)}\right] = \left(f(\hat{x}) - v^0\right) + \left(v^0 - \mathbb{E}[\hat{v}_N]\right),$$

and that $v^0 - \mathbb{E}[\hat{v}_N] > 0$.

The bias $v^0 - \mathbb{E}[\hat{v}_N]$ is positive and (under mild regularity conditions)

$$\lim_{N\to\infty} N^{1/2}\left(v^0 - \mathbb{E}[\hat{v}_N]\right) = \mathbb{E}\left[\max_{x\in S^0} Y(x)\right],$$

where S^0 is the set of optimal solutions of the true problem, $(Y(x_1), ..., Y(x_k))$ has a multivariate normal distribution with zero mean vector and covariance matrix given by the covariance matrix of the random vector $(F(x_1,\xi), ..., F(x_k,\xi))$. For ill-conditioned problems this bias is of order $O(N^{-1/2})$ and can be large if the ε-optimal solution set S^ε is large for some small $\varepsilon \geq 0$.

Sample size estimates (by Large Deviations type bounds)

Consider an iid sequence $Y_1, \ldots, Y_N$ of replications of a real valued random variable Y, and let $Z_N := N^{-1} \sum_{i=1}^{N} Y_i$ be the corresponding sample average. Then for any real numbers a and $t > 0$ we have that $\text{Prob}(Z_N \geq a) = \text{Prob}(e^{tZ_N} \geq e^{ta})$, and hence, by Markov inequality

$$\text{Prob}(Z_N \geq a) \leq \quad e^{-ta}\mathbb{E}[e^{tZ_N}] = e^{-ta}[M(t/N)]^N,$$

where $M(t) := \mathbb{E}[e^{tY}]$ is the *moment generating function* of Y. Suppose that Y has finite mean $\mu := \mathbb{E}[Y]$ and let $a \geq \mu$. By taking the logarithm of both sides of the above inequality, changing variables $t' = t/N$ and minimizing over $t' > 0$, we obtain

$$\frac{1}{N} \log \left[\text{Prob}(Z_N \geq a) \right] \leq -I(a), \tag{26}$$

where $I(z) := \sup_{t \in \mathbb{R}} \{tz - \Lambda(t)\}$ is the conjugate of the logarithmic moment generating function $\Lambda(t) := \log M(t)$.

Suppose that $|\mathcal{X}| < \infty$, i.e., the set $\mathcal{X}$ is finite, and: (i) for every $x \in \mathcal{X}$ the expected value $f(x) = \mathbb{E}[F(x, \boldsymbol{\xi})]$ is finite, (ii) there are constants $\sigma > 0$ and $a \in (0, +\infty]$ such that

$$M_x(t) \leq \exp\{\sigma^2 t^2/2\}, \ \ \forall t \in [-a, a], \ \forall x \in \mathcal{X},$$

where $M_x(t)$ is the moment generating function of the random variable $F(u(x), \boldsymbol{\xi}) - F(x, \boldsymbol{\xi}) - \mathbb{E}[F(u(x), \boldsymbol{\xi}) - F(x, \boldsymbol{\xi})]$ and $u(x)$ is a point of the optimal set S^0. Choose $\varepsilon > 0$, $\delta \geq 0$ and $\alpha \in (0, 1)$ such that $0 < \varepsilon - \delta \leq a\sigma^2$. Then for sample size

$$N \geq \frac{2\sigma^2}{(\varepsilon - \delta)^2} \log\left(\frac{|X|}{\alpha}\right)$$

we are guaranteed, with probability at least $1 - \alpha$, that any δ-optimal solution of the SAA problem is an ε-optimal solution of the true problem, i.e., $\text{Prob}(\widehat{S}_N^\delta \subset S^\varepsilon) \geq 1 - \alpha$.

Let $\mathcal{X} = \{x_1, x_2\}$ with $f(x_2) - f(x_1) > \varepsilon > 0$ and suppose that random variable $F(x_2, \boldsymbol{\xi}) - F(x_1, \boldsymbol{\xi})$ has normal distribution with mean $\mu = f(x_2) - f(x_1)$ and variance σ^2. By solving the corresponding SAA problem we make the correct decision (that x_1 is the minimizer) if $\hat{f}_N(x_2) - \hat{f}_N(x_1) > 0$. Probability of this event is $\Phi(\mu\sqrt{N}/\sigma)$. Therefore we need the sample size $N > z_\alpha^2 \sigma^2 / \varepsilon^2$ in order for our decision to be correct with probability at least $1 - \alpha$.

In order to solve the corresponding optimization problem we need to test $H_0 : \mu \leq 0$ versus $H_a : \mu > 0$. Assuming that σ^2 is known, by Neyman-Pearson Lemma, the uniformly most powerful test is: "reject H_0 if $\hat{f}_N(x_2) - \hat{f}_N(x_1)$ is bigger than a specified critical value".

Now let $\mathcal{X} \subset \mathbb{R}^n$ be a set of finite diameter $D := \sup_{x',x\in\mathcal{X}} \|x' - x\|$. Suppose that: (i) for every $x \in \mathcal{X}$ the expected value $f(x) = \mathbb{E}[F(x, \boldsymbol{\xi})]$ is finite, (ii) there is a constant $\sigma > 0$ such that

$$M_{x',x}(t) \le \exp\{\sigma^2 t^2/2\}, \ \ \forall t \in \mathbb{R}, \ \forall x', x \in \mathcal{X},$$

where $M_{x',x}(t)$ is the moment generating function of the random variable $F(x', \boldsymbol{\xi}) - F(x, \boldsymbol{\xi}) - \mathbb{E}[F(x', \boldsymbol{\xi}) - F(x, \boldsymbol{\xi})]$, (iii) there exists $\kappa : \Xi \to \mathbb{R}_+$ such that its moment generating function is finite valued in a neighborhood of zero and

$$\left| F(x', \xi) - F(x, \xi) \right| \le \kappa(\xi) \|x' - x\|, \ \ \forall \xi \in \Xi, \ \forall x', x \in \mathcal{X}.$$

Choose $\varepsilon > 0$, $\delta \in [0, \varepsilon)$ and $\alpha \in (0, 1)$. Then for sample* size

$$N \ge \frac{O(1)\sigma^2}{(\varepsilon - \delta)^2} \left[n \log \left(\frac{O(1)DL}{(\varepsilon - \delta)^2} \right) + \log \left(\frac{2}{\alpha} \right) \right],$$

we are guaranteed that $\text{Prob}\left(\widehat{S}_N^\delta \subset S^\varepsilon \right) \ge 1 - \alpha$.

*$O(1)$ denotes a generic constant independent of the data.

In particular, if $\kappa(\xi) \equiv L$, then the estimate takes the form

$$N \geq O(1) \left(\frac{LD}{\varepsilon - \delta}\right)^2 \left[n \log \left(\frac{O(1)DL}{\varepsilon - \delta}\right) + \log \left(\frac{1}{\alpha}\right)\right].$$

Suppose further that for some $c > 0$, $\gamma \geq 1$ and $\bar{\varepsilon} > \varepsilon$ the following growth condition holds

$$f(x) \geq v^0 + c[\text{dist}(x, S^0)]^\gamma, \ \ \forall x \in S^{\bar{\varepsilon}},$$

and that the problem is convex. Then, for $\delta \in [0, \varepsilon/2]$, we have the following estimate of the required sample size:

$$N \geq \left(\frac{O(1)LD}{c^{1/\gamma}\varepsilon^{(\gamma-1)/\gamma}}\right)^2 \left[n \log \left(\frac{O(1)\bar{D}L}{\varepsilon}\right) + \log \left(\frac{1}{\alpha}\right)\right],$$

where $\bar{D}$ is the diameter of $S^{\bar{\varepsilon}}$. In particular, if $S^0 = \{x^0\}$ is a singleton and $\gamma = 1$, we have the estimate (independent of ε):

$$N \geq O(1)c^{-2}L^2 \left[n \log(O(1)c^{-1}L) + \log(\alpha^{-1})\right].$$

Example Let $F(x,\xi) := \|x\|^{2k} - 2k\left(\xi^{\mathsf{T}}x\right)$, where $k \in \mathbb{N}$ and

$$\mathcal{X} := \{x \in \mathbb{R}^n : \|x\| \le 1\}.$$

Suppose, that $\xi \sim N(0, \sigma^2 I_n)$. Then $f(x) = \|x\|^{2k}$, and for $\varepsilon \in [0,1]$, the set of ε-optimal solutions of the true problem is

$$\{x : \|x\|^{2k} \le \varepsilon\}.$$

Let $\bar{\xi}_N := (\xi^1 + ... + \xi^N)/N$. The corresponding sample average function is

$$\hat{f}_N(x) = \|x\|^{2k} - 2k\left(\bar{\xi}_N^{\mathsf{T}}x\right),$$

and $\hat{x}_N = \|\bar{\xi}_N\|^{-\gamma}\bar{\xi}_N$, where $\gamma := \frac{2k-2}{2k-1}$ if $\|\bar{\xi}_N\| \le 1$, and $\gamma = 1$ if $\|\bar{\xi}_N\| > 1$. Therefore, for $\varepsilon \in (0,1)$, the optimal solution of the SAA problem is an ε-optimal solution of the true problem iff $\|\bar{\xi}_N\|^{\nu} \le \varepsilon$, where $\nu := \frac{2k}{2k-1}$.

