Dealing with Uncertainty *in Decision Making Models*

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ENSTA, Paris, Summer 1964

A product mix problem

Choose $x_j \geq 0,\ j = 1, \ldots, 4,$ to maximize:

$$\sum_{j=1}^{4} c_j x_j = 12x_1 + 25x_2 + 21x_3 + 40x_4$$

such that (constraints)

$$t_{c1}x_1 + t_{c2}x_2 + t_{c3}x_3 + t_{c4}x_4 \leq d_c \quad \text{(carpentry)}$$

$$t_{f1}x_1 + t_{f2}x_2 + t_{f3}x_3 + t_{f4}x_4 \leq d_f \quad \text{(finishing)}$$

$d_c (d_f) =$ total time available for carpentry (finishing)

Linear Programming Sol'n

$$\max\langle c,x\rangle \text{ such that } Tx \le d,\ x \in \mathbb{R}^n_+ \qquad \text{with}$$

$$T = \begin{bmatrix} t_{c1} & t_{c2} & t_{c3} & t_{c4} \\ t_{f1} & t_{f2} & t_{f3} & t_{f4} \end{bmatrix} = \begin{bmatrix} 4 & 9 & 7 & 10 \\ 1 & 1 & 3 & 40 \end{bmatrix}, \quad \begin{bmatrix} d_c \\ d_f \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$$

Optimal: $x^d = (1{,}333.33,\ 0,\ 0,\ 66.67)$

but ... "reality": $t_{cj} = t_{cj} + \eta_{cj}, \quad t_{fj} = t_{fj} + \eta_{fj}$

entry	values			
$d_c + \zeta_c$	5,873	5,967	6,033	6,127
$d_f + \zeta_f$	3.936	3.984	4,016	4,064

10 random variables $\Rightarrow L = 1{,}048{,}576$ possible pairs $(T^l,\ d^l)$

Linear Programming Sol'n

$\max\langle c,x\rangle$ such that $Tx \leq d,\ x \in \mathbb{R}^n_+$ with

$$T = \begin{bmatrix} t_{c1} & t_{c2} & t_{c3} & t_{c4} \\ t_{f1} & t_{f2} & t_{f3} & t_{f4} \end{bmatrix} = \begin{bmatrix} 4 & 9 & 7 & 10 \\ 1 & 1 & 3 & 40 \end{bmatrix}, \quad \begin{bmatrix} d_c \\ d_f \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$$

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Taking recourse into account

What if $\sum_{j=1}^{4}(t_{cj}+\eta_{cj})x_j > d_{cj}+\varsigma_{cj}$?? $\Rightarrow$ overtime

With $\xi = (\eta_{\{\cdot,\cdot\}},\zeta_{\{\cdot\}})$, recourse : $y = (y_c(\xi), y_f(\xi))$ at cost (q_c, q_f)

$$
\begin{array}{lllllll}
\max & \langle c,x\rangle & -p_1\langle q,y^1\rangle & -p_2\langle q,y^2\rangle & \cdots - & p_L\langle q,y^L\rangle & & \\
\text{s.t.} & T^1x & -y^1 & & & & \le & d^1 \\
& T^2x & & -y^2 & & & \le & d^2 \\
& \vdots & & & \ddots & & & \vdots \\
& T^Lx & & & & -y^L & \le & d^L \\
& x\ge 0, & y^1\ge 0, & y^2\ge 0, & \cdots & y^L\ge 0. & &
\end{array}
$$

Structured large scale l.p. $(L \approx 10^6)$

Equivalent Deterministic Program

not done

Define $\Xi = \left\{ \xi = (\eta, \zeta) \right\}, \ p_\xi = [\boldsymbol{\xi} = \xi]$

$$Q(\xi, x) = \max \left\{ \langle -q, y \rangle \middle| T_\xi x - y \geq d_\xi, y \geq 0 \right\}$$

EQ concave

$$EQ(x) = E\{Q(\boldsymbol{\xi}, x)\} = \sum_{\xi \in \Xi} p_\xi Q(\xi, x)$$

$$(DEQ) \quad \max \langle c, x \rangle + EQ(x) \text{ such that } x \in \mathbb{R}^n_+$$

non-smooth convex optimization problem

Robust Solutions !!!

DEQ Optimal : $x^* = (257,\ 0,\ 665.2,\ 33.8)$

while $x^d = (1{,}333.33,\ 0,\ 0,\ 66.67)$

Expected profit x^*: \$18.051, x^d : \$17,942

x^d not close to optimal (- 6.5%)

x^d isn't pointing in the right direction

x^* robust, considered all 10^6 possibilities.