We have that $\bar{\xi}_N \sim N(0, \sigma^2 N^{-1} I_n)$, and hence $N\|\bar{\xi}_N\|^2/\sigma^2$ has the chi-square distribution with n degrees of freedom. Consequently, the probability that $\|\bar{\xi}_N\|^\nu > \varepsilon$ is equal to the probability

$$\text{Prob}\left(\chi^2_n > N\varepsilon^{2/\nu}/\sigma^2\right).$$

Moreover, $\mathbb{E}[\chi^2_n] = n$ and the probability $\text{Prob}(\chi^2_n > n)$ increases and tends to 1/2 as n increases. Consequently, for $\alpha \in (0, 0.3)$ and $\varepsilon \in (0, 1)$, for example, the sample size N should satisfy

$$N > \frac{n\sigma^2}{\varepsilon^{2/\nu}} \tag{27}$$

in order to have the property: "with probability $1 - \alpha$ an (exact) optimal solution of the SAA problem is an ε-optimal solution of the true problem". Note that $\nu \to 1$ as $k \to \infty$.

Stochastic Approximation (SA) approach

An alternative approach is going back to Robbins and Monro (1951) and is known as the Stochastic Approximation (SA) method. Assume that the true problem is *convex*, i.e., the set $\mathcal{X} \subset \mathbb{R}^n$ is convex (and closed and bounded) and function $F(\cdot, \xi) : \mathcal{X} \to \mathbb{R}$ is convex for all $\xi \in \Xi$.

Also assume existence of the following *stochastic oracle*: given $x \in \mathcal{X}$ and a random realization $\xi \in \Xi$, the oracle returns the quantity $F(x, \xi)$ and a *stochastic subgradient* – a vector $G(x, \xi)$ such that $g(x) := \mathbb{E}[G(x, \xi)]$ is well defined and is a subgradient of $f(\cdot)$ at x, i.e., $g(x) \in \partial f(x)$. For example, use $G(x, \xi) \in \partial_x F(x, \xi)$.

Classical SA algorithm

$$x_{j+1} = \Pi_X(x_j - \gamma_j G(x_j, \xi^j)),$$

where $\gamma_j = \theta/j$, $\theta > 0$, and $\Pi_X(x) = \arg\min_{z \in X} \|x - z\|_2$ is the orthogonal (Euclidean) projection onto $\mathcal{X}$. Theoretical bound (assuming $f(\cdot)$ is *strongly convex and differentiable*)

$$\mathbb{E}\left[f(x_j) - v^0\right] = O(j^{-1}),$$

for an optimal choice of constant θ (here v^0 is the optimal value of the true problem). This algorithm is very sensitive to choice of θ, does not work well in practice.

As a simple example consider $f(x) = \frac{1}{2}cx^2$, with $c = 0.2$ and $\mathcal{X} = [-1, 1] \subset \mathbb{R}$ and assume that there is no noise, i.e., $G(x, \xi) \equiv \nabla f(x)$. Suppose, further, that we take $\theta = 1$ (i.e., $\gamma_j = 1/j$), which will be the optimal choice for $c = 1$. Then the iteration process becomes

$$x_{j+1} = x_j - f'(x_j)/j = \left(1 - \tfrac{1}{5j}\right) x_j,$$

and hence starting with $x_1 = 1$,

$$\begin{array}{rl} x_j & = \prod_{s=1}^{j-1} \left(1 - \frac{1}{5s}\right) = \exp\left\{-\sum_{s=1}^{j-1} \ln\left(1 + \frac{1}{5s-1}\right)\right\} \\ & > \exp\left\{-\sum_{s=1}^{j-1} \frac{1}{5s-1}\right\} > \exp\left\{-\left(0.25 + \int_1^{j-1} \frac{1}{5t-1} dt\right)\right\} \\ & > \exp\left\{-0.25 + 0.2 \ln 1.25 - \frac{1}{5} \ln j\right\} > 0.8 j^{-1/5}. \end{array}$$

That is, the convergence is extremely slow. For example for $j = 10^9$ the error of the iterated solution is greater than 0.015. On the other hand for the optimal stepsize factor of $\theta = 1/c = 5$, the optimal solution $\bar{x} = 0$ is found in one iteration.

Robust SA approach (with averaging) (B. Polyak, 1990): consider

$$\tilde{x}_j = \sum_{t=1}^{j} \nu_t x_t, \quad \text{where } \nu_t = \frac{\gamma_t}{\sum_{\tau=1}^{j} \gamma_\tau}.$$

Let $D_{\mathcal{X}} = \max_{x \in \mathcal{X}} \|x - x_1\|_2$, and assume that

$$\mathbb{E}\left[\|G(x,\xi)\|_2^2\right] \le M^2, \quad \forall x \in \mathcal{X},$$

for some constant $M > 0$. Then

$$\mathbb{E}\left[f(\tilde{x}_j) - v^0\right] \le \frac{D_{\mathcal{X}}^2 + M^2 \sum_{t=1}^{j} \gamma_t^2}{2 \sum_{t=1}^{j} \gamma_t}.$$

For $\gamma_t = \frac{\theta D_{\mathcal{X}}}{M\sqrt{t}}$, after N iterations, we have

$$\mathbb{E}\left[f(\tilde{x}_N) - v^0\right] \le \frac{\max\left\{\theta, \theta^{-1}\right\} M (D_{\mathcal{X}}^2 + \log N)}{2 D_{\mathcal{X}} \sqrt{N}}.$$

Constant step size variant: fixed in advance sample size (number of iterations) N and step size $\gamma_j \equiv \gamma$, $j = 1, ..., N$: $\tilde{x}_N = \frac{1}{N}\sum_{j=1}^N x_j$. Theoretical bound

$$\mathbb{E}[f(\tilde{x}_N) - v^0] \le \frac{D_{\mathcal{X}}^2}{2\gamma N} + \frac{\gamma M^2}{2}.$$

For optimal (up to factor θ) $\gamma = \frac{\theta D_{\mathcal{X}}}{M\sqrt{N}}$ we have

$$\mathbb{E}\left[f(\tilde{x}_N) - v^0\right] \le \frac{D_{\mathcal{X}} M}{2\theta\sqrt{N}} + \frac{\theta D_{\mathcal{X}} M}{2\sqrt{N}} \le \frac{\kappa D_{\mathcal{X}} M}{\sqrt{N}},$$

where $\kappa = \max\{\theta, \theta^{-1}\}$. By Markov inequality it follows that

$$\text{Prob}\left\{f(\tilde{x}_N) - v^0 > \varepsilon\right\} \le \frac{\kappa D_{\mathcal{X}} M}{\varepsilon\sqrt{N}}.$$

Hence the sample size $N \ge \frac{\kappa^2 D_{\mathcal{X}}^2 M^2}{\varepsilon^2 \alpha^2}$ guarantees that $\text{Prob}\left\{f(\tilde{x}_N) - v^0 > \varepsilon\right\} \le \alpha$.

Mirror Decent SA method (A. Nemirovski)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $\omega : \mathcal{X} \to \mathbb{R}$ be continuously differentiable strongly convex on $\mathcal{X}$ with respect to $\|\cdot\|$ function, i.e., for $x, x' \in \mathcal{X}$:

$$\omega(x') \geq \omega(x) + (x' - x)^{\mathsf{T}} \nabla \omega(x) + \tfrac{1}{2} c \|x' - x\|^2.$$

Prox mapping $P_x : \mathbb{R}^n \to \mathcal{X}$:

$$P_x(y) = \arg\min_{z \in X} \left\{ \omega(z) + (y - \nabla \omega(x))^{\mathsf{T}} z \right\}.$$

For $\omega(x) = \frac{1}{2} x^T x$ we have that $P_x(y) = \Pi_{\mathcal{X}}(x - y)$. Set

$$x_{j+1} = P_{x_j}(\gamma_j G(x_j, \xi^j)),$$

$$\tilde{x}_j = \sum_{t=1}^{j} \nu_t x_t, \quad \text{where } \nu_t = \frac{\gamma_t}{\sum_{\tau=1}^{j} \gamma_\tau}.$$

Then

$$\mathbb{E}\left[f(\tilde{x}_j) - v^0\right] \le \frac{D^2_{\omega,\mathcal{X}} + (2c)^{-1} M_*^2 \sum_{t=1}^{j} \gamma_t^2}{2\sum_{t=1}^{j} \gamma_t},$$

where M_* is a positive constant such that

$$\mathbb{E}\left[\|G(x,\xi)\|_*^2\right] \le M_*^2, \ \ \forall x \in \mathcal{X},$$

$\|x\|_* = \sup_{\|z\|\le 1} x^T z$ is the dual norm of the norm $\|\cdot\|$, and

$$D_{\omega,\mathcal{X}} = \left[\max_{z\in X} \omega(z) - \min_{x\in X} \omega(x)\right]^{1/2}.$$

For constant step size $\gamma_j = \gamma$, $j = 1, ..., N$, with optimal (up to factor $\theta > 0$) stepsize $\gamma = \frac{\theta D_{\omega,\mathcal{X}}}{M_*}\sqrt{\frac{2c}{N}}$, we have

$$\mathbb{E}\left[f(\tilde{x}_N) - v^0\right] \le \frac{\max\{\theta, \theta^{-1}\}\sqrt{2} D_{\omega,\mathcal{X}} M_*}{\sqrt{cN}}.$$

Large Deviations type bounds.

Suppose that

$$\mathbb{E}\left[\exp\left\{\|G(x,\xi)\|_*^2/M_*^2\right\}\right] \leq \exp\{1\}, \ \ \forall x \in \mathcal{X}.$$

Then for the constant stepsize policy and any $\Omega \geq 1$:

$$\mathrm{Prob}\left\{f(\tilde{x}_N) - v^0 > \frac{\sqrt{2}\max\{\theta,\theta^{-1}\}M_* D_{\omega,\mathcal{X}}(12+2\Omega)}{\sqrt{cN}}\right\} \leq 2\exp\{-\Omega\}.$$

It follows that for $\varepsilon > 0$,$\alpha \in (0,1)$ and

$$N_{SA} = O(1)\varepsilon^{-2} D_{\omega,\mathcal{X}}^2 M_*^2 \log^2(1/\alpha),$$

we are guaranteed that $\mathrm{Prob}\left\{f(\tilde{x}_N) - v^0 > \varepsilon\right\} \leq \alpha$.

This can be compared with a corresponding estimate of the sample size for the SAA method:

$$N_{\mathrm{SAA}} = O(1)\varepsilon^{-2} R^2 M_*^2 \Big[\log(1/\alpha) + n\log\left(RM_*/\varepsilon\right)\Big].$$

Example

Let $\mathcal{X} = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1,\ x \geq 0\}$ be a standard simplex. Consider two setups for the Mirror Descent SA: *Euclidean setup*, where $\|\cdot\| = \|\cdot\|_2$ and $\omega(x) = \frac{1}{2}x^T x$, and ℓ_1-*setup*, where $\|\cdot\| = \|\cdot\|_1$ with $\|\cdot\|_* = \|\cdot\|_\infty$, and ω is the *entropy* function

$$\omega(x) = \sum_{i=1}^n x_i \ln x_i.$$

For the constant stepsize policy, the Euclidean setup leads to

$$\mathbb{E}\left[f(\tilde{x}_N) - v^0\right] \leq O(1) \max\left\{\theta, \theta^{-1}\right\} M N^{-1/2},$$

with $M^2 = \sup_{x\in\mathcal{X}} \mathbb{E}\left[\|G(x,\xi)\|_2^2\right]$ (note that the Euclidean diameter of $\mathcal{X}$ is $\sqrt{2}$).

The ℓ_1-setup corresponds to $D_{\omega,\mathcal{X}} = \sqrt{\ln n}$, $c = 1$ and $x_1 = \arg\min_X \omega = n^{-1}(1,...,1)^T$. The associated Mirror Descent SA is easily implementable: the prox mapping can be computed in $O(n)$ operations according to the explicit formula:

$$[P_x(y)]_i = \frac{x_i e^{-y_i}}{\sum_{k=1}^n x_k e^{-y_k}}, \quad i = 1, ..., n.$$

The efficiency estimate guaranteed with the ℓ_1-setup is

$$\mathbb{E}\left[f(\tilde{x}_N) - v^0\right] \le O(1) \max\left\{\theta, \theta^{-1}\right\} \sqrt{\log n} M_* N^{-1/2},$$

with $M_*^2 = \sup_{x \in X} \mathbb{E}\left[\|G(x,\xi)\|_\infty^2\right]$, To compare the Euclidean and ℓ_1-setups, observe that $M_* \le M$, and the ratio M_*/M can be as small as $n^{-1/2}$. When X is a standard simplex of large dimension, we have strong reasons to prefer the ℓ_1-setup to the usual Euclidean one.

Bounds by Mirror Decent SA method.

For iterates

$$x_{j+1} = P_{x_j}(\gamma_j G(x_j, \xi^j)).$$

Consider

$$f^N(x) := \sum_{j=1}^{N} \nu_j [f(x_j) + g(x_j)^{\mathsf{T}}(x - x_j)],$$

where $f(x) = \mathbb{E}[F(x, \boldsymbol{\xi})]$, $g(x) = \mathbb{E}[G(x, \boldsymbol{\xi})]$ and $\nu_j := \gamma_j / (\sum_{j=1}^{N} \gamma_j)$.
Since $g(x) \in \partial f(x)$, it follows that

$$f_*^N := \min_{x \in X} f^N(x) \le v^0.$$

Also $v^0 \le f(\tilde{x}_N)$ and by convexity of f,

$$f(\tilde{x}_N) \le f^{*N} := \sum_{j=1}^{N} \nu_j f(x_j).$$

That is, for any realization of the random sample $\xi^1, ..., \xi^N$,

$$f_*^N \leq v^0 \leq f^{*N}.$$

Computational (observable) counterparts of these bounds:

$$\underline{f}^N := \min_{x \in X} \sum_{j=1}^{N} \nu_j [F(x_j, \xi^j) + G(x_j, \xi^j)^{\mathsf{T}}(x - x_j)],$$

$$\overline{f}^N := \sum_{j=1}^{N} \nu_j F(x_j, \xi^j).$$

We have that $\mathbb{E}\left[f^{*N}\right] = \mathbb{E}\left[\overline{f}^N\right]$, and

$$\mathbb{E}\left[\underline{f}^N\right] \leq v^0 \leq \mathbb{E}\left[\overline{f}^N\right].$$

Complexity of multistage stochastic programming

Conditional sampling. Let ξ_2^i, $i = 1, ..., N_1$, be an iid random sample of $\boldsymbol{\xi}_2$. Conditional on $\boldsymbol{\xi}_2 = \xi_2^i$, a random sample ξ_3^{ij}, $j = 1, ..., N_2$, is generated and etc. The obtained scenario tree is considered as a sample approximation of the true problem. Note that the total number of scenarios $N = \prod_{t=1}^{T-1} N_t$ and each scenario in the generated tree is considered with the same probability $1/N$. Note also that in the case of stagewise independence of the corresponding random process, we have two possible strategies. We can generate a different (independent) sample ξ_3^{ij}, $j = 1, ..., N_2$, for every generated node ξ_2^i, or we can use the same sample ξ_3^j, $j = 1, ..., N_2$, for every ξ_2^i. In the second case we preserve the stagewise independence condition for the generated scenario tree.

For $T = 3$, under certain regularity conditions, for $\varepsilon > 0$ and $\alpha \in (0, 1)$, and the sample sizes N_1 and N_2 satisfying

$$O(1)\left[\left(\frac{D_1 L_1}{\varepsilon}\right)^{n_1} \exp\left\{-\frac{O(1) N_1 \varepsilon^2}{\sigma_1^2}\right\} + \left(\frac{D_2 L_2}{\varepsilon}\right)^{n_2} \exp\left\{-\frac{O(1) N_2 \varepsilon^2}{\sigma_2^2}\right\}\right] \leq \alpha,$$

we have that any first-stage $\varepsilon/2$-optimal solution of the SAA problem is an ε-optimal first-stage solution of the true problem with probability at least $1 - \alpha$. (Here $D_1, D_2, L_1, L_2, \sigma_1, \sigma_2$ are certain analogues of similar constants in the sample size estimate of two stage problem.)