NewsVendor Problem

$$\max \; -cx + (c+r)y, \; x \geq 0, \quad 0 \leq y \leq \min\{x, \xi\}$$

$$\Xi = [0, 150]$$

$$\text{c} = 10, \; r = 15$$

Pick $\xi^1, \ldots, \xi^L$ (*scenarios*), and find:

$$(x^l, y^l) \in \operatorname*{argmin}_{x \geq 0, y \geq 0} \left\{ -cx + (c+r)y \,\middle|\, y \leq \min\left[\xi^l, x\right] \right\}$$

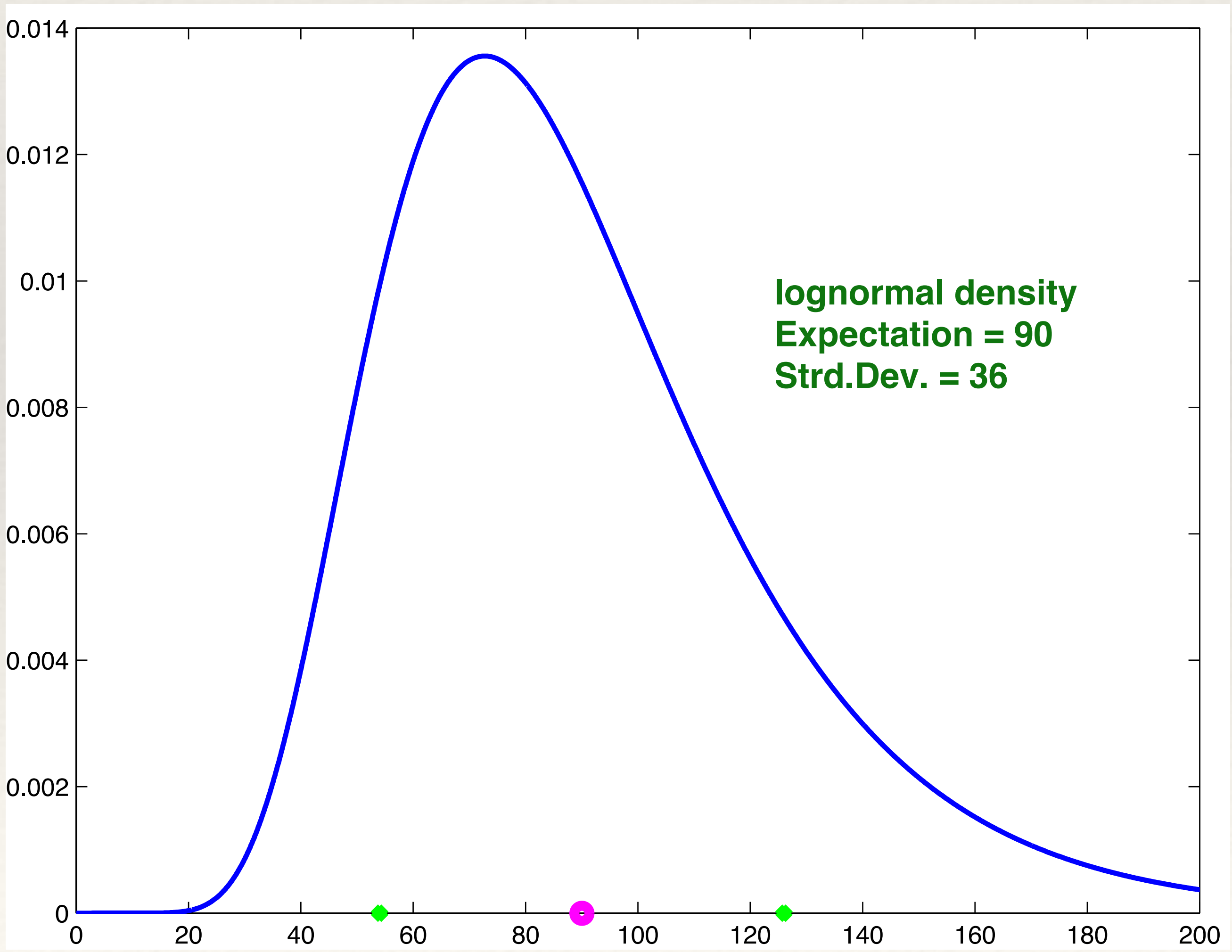
Wait-and-see sol'ns: $x^l = \xi^l$. "Reconciliation"

no help in choosing x^* optimal!