In particular, suppose that $N_1 = N_2$ and take $L := \max\{L_1, L_2\}$, $D := \max\{D_1, D_2\}$, $\sigma^2 := \max\{\sigma_1^2, \sigma_2^2\}$ and $n := \max\{n_1, n_2\}$. Then the required sample size $N_1 = N_2$:

$$N_1 \geq \frac{O(1)\sigma^2}{\varepsilon^2}\left[n \log\left(\frac{O(1)DL}{\varepsilon}\right) + \log\left(\frac{1}{\alpha}\right)\right],$$

with total number of scenarios $N = N_1^2$.

That is, the total number of scenarios needed to solve a T-stage stochastic program with a reasonable accuracy by the SAA method grows exponentially with increase of the number of stages T. Another way of putting this is that the number of scenarios needed to solve T-stage problem would grow as $O\left(\varepsilon^{-2(T-1)}\right)$ with decrease of the error level $\varepsilon > 0$.

This indicates that from the point of view of the **number of scenarios**, complexity of multistage programming problems grows exponentially with increase of the number of stages. Furthermore, even if the SAA problem can be solved, its solution does not define a policy for the true problem and of use may be only the computed first stage solution.

Risk-averse approach

Consider the following formulation of the financial planning problem:

$$\operatorname*{Max}_{x \geq 0} \ \mathbb{E}[W_1] - \lambda \mathbb{D}[W_1] \quad \text{s.t.} \ \sum_{i=1}^{n} x_i = W_0, \tag{28}$$

where $\mathbb{D}[W_1]$ is a measure of dispersion (variability) of W_1 and $\lambda \geq 0$ represents a compromise weight between maximizing returns and minimizing risk of investment.

Markowitz (1952) approach: take $\mathbb{D}[W_1] := \text{Var}[W_1]$. In the present (linear) case

$$\text{Var}[W_1] = \text{Var}\left[\sum_{i=1}^{n} \xi_i x_i\right] = x^T \Sigma x,$$

where Σ is the covariance matrix of ξ.

Min-max approach to stochastic programming:

$$\operatorname*{Min}_{x\in\mathcal{X}} \left\{ f(x) := \sup_{\mu\in\mathfrak{A}} \mathbb{E}_\mu[F(x,\omega)] \right\},$$

where $F : \mathbb{R}^n \times \Omega \to \mathbb{R}$ and $\mathfrak{A}$ is a set of probability measures (distributions) on the sample space $(\Omega, \mathcal{F})$.

Optimization of mean-risk models:

$$\operatorname*{Min}_{x\in\mathcal{X}} \rho[F_x(\omega)],$$

where $\rho : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ is a mean-risk function, $\mathcal{Z}$ is a (linear) space of "allowable" functions $Z(\omega)$ and $F_x(\cdot) = F(x,\cdot) \in \mathcal{Z}$ for all $x \in X$.

Markowitz's approach: $\rho(Z) := \mathbb{E}[Z] + c\text{Var}[Z], \quad Z \in \mathcal{Z}$, where $c > 0$ is a weight constant (note that here we deal with minimization, rather than maximization, problem).

Axiomatic approach (coherent measures of risk), by Artzner, Delbaen, Eber, Heath (1999):

(A1) Convexity:

$$\rho(\alpha Z_1 + (1-\alpha)Z_2) \le \alpha\rho(Z_1) + (1-\alpha)\rho(Z_2)$$

for all $Z_1, Z_2 \in \mathcal{Z}$ and $\alpha \in [0, 1]$.

(A2) Monotonicity: If $Z_1, Z_2 \in \mathcal{Z}$ and $Z_2 \ge Z_1$, then $\rho(Z_2) \ge \rho(Z_1)$.

(A3) Translation Equivariance: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.

(A4) Positive Homogeneity:

$$\rho(\alpha Z) = \alpha\rho(Z), \;\; Z \in \mathcal{Z}, \; \alpha > 0.$$

Space $\mathcal{Z}$ is paired with a linear space $\mathcal{Y}$ of finite signed measures on $(\Omega, \mathcal{F})$ such that the scalar product (bilinear form)

$$\langle \mu, Z \rangle := \int_\Omega Z(\omega) d\mu(\omega)$$

is well defined for all $Z \in \mathcal{Z}$ and $\mu \in \mathcal{Y}$. Typical examples $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$ and $\mathcal{Y} := L_q(\Omega, \mathcal{F}, P)$, where $p, q \in [1, +\infty]$ such that $1/p + 1/q = 1$, and P is a probability (reference) measure on $(\Omega, \mathcal{F})$.

Dual representation of risk functions

By Fenchel-Moreau theorem if ρ is convex (assumption (A1)) and lower semicontinuous, then

$$\rho(Z) = \sup_{\mu \in \mathfrak{A}} \{\langle \mu, Z \rangle - \rho^*(\mu)\} ,$$

where

$$\rho^*(\mu) := \sup_{Z\in\mathcal{Z}} \{\langle \mu, Z\rangle - \rho(Z)\},$$
$$\mathfrak{A} := \mathrm{dom}(\rho^*) = \{\mu \in \mathcal{Y} : \rho^*(\mu) < +\infty\}.$$

It is possible to show that condition (A2) (monotonicity) holds iff $\mu \succeq 0$ for every $\mu \in \mathfrak{A}$. Condition (A3) (translation equivariance) holds iff $\mu(\Omega) = 1$ for every $\mu \in \mathfrak{A}$. If ρ is positively homogeneous, then $\rho^*(\mu) = 0$ for every $\mu \in \mathfrak{A}$. If conditions (A1)–(A4) hold, then $\mathfrak{A}$ is a set of probability measures and

$$\rho(Z) = \sup_{\mu\in\mathfrak{A}} \mathbb{E}_\mu[Z].$$

Consequently, problem $\mathrm{Min}_{x\in\mathcal{X}}\, \rho[F(x,\omega)]$ is equivalent to the min-max problem.

Average Value-at-Risk (also called Conditional Value-at-Risk)

Chance (probabilistic) constraint:

$$\text{Prob}\Big\{C(x,\xi) > \tau\Big\} \le \alpha. \tag{29}$$

Constraint (29) can be written as

$$\mathbb{E}\left[\mathbb{1}_{(0,\infty)}(Z_x)\right] \le \alpha,$$

where $Z_x := C(x,\xi) - \tau$. Let $\psi : \mathbb{R} \to \mathbb{R}_+$ be nondecreasing, convex function such that $\psi(\cdot) \ge \mathbb{1}_{(0,\infty)}(\cdot)$. We have that

$$\inf_{t>0} \mathbb{E}[\psi(tZ_x)] \ge \mathbb{E}\left[\mathbb{1}_{(0,\infty)}(Z_x)\right],$$

and hence

$$\inf_{t>0} \mathbb{E}[\psi(tZ_x)] \le \alpha \tag{30}$$

is a conservative approximation of the chance constraint (29).

The choice $\psi^*(z) := [1+z]_+$ gives best conservative approximation. For this choice of ψ, (30) is equivalent to

$$\underbrace{\inf_{t\in\mathbb{R}} \left\{t + \alpha^{-1}\mathbb{E}[Z_x - t]_+\right\}}_{\mathsf{AV@R}_\alpha(Z_x)} \le 0. \tag{31}$$

Note that the minimum in the left hand side of (31) is attained at $t^* = \mathsf{V@R}_\alpha(Z_x)$, where

$$\mathsf{V@R}_\alpha(Z) = H_Z^{-1}(1-\alpha) := \inf\left\{t : H_Z(t) \ge 1-\alpha\right\},$$

with $H_Z(t) := \mathrm{Prob}(Z \le t)$ being the cdf of Z.

Constraint $\text{Prob}(Z \le 0) \ge 1-\alpha$ is equivalent to $\text{V@R}_\alpha(Z) \le 0$. Therefore $\text{AV@R}_\alpha(C(x,\xi)) \le \tau$ is a conservative approximation of the chance constraint (29).

Note that $\rho(Z) = \text{AV@R}_\alpha(Z)$ is a coherent risk measure. It is natural to take here $\mathcal{Z} = L_1(\Omega, \mathcal{F}, P)$ and $\mathcal{Z}^* = L_\infty(\Omega, \mathcal{F}, P)$. Dual representation

$$\text{AV@R}_\alpha(Z) = \sup_{\zeta \in \mathfrak{A}} \int_\Omega Z(\omega)\zeta(\omega)dP(\omega),$$

where

$$\mathfrak{A} = \left\{ \zeta \in \mathcal{Z}^* : 0 \le \zeta(\omega) \le \alpha^{-1} \; a.e.\; \omega \in \Omega,\; \int_\Omega \zeta(\omega)dP(\omega) = 1 \right\}.$$

Suppose that $Z^1, ..., Z^N$ is an iid sample of random variable Z. Then by replacing the true probability distribution P of Z by its empirical* estimate $P_N = \sum_{j=1}^N \delta(Z^j)$ we obtain sample estimate of $\theta := \mathsf{AV@R}_\alpha(Z)$:

$$\widehat{\theta}_N = \inf_{t\in\mathbb{R}} \left\{ t + \alpha^{-1} N^{-1} \sum_{j=1}^N [Z^j - t]_+ \right\}.$$

By the Delta theorem (Theorem 2) we have

$$\widehat{\theta}_N = \inf_{t\in[t^*,t^{**}]} \left\{ t + \alpha^{-1} N^{-1} \sum_{j=1}^N [Z^j - t]_+ \right\} + o_p(N^{-1/2}),$$

where $[t^*, t^{**}]$ is the set of $1-\alpha$ quantiles of the distribution of the random vector Z. If the $(1-\alpha)$-quantile $t^* = \mathsf{V@R}_\alpha(Z)$ is unique, then

$$\widehat{\theta}_N = \mathsf{V@R}_\alpha(Z) + \alpha^{-1} Y_N + o_p(N^{-1/2}),$$

where $Y_N := N^{-1} \sum_{j=1}^N [Z^j - t^*]_+$. This approximation can be reasonable when N is significantly bigger than $1/\alpha$.

*By $\delta(z)$ we denote measure of mass one at the point z.

Example Mean-variance risk function ($c > 0$):

$$\rho(Z) := \mathbb{E}[Z] + c\,\mathsf{Var}[Z], \;\; Z \in \mathcal{Z} := L_2(\Omega, \mathcal{F}, P).$$

Dual representation:

$$\rho(Z) = \sup_{\zeta \in \mathcal{Z}, \mathbb{E}[\zeta]=1} \Big\{ \langle \zeta, Z \rangle - (4c)^{-1} \mathsf{Var}[\zeta] \Big\}.$$

Satisfies (A1) and (A3), does not satisfy (A2) and (A4).

Example Mean-upper-semideviation risk function of order $p \in [1, +\infty)$, $Z \in \mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $c \geq 0$ and

$$\rho(Z) := \mathbb{E}[Z] + c\Big(\mathbb{E}_P\Big\{[Z - \mathbb{E}_P[Z]]_+^p\Big\}\Big)^{1/p}.$$

Here ρ satisfies (A1),(A3),(A4), and also (A2) (monotonicity) if $c \leq 1$. The max-representation

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \int_\Omega Z(\omega)\zeta(\omega)dP(\omega)$$

holds with $\mathfrak{A} = \{\zeta : \zeta = 1 + h - \int_\Omega h dP, \; \|h\|_q \leq c, \; h \geq 0\}$.

Let $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$. It is said that a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is *law invariant* with respect to the reference probability measure P, if for all $Z_1, Z_2 \in \mathcal{Z}$ we have the implication

$$\left\{ Z_1 \stackrel{\mathcal{D}}{\sim} Z_2 \right\} \Rightarrow \left\{ \rho(Z_1) = \rho(Z_2) \right\}.$$

Theorem 3 (Kusuoka) *Suppose that the probability space* $(\Omega, \mathcal{F}, P)$ *is nonatomic and let* $\rho : \mathcal{Z} \to \mathbb{R}$ *be a law invariant coherent risk measure. Then there exists a set* $\mathfrak{M}$ *of probability measures on the interval* $(0, 1]$ *(equipped with its Borel sigma algebra) such that*

$$\rho(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha), \ \ \forall Z \in \mathcal{Z}. \tag{32}$$

Two-stage stochastic programs.

Suppose that the function $F(x,\omega)$ is given by the optimal value of the second stage problem:

$$\operatorname*{Min}_{y\in\mathfrak{X}(x,\omega)} g(x,y,\omega),$$

where $g : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}$ is a random lower semicontinuous function and $\mathfrak{X} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ is a closed valued measurable multifunction. Then, for a fixed $x \in \mathcal{X}$,

$$\rho(F(x,\omega)) = \inf\nolimits_{y(\cdot)\in\mathfrak{X}(x,\cdot)} \rho[g(x,y(\omega),\omega)],$$

and hence the the first stage problem is equivalent to the problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{x\in\mathcal{X},\, y(\cdot)} & \rho[g(x,y(\omega),\omega)] \\ \text{subject to} & y(\omega) \in \mathfrak{X}(x,\omega) \text{ a.e. } \omega \in \Omega. \end{array}$$

How this can be extended to a dynamic process (multistage programming)?

Tree representation. Notation: Ω_t set of nodes at stage $t = 1,...,T$, $K_t = |\Omega_t|$, $C_a \subset \Omega_{t+1}$ set of children nodes of $a \in \Omega_t$. With every node $a \in \Omega_t$ we can associate risk function

$$\rho^a : \mathbb{R}^{|C_a|} \to \mathbb{R},\ a \in \Omega_t,\ t = 1,...,T-1.$$

For example, let $p^a \in \mathbb{R}^{|C_a|}$ be conditional probability vector of moving from node a to its children nodes, and

$$\rho^a(Z) := \mathbb{E}_{p^a}[Z],\ a \in \Omega_t,\ t = 1,...,T-1,$$

or

$$\rho^a(Z) := \mathbb{E}_{p^a}[Z] + c_a \mathbb{E}_{p^a}\Big[Z - \mathbb{E}_{p^a}[Z]\Big]_+,\ c_a \in [0,1].$$

Note that $\mathbb{R}^{K_{t+1}} = \mathbb{R}^{|C_{a_1}|} \times \cdots \times \mathbb{R}^{|C_{a_{K_t}}|}$, where $\{a_1,...,a_{K_t}\} = \Omega_t$. Define

$$\rho_{t+1} := (\rho^{a_1},...,\rho^{a_{K_t}}) : \mathbb{R}^{K_{t+1}} \to \mathbb{R}^{K_t},\ \ t = 1,...,T-1.$$

With the considered tree is associated sequence of (finite) sigma algebras $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T$, with $\mathcal{F}_T = 2^{\Omega_T}$ and $\mathcal{F}_1 = \{\emptyset, \Omega_T\}$. Let $\mathcal{Z}_t$ be the space of all $\mathcal{F}_t$-measurable functions $Z : \Omega_T \to \mathbb{R}$, i.e., $Z(\cdot)$ is constant on every C_a, $a \in \Omega_t$. The space $\mathcal{Z}_t$ can be identified with $\mathbb{R}^{K_t}$. It is said that $\rho_{t+1} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ is a conditional risk mapping if the following conditions hold

(A'1) Convexity:

$$\rho_{t+1}(\tau Z_1 + (1-\tau)Z_2) \preceq \tau \rho_{t+1}(Z_1) + (1-\tau)\rho_{t+1}(Z_2)$$

for all $Z_1, Z_2 \in \mathcal{Z}_{t+1}$ and $\tau \in [0,1]$.

(A'2) Monotonicity: If $Z_1, Z_2 \in \mathcal{Z}_{t+1}$ and $Z_2 \succeq Z_1$, then $\rho_{t+1}(Z_2) \succeq \rho_{t+1}(Z_1)$.

(A'3) Translation Equivariance: If $Y \in \mathcal{Z}_t$ and $Z \in \mathcal{Z}_{t+1}$, then $\rho_{t+1}(Z+Y) = \rho_{t+1}(Z) + Y$,

(A'4) Positive Homogeneity:

$$\rho_{t+1}(\tau Z) = \tau \rho_{t+1}(Z), \;\; Z \in \mathcal{Z}_{t+1}, \; \tau > 0.$$

We have that with each coherent risk measure ρ^a, $a \in \Omega_t$ is associated set $\mathfrak{A}_{t+1}(a) \subset \mathbb{R}^{K_{t+1}}$ of probability vectors such that

$$\rho^a(Z) = \max_{p \in \mathfrak{A}_{t+1}(a)} \mathbb{E}_p[Z].$$

Let $\nu = (\nu_a)_{a \in \Omega_t}$ be a probability distribution on Ω_t, and

$$\mathfrak{C}_{t+1} := \left\{ \mu = \sum_{a \in \Omega_t} \nu_a p^a : p^a \in \mathfrak{A}_{t+1}(a) \right\}.$$

We have that

$$\mathbb{E}_\mu[Z|\mathcal{F}_t] = \mathbb{E}_{p^a}[Z], \; Z \in \mathcal{Z}_{t+1},$$

and it follows that

$$\rho_{t+1}(Z) = \max_{\mu \in \mathfrak{C}_{t+1}} \mathbb{E}_\mu[Z|\mathcal{F}_t].$$

Risk averse multistage programming (nested formulation):

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} \; f_1(x_1) + \rho_2 \Big[\inf_{x_2 \in \mathcal{X}_2(x_1,\omega)} f_2(x_2,\omega) + \dots$$
$$+ \rho_{T-1} \Big[\inf_{x_{T-1} \in \mathcal{X}_T(x_{T-2},\omega)} f_{T-1}(x_{T-1},\omega) \tag{33}$$
$$+ \rho_T [\inf_{x_T \in \mathcal{X}_T(x_{T-1},\omega)} f_T(x_T,\omega)] \Big] \Big].$$

Here ω is an element of $\Omega := \Omega_T$, the objective functions $f_t : \mathbb{R}^{n_{t-1}} \times \Omega \to \mathbb{R}$ are real valued functions and $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \Omega \rightrightarrows \mathbb{R}^{n_t}$, $t = 2, \dots, T$, are multifunctions such that $f_t(x_t, \cdot)$ and $\mathcal{X}_t(x_{t-1}, \cdot)$ are $\mathcal{F}_t$-measurable for all x_t and x_{t-1}. Note that if the corresponding risk measures ρ^a are defined as conditional expectations, then the above multistage problem becomes a risk neutral multistage problem.

Consider function $\varrho : \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_T \to \mathbb{R}$ defined as

$$\varrho(Z_1, \dots, Z_T) := Z_1 + \rho_2\Big[Z_2 + \cdots + \rho_{T-1}\big[Z_{T-1} + \rho_T[Z_T]\big]\Big]. \quad (34)$$

By condition (A'3) we have that

$$\rho_{T-1}\big[Z_{T-1} + \rho_T[Z_T]\big] = \rho_{T-1} \circ \rho_T\big[Z_{T-1} + Z_T\big].$$

By continuing this process we obtain that

$$\varrho(Z_1, \dots, Z_T) = \bar{\rho}(Z_1 + \cdots + Z_T),$$

where $\bar{\rho} := \rho_2 \circ \cdots \circ \rho_T$. We refer to $\bar{\rho} : \mathcal{Z}_T \to \mathbb{R}$ as the composite risk measure. That is,

$$\bar{\rho}(Z_1 + \cdots + Z_T) = Z_1 + \rho_2\Big[Z_2 + \cdots + \rho_{T-1}\big[Z_{T-1} + \rho_T[Z_T]\big]\Big],$$

defined for $Z_t \in \mathcal{Z}_t$, $t = 1, \dots, T$. Conditions (A'1)–(A'4) imply that $\bar{\rho}$ is a coherent risk measure.

We can write the risk averse multistage programming problem

$$\begin{array}{ll} \underset{x_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T}{\text{Min}} & \bar{\rho}\Big[f_1(x_1) + f_2(\boldsymbol{x}_2(\omega), \omega) + \cdots + f_T(\boldsymbol{x}_T(\omega), \omega)\Big] \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ \boldsymbol{x}_t(\omega) \in \mathcal{X}_t(\boldsymbol{x}_{t-1}(\omega), \omega),\ t = 2, \ldots, T. \end{array}$$

Dynamic Programming Equations.

For the last period T we have

$$Q_T(x_{T-1}, \omega) := \inf_{x_T \in \mathcal{X}_T(x_{T-1}, \omega)} f_T(x_T, \omega),$$

$$\mathcal{Q}_T(x_{T-1}, \omega) := \rho_T[Q_T(x_{T-1}, \omega)],$$

and for $t = T-1,\dots,2$,

$$\mathcal{Q}_t(x_{t-1},\omega) := \rho_t\left[Q_t(x_{t-1},\omega)\right],$$

where

$$Q_t(x_{t-1},\omega) := \inf_{x_t \in \mathcal{X}_t(x_{t-1},\omega)} \Big\{ f_t(x_t,\omega) + \mathcal{Q}_{t+1}(x_t,\omega) \Big\}.$$

Finally, at the first stage we solve the problem

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} f_1(x_1) + \rho_2[Q_2(x_1,\omega)].$$

By using the conditional max-representation, we can write the dynamic programming equations in the form

$$Q_t(x_{t-1},\omega) = \inf_{x_t \in \mathcal{X}_t(x_{t-1},\omega)} \Big\{ f_t(x_t,\omega) + \sup_{\mu \in \mathfrak{C}_{t+1}} \mathbb{E}_\mu\Big[Q_{t+1}(x_t)\Big|\mathcal{F}_t\Big](\omega) \Big\}.$$

Time consistency of stochastic programming problems

Consider a multiperiod stochastic program

$$\begin{array}{cl}\underset{x_1, x_2(\cdot), \ldots, x_T(\cdot)}{\text{Min}} & \varrho\Big(f_1(x_1), f_2(x_2(\xi_{[2]}), \xi_2), \ldots, f_T\left(x_T(\xi_{[T]}), \xi_T\right)\Big) \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t),\ t = 2, \ldots, T,\end{array}$$

where $\varrho : \mathcal{Z}_1 \times \mathcal{Z}_2 \times \cdots \times \mathcal{Z}_T \to \mathbb{R}$ is a multiperiod risk measure.

Is it time consistent? For example is

$$\begin{array}{cl}\underset{x_1, x_2(\cdot), \ldots, x_T(\cdot)}{\text{Min}} & \mathsf{AV@R}_\alpha\Big(f_1(x_1) + \cdots + f_T\left(x_T(\xi_{[T]}), \xi_T\right)\Big) \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t),\ t = 2, \ldots, T,\end{array}$$

time consistent? Note that for $\alpha \in (0, 1)$,

$$\mathsf{AV@R}_\alpha(\cdot) \neq \mathsf{AV@R}_{\alpha|\xi_1}\Big(\mathsf{AV@R}_{\alpha|\xi_{[2]}}\big(\cdots \mathsf{AV@R}_{\alpha|\xi_{[T-1]}}(\cdot)\big)\Big).$$

Principle of conditional optimality.
At every state of the system, optimality of our decisions should not depend on scenarios which we already know cannot happen in the future.

Let us consider the following example. Let $\varrho(Z_1,...,Z_T) := Z_1 + \sum_{t=2}^T \mathsf{AV@R}_\alpha(Z_t)$. We can write

$$\varrho(Z_1,...,Z_T) := \inf_{r_2,...,r_T} \mathbb{E}\left\{Z_1 + \sum_{t=2}^T \left(r_t + \alpha^{-1}[Z_t - r_t]_+\right)\right\}.$$

The objective function of the corresponding multiperiod optimization problem can be written as

$$f_1(x_1) + \sum_{t=2}^T r_t + \mathbb{E}\left\{\sum_{t=2}^T \alpha^{-1}[f_t(x_t,\xi_t) - r_t]_+\right\},$$

with additional variables $r = (r_2,...,r_T)$. The problem can be viewed as a standard multistage stochastic program with (r, x_1) being first stage decision variables.

Dynamic programming equations: the cost-to-go function $Q_t(x_{t-1}, r_t, ..., r_T, \xi_{[t]})$ is given by the optimal value of the problem

$$\operatorname*{Min}_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \alpha^{-1} \left[f_t(x_t, \xi_t) - r_t\right]_+ + \mathbb{E}_{|\xi_{[t]}} \left[Q_{t+1}(x_t, r_t, ..., r_T, \xi_{[t]})\right].$$

Although it is possible to write dynamic programming equations for this problem, note that decision variables $r_2, ..., r_T$ are decided at the first stage and their optimal values depend on all scenarios starting at the root node at stage $t = 1$. Consequently optimal decisions at later stages depend on scenarios other than following a considered node. That is, here *the principle of conditional optimality does not hold.*

A well known quotation of Bellman: *"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."*

In order to make this precise we have to define what do we mean by saying that an optimal policy remains optimal at every stage of the process conditional on an observed realization of the data process.

We assume that with the multiperiod problem is associated a sequence of mappings $\varrho_{t,T} : \mathcal{Z}_t \times \cdots \times \mathcal{Z}_T \to \mathcal{Z}_t$, $t = 1, ..., T$. We say that $\{\varrho_{t,T}\}$ is a sequence of *multiperiod mappings* if it satisfies the condition of monotonicity:

if $Z, Z' \in \mathcal{Z}_t \times \cdots \times \mathcal{Z}_T$ and $Z \succeq Z'$, then $\varrho_{t,T}(Z) \succeq \varrho_{t,T}(Z')$.

Examples.

Conditional expectations:

$$\varrho_{t,T}(Z_t, ..., Z_T) := \mathbb{E}_{|\xi_{[t]}}(Z_t + ... + Z_T).$$

Conditional max-risk measures (recall that $\mathsf{AV@R}_0(\cdot) = \operatorname{ess\,sup}(\cdot)$):

$$\varrho_{t,T}(Z_t, ..., Z_T) := \mathsf{AV@R}_{0|\xi_{[t]}}(Z_t + ... + Z_T).$$

For a law invariant coherent (convex) risk measure $\rho(\cdot)$:

$$\varrho_{t,T}(Z_t, ..., Z_T) := \rho_{|\xi_{[t]}}(Z_t + ... + Z_T).$$

Nested form:

$$\begin{array}{rcl} \varrho_{t,T}(Z_t, ..., Z_T) & := & \rho_{|\xi_{[t]}}\Big(\cdots \rho_{|\xi_{[T-1]}}(Z_t + ... + Z_T)\Big) \\ & = & Z_t + \rho_{|\xi_{[t]}}\Big(Z_{t+1} + \cdots + \rho_{|\xi_{[T-1]}}(Z_T)\Big) \end{array}$$

Recall that only conditional expectations and conditional max-risk measures can be represented in the nested form of coherent risk measures.

For $t = 2, ..., T$, consider problems

$$
(P_t): \quad \begin{array}{cl} \underset{x_t(\cdot),\dots,x_T(\cdot)}{\text{Min}} & \varrho_{t,T}\Big(f_t(x_t(\xi_{[t]}),\xi_t),\dots,f_T\left(x_T(\xi_{[T]}),\xi_T\right)\Big) \\ \text{s.t.} & x_\tau(\xi_{[\tau]}) \in \mathcal{X}_\tau(x_{\tau-1}(\xi_{[\tau-1]}),\xi_\tau),\ \tau = t,\dots,T, \end{array}
$$

conditional on $\xi_{[t]}$ and x_{t-1}. At $t = 1$ the problem (P_1) is

$$
\begin{array}{cl} \underset{x_1,x_2(\cdot),\dots,x_T(\cdot)}{\text{Min}} & \varrho_{1,T}\Big(f_1(x_1), f_2(x_2(\xi_{[2]}),\xi_2),\dots,f_T\left(x_T(\xi_{[T]}),\xi_T\right)\Big) \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}),\xi_t),\ t = 2,\dots,T, \end{array}
$$

with $\varrho_{1,T} : \mathcal{Z}_1 \times \mathcal{Z}_2 \times \cdots \times \mathcal{Z}_T \to \mathbb{R}$ being a multiperiod risk measure.

We say that an optimal policy $\bar{x}_1, \bar{x}_2(\xi_{[2]}), ..., \bar{x}_T(\xi_{[T]})$ of problem (P_1) is *time consistent* if the policy $\bar{x}_t(\xi_{[t]}), ..., \bar{x}_T(\xi_{[T]})$ is optimal for the problem (P_t), $t = 2, ..., T$, conditional on $\xi_{[t]}$ and $x_{t-1} = \bar{x}_{t-1}(\xi_{[t-1]})$.

For the nested conditional risk measures

$$\varrho_{t,T}(Z_t, ..., Z_T) = Z_t + \rho_{|\xi_{[t]}}\Big(Z_{t+1} + \cdots + \rho_{|\xi_{[T-1]}}(Z_T)\Big),$$

every optimal policy of (P_1) is time consistent.

(C1) For all $1 \leq \tau < \theta \leq T$ and $Z, Z' \in \mathcal{Z}_\tau \times \cdots \times \mathcal{Z}_T$, the conditions

$$Z_t = Z'_t,\ t = \tau, ..., \theta - 1, \text{ and } \varrho_{\theta,T}(Z_\theta, ..., Z_T) \preceq \varrho_{\theta,T}(Z'_\theta, ..., Z'_T)$$

imply that

$$\varrho_{\tau,T}(Z_\tau, ..., Z_T) \preceq \varrho_{\tau,T}(Z'_\tau, ..., Z'_T).$$

Proposition 1 *Suppose that condition* (C1) *holds and the multi-period problem* (P_1) *has* unique *optimal solution policy* $\bar{x}_1, \bar{x}_2(\xi_{[2]}), ..., \bar{x}_T(\xi_{[T]})$. *Then the policy* $\bar{x}_t(\xi_{[t]}), ..., \bar{x}_T(\xi_{[T]})$ *is optimal for the problem* (P_t), $t = 2, ..., T$, *conditional on* $\xi_{[t]}$ *and* $x_{t-1} = \bar{x}_{t-1}$.

In case problem (P_1) has more than one optimal solution we need a stronger condition to ensure that an optimal solution of (P_1) is conditionally optimal for (P_t), $t = 2, ..., T$:

(C1*) For all $1 \le \tau < \theta \le T$ and $Z, Z' \in \mathcal{Z}_\tau \times \cdots \times \mathcal{Z}_T$, the conditions

$$Z_t = Z'_t,\ t = \tau, ..., \theta - 1, \text{ and } \varrho_{\theta,T}(Z_\theta, ..., Z_T) \prec \varrho_{\theta,T}(Z'_\theta, ..., Z'_T)$$

imply that

$$\varrho_{\tau,T}(Z_\tau, ..., Z_T) \prec \varrho_{\tau,T}(Z'_\tau, ..., Z'_T).$$

Consider the following conditions.

(C2) For $t = 1, ..., T$, it holds that $\varrho_{t,T}(0, ..., 0) = 0$, and for all $(Z_1, ..., Z_T) \in \mathcal{Z}$ and $t = 1, ..., T-1$, it holds that

$$\varrho_{t,T}(Z_t, Z_{t+1}, ..., Z_T) = Z_t + \varrho_{t,T}(0, Z_{t+1}, ..., Z_T).$$

(C3) For all $1 \leq \tau < \theta \leq T$ and $Z \in \mathcal{Z}$ it holds that

$$\varrho_{\tau,T}(Z_\tau, ..., Z_\theta, ..., Z_T) = \varrho_{\tau,\theta}\Big(Z_\tau, ..., Z_{\theta-1}, \varrho_{\theta,T}(Z_\theta, ..., Z_T)\Big).$$

Multiperiod coherent mappings of the nested form satisfy conditions (C1),(C2) and (C3), but not necessarily (C1*). We have the following relations between these conditions.

Proposition 2 *Let* $\{\varrho_{t,T}\}$ *be a sequence of multiperiod mappings. Then the following holds.* (i) *Conditions* (C1) *and* (C2) *imply condition* (C3). (ii) *Condition* (C3) *implies condition* (C1).

It is tempting to impose the following condition on the sequence of multiperiod mappings:

For all $1 \le \tau < \theta \le T$ and $Z, Z' \in \mathcal{Z}_\tau \times \cdots \times \mathcal{Z}_T$, the conditions

$$Z_t = Z'_t,\ t = \tau, ..., \theta - 1, \text{ and } \varrho_{\tau,T}(Z_\tau, ..., Z_T) \preceq \varrho_{\tau,T}(Z'_\tau, ..., Z'_T)$$

imply that

$$\varrho_{\theta,T}(Z_\theta, ..., Z_T) \preceq \varrho_{\theta,T}(Z'_\theta, ..., Z'_T).$$

This condition will impliy the desired property of the sequence of multiperiod mappings. Unfortunately this condition does not hold even for a sequence of conditional expectation mappings

$$\varrho_{t,T}(Z_t, ..., Z_T) := \mathbb{E}_{|\xi_{[t]}}[Z_t + ... + Z_T].$$

Suppose that conditions (C1) and (C2) hold. Then condition (C3) follows, and hence we can write

$$\begin{array}{rcl} \varrho_{1,T}(Z_1,...,Z_T) & = & \varrho_{1,T-1}\Big(Z_1,...,Z_{T-2},\varrho_{T-1,T}(Z_{T-1},Z_T)\Big) \\ & = & \varrho_{1,T-1}\Big(Z_1,...,Z_{T-2},Z_{T-1}+\varrho_{T-1,T}(0,Z_T)\Big). \end{array}$$

By continuing this process backwards we obtain the nested representation of $\varrho_{1,T}$:

$$\varrho_{1,T}(Z_1,...,Z_T) = Z_1 + \rho_2\Big[Z_2 + \cdots + \rho_{T-1}\Big[Z_{T-1} + \rho_T[Z_T]\Big]\Big],$$

where $\rho_{t+1} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ are one step mappings defined as

$$\rho_{t+1}(Z_{t+1}) := \varrho_{t,t+1}(0, Z_{t+1}) = \varrho_{t,T}(0, Z_{t+1}, 0, ..., 0).$$

Clearly the mappings ρ_{t+1} inherit such properties of $\varrho_{t,T}$ as monotonicity, convexity and positive homogeneity.

Risk Averse Multistage Portfolio Selection

A nested formulation of multistage portfolio selection* can be written as (recall that e denotes vector of ones)

$$\begin{array}{ll}\text{Min} & \left\{\bar{\rho}(W_T) := \rho_1\Big[\cdots\rho_{T-1|W_{T-2}}\Big[\rho_{T|W_{T-1}}[W_T]\Big]\Big]\right\} \\ \text{s.t.} & W_{t+1} = \xi_{t+1}^{\mathsf{T}} x_t,\ e^{\mathsf{T}} x_t = W_t,\ x_t \geq 0,\ t = 0,\dots,T-1.\end{array}$$

If we set $\rho_{t|W_{t-1}} := \mathbb{E}_{|W_{t-1}}$, $t = 1,...,T$, then $\bar{\rho}(W_T) = \mathbb{E}[W_T]$. Now let, for example,

$$\rho_{t|W_{t-1}}(\cdot) := (1-\beta)\mathbb{E}_{|W_{t-1}}(\cdot) + \beta \mathsf{AV@R}_\alpha(\cdot|W_{t-1}),$$

$\alpha \in (0,1)$, $\beta \in (0,1)$. Suppose that the random process ξ_t is *stagewise independent*.

*Note that we formulated here the problem as a minimization rather than maximization problem.

Let us write dynamic programming equations. At the last stage we have to solve problem

$$\begin{array}{cl} \underset{x_{T-1}\geq 0, W_T}{\text{Min}} & \rho_{T|W_{T-1}}[W_T] \\ \text{s.t.} & W_T = \xi_T^{\mathsf{T}} x_{T-1},\ e^{\mathsf{T}} x_{T-1} = W_{T-1}. \end{array} \tag{35}$$

Since W_{T-1} is a function of $\xi_{[T-1]}$, by the stagewise independence we have that ξ_T, and hence W_T, are independent of W_{T-1}. Therefore we have in (35) that $\rho_{T|W_{T-1}}[W_T] = \rho[W_T]$, where

$$\rho(\cdot) = (1-\beta)\mathbb{E}(\cdot) + \beta \mathsf{AV@R}_\alpha(\cdot) \tag{36}$$

is the corresponding (unconditional) risk measure.

It follows by positive homogeneity of $\rho(\cdot)$ that the optimal value of (35) is $Q_{T-1}(W_{T-1}) = W_{T-1}\nu_{T-1}$, where ν_{T-1} is the optimal value of

$$\begin{array}{cl} \underset{x_{T-1}\geq 0, W_T}{\text{Min}} & \rho[W_T] \\ \text{s.t.} & W_T = \xi_T^{\mathsf{T}} x_{T-1},\ e^{\mathsf{T}} x_{T-1} = 1, \end{array} \tag{37}$$

and an optimal solution of (35) is $\bar{x}_{T-1}(W_{T-1}) = W_{T-1}x^*_{T-1}$, where x^*_{T-1} is an optimal solution of (37).
And so on we obtain that the optimal policy $\bar{x}_t(W_t)$ here is *myopic*. That is, $\bar{x}_t(W_t) = W_t x_t^*$, where x_t^* is an optimal solution of

$$\begin{array}{cl} \underset{x_t\geq 0, W_{t+1}}{\text{Min}} & \rho[W_{t+1}] \\ \text{s.t.} & W_{t+1} = \xi_{t+1}^{\mathsf{T}} x_t,\ e^{\mathsf{T}} x_t = 1. \end{array} \tag{38}$$

Note that the composite risk measure $\bar{\rho}$ is quite complicated here.

An alternative, multiperiod risk averse approach can be formulated as

$$\begin{array}{ll}\text{Min} & \left\{\rho[W_T] = (1-\beta)\mathbb{E}[W_T] + \beta\Big(r + \alpha^{-1}\mathbb{E}[W_T - r]_+\Big)\right\} \\ \text{s.t.} & W_{t+1} = \xi_{t+1}^{\mathsf{T}} x_t,\ e^{\mathsf{T}} x_t = W_t,\ x_t \geq 0,\ t = 0, \dots, T-1.\end{array} \tag{39}$$

Here $r \in \mathbb{R}$ is the (additional) first stage decision variable. After r is decided, at the first stage, the problem comes to minimizing $\mathbb{E}[U(W_T)]$ at the last stage, where

$$U(W) := (1-\beta)W + \beta\alpha^{-1}[W - r]_+$$

can be viewed as a disutility function.

It is possible to write dynamic programming equations for this problem. Note that here r is the *first stage* decision variable, the optimal policy is not myopic and the property of time consistency is not satisfied

Robust Inventory Model

Consider the following robust formulation of inventory model

$$\begin{array}{c} \underset{x_t \geq y_t}{\text{Min}} \; \sum_{t=1}^{T} \rho^{\max}\Big\{c_t(x_t - y_t) + \psi_t(x_t, d_t)\Big\} \\ \text{s.t. } y_{t+1} = x_t - D_t, \; t = 1, ..., T-1, \end{array} \tag{40}$$

where $\psi_t(x_t, d_t) := b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+$, y_1 is a given initial inventory level, $D_1, ..., D_T$ is a demand process, and c_t, b_t, h_t are the ordering, backorder penalty, and holding costs per unit, respectively, at time t. We assume that $b_t > c_t > 0$ and $h_t \geq 0$, $t = 1, ..., T$, and that the support of the distribution of random vector $(D_1, ..., D_T)$ is bounded. The optimization in (40) is performed over feasible policies $x_t = x_t(D_{[t-1]})$, $t = 2, ..., T$, with x_1 being deterministic.

The dynamic programming equations for this problem can be written as follows. At stages $t = T, \dots, 2$ the cost-to-go function $Q_t(y_t, D_{[t-1]})$ is given by the optimal value of the problem

$$\operatorname*{Min}_{x_t \geq y_t} c_t(x_t - y_t) + \rho^{\max}_{|D_{[t-1]}}\Big[\psi_t(x_t, D_t) + Q_{t+1}\left(x_t - D_t, D_{[t]}\right)\Big],$$

with $Q_{T+1}(\cdot,\cdot) \equiv 0$. Finally, at the first stage we need to solve problem

$$\operatorname*{Min}_{x_1 \geq y_1} c_1(x_1 - y_1) + \rho^{\max}\Big[\psi_1(x_1, D_1) + Q_2\left(x_1 - D_1, D_1\right)\Big].$$

Suppose now that the support of the distribution of $(D_1, ..., D_T)$ is given by the direct product $\mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ for some (bounded) sets $\mathcal{D}_t \subset \mathbb{R}$, $t = 1, ..., T$. Then the cost-to-go function at the last stage is

$$Q_T(y_T) = \inf_{x_T \geq y_T} \left\{ c_T(x_T - y_T) + \sup_{d_T \in \mathcal{D}_T} \psi_T(x_T, d_T) \right\}.$$

And so on for $t = T-1, ..., 2$ the dynamic programming equations can be written as

$$Q_t(y_t) = \inf_{x_t \geq y_t} \left\{ c_t(x_t - y_t) + \sup_{d_t \in \mathcal{D}_t} \Big[\psi_t(x_t, d_t) + Q_{t+1}\left(x_t - d_t\right) \Big] \right\}.$$

Note that here the cost-to-go function $Q_t(y_t)$, $t = 2, \dots, T$, is independent of $d_{[t-1]}$, and by convexity arguments the optimal policy $\bar{x}_t = \bar{x}_t(d_{[t-1]})$ is a basestock policy. That is, $\bar{x}_t = \max\{y_t, x_t^*\}$, where x_t^* is an optimal solution of

$$\operatorname*{Min}_{x_t} \left\{ c_t x_t + \sup_{d_t \in \mathcal{D}_t} \Big[\psi_t(x_t, D_t) + Q_{t+1}\left(x_t - D_t\right) \Big] \right\}, \qquad (41)$$

and $y_t = \bar{x}_{t-1} - D_{t-1}$, $t = 2, ..., T$, with y_1 being given.