ξ : Estimated Density h

ξ log-normal: $h(z) = \left(z\tau\sqrt{2\pi}\right)^{-1} e^{-\frac{(\ln z-\theta)^2}{2\tau^2}}$

$\theta = 4.43,\ \tau = 0.38;\ H(z) = \int_0^z h(s)\,ds$



Maximize expected return

$$\max \; -cx + E\{(c+r)y_{\xi}\}$$

$$\text{such that } \; x \geq 0, \; 0 \leq y_{\xi} \leq \min[\xi, x]$$

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$$\text{such that } \; x \geq 0, \;\; 0 \leq y_\xi \leq \min[\xi, x]$$

DEQ: $\max \; -cx \; + EQ(x) =$

$$-cx \; + \left((c+r) \int_0^x \xi H(d\xi) + \int_x^\infty x H(d\xi) \right)$$

sol'n: $x^* = H^{-1}\left(\frac{r}{c+r}\right) = H^{-1}(0.6) = 99.2; \quad c = 10, r = 15$

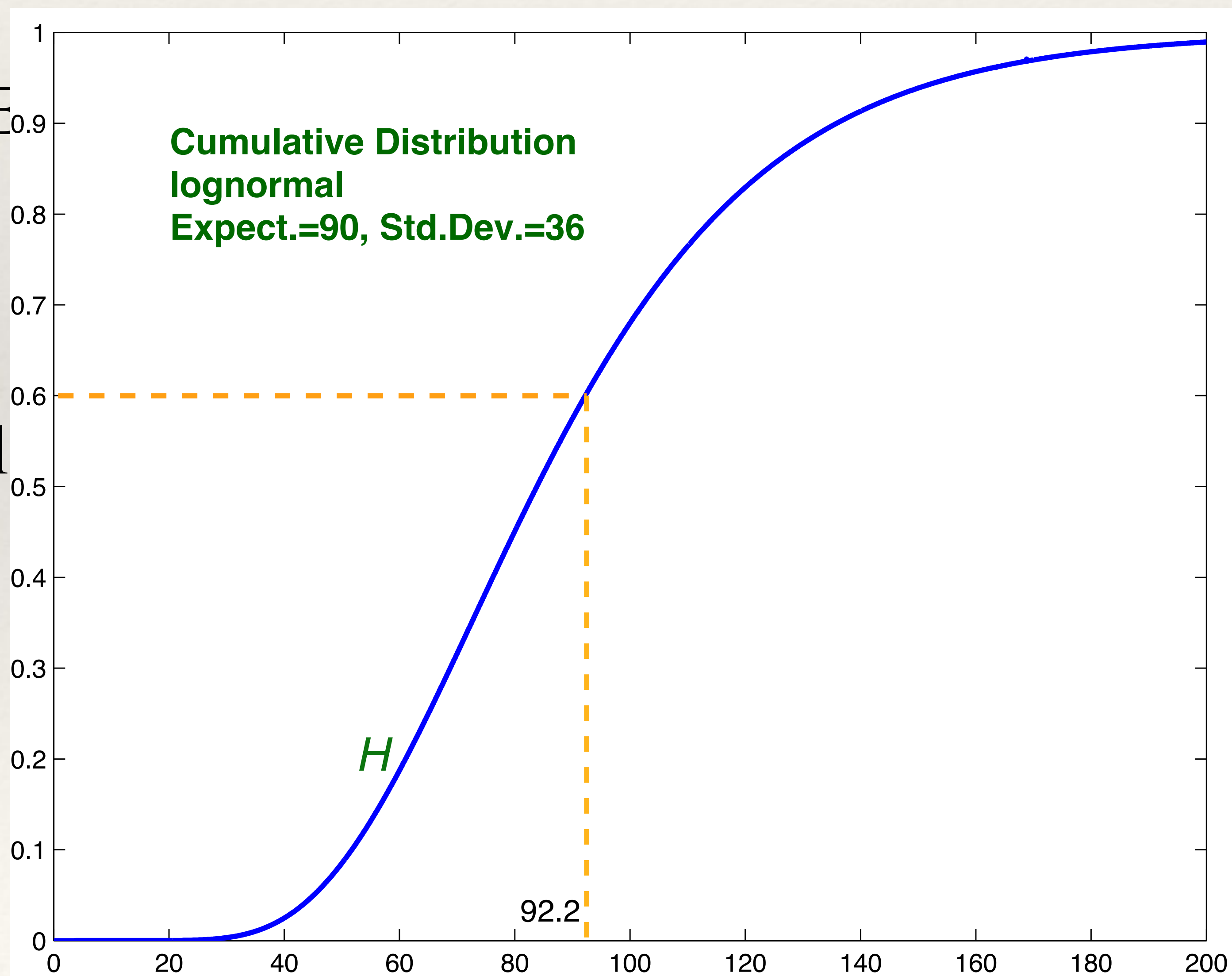
Maximize expected return

$$\max \ -cx + E\{(c+r)y_\xi\}$$

$$\text{such that } \ x \geq 0, \ 0 \leq y_\xi \leq \min[\xi, x]$$

DE

$\xi)\Big)$

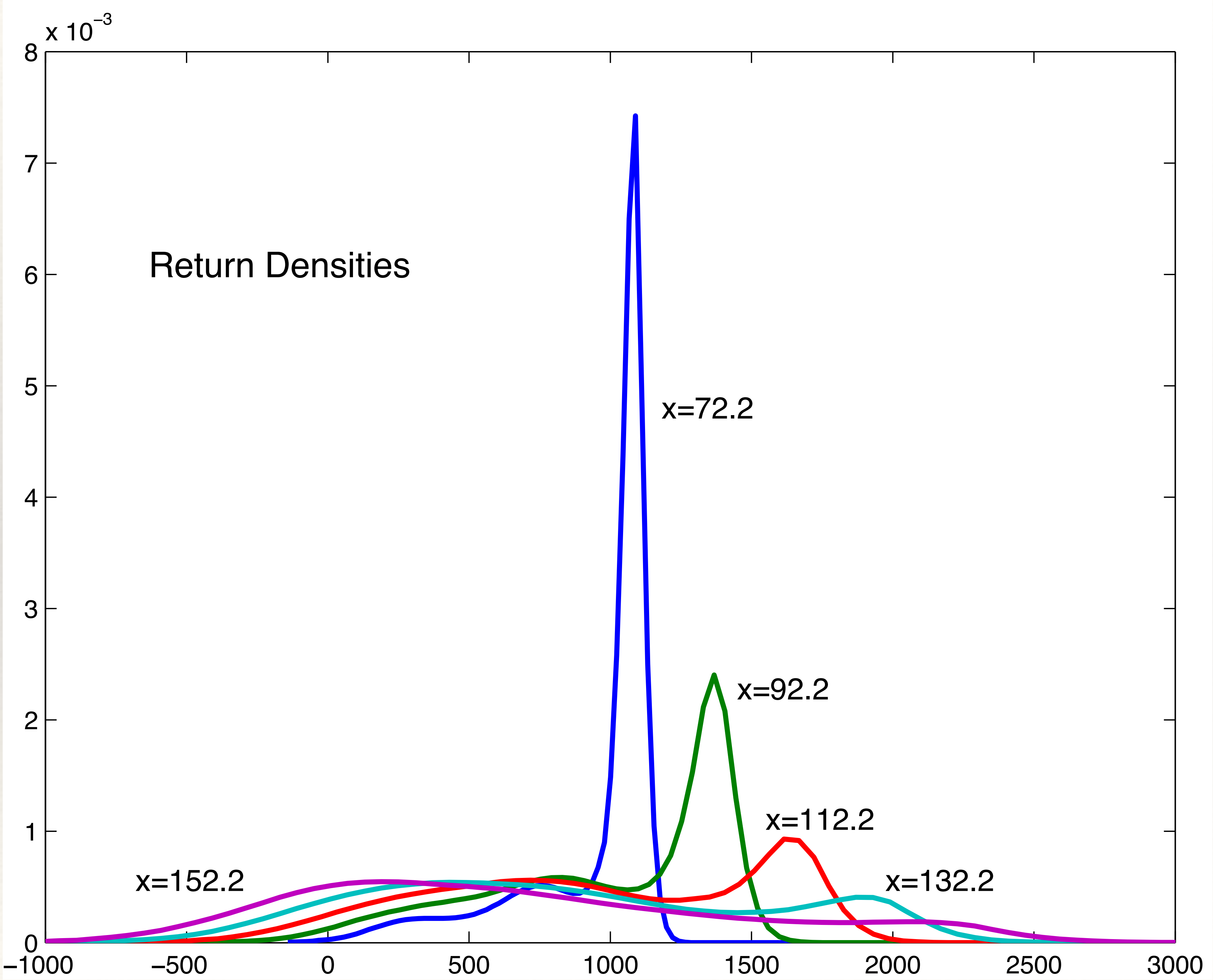


sol

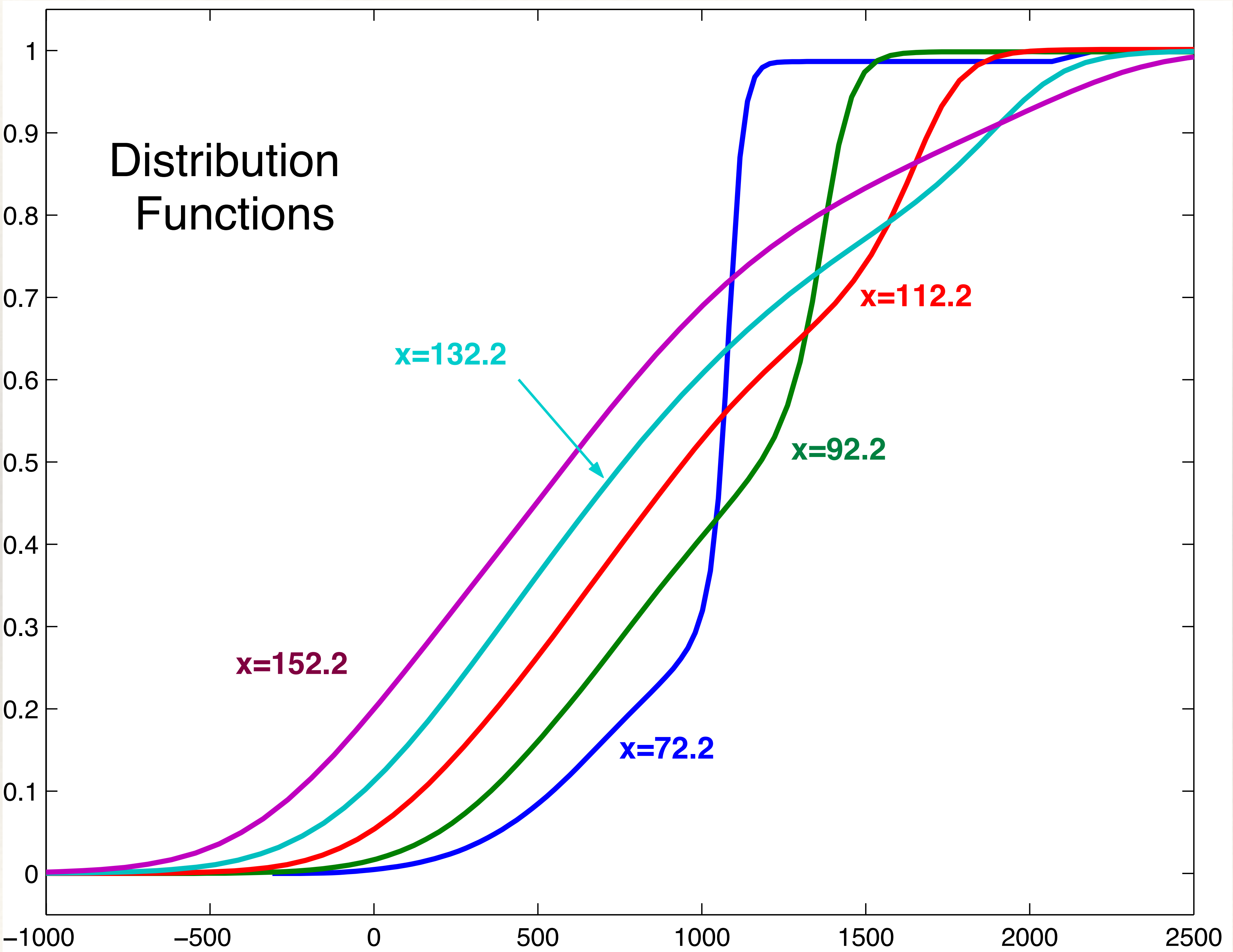
$10, r = 15$

… but is maximum expected return the "real" objective?

The “returns” densities



Choosing: “returns” distribution



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Decision criteria: from a distribution la″ number

- maximize expected return (scaled?)
- max. E{return} & minimize customers lost
- minimize Value-at-Risk (V@A, CV@R)
- minimize probability of any loss
- minimize a “safeguarding” measure, ...

$$\max E\left\{f(\xi,x)\right\} \Rightarrow \max E\left\{u\left(f(\xi,x)\right)\right\}$$

V@E: Value-at-Risk

$F(v;x) = \text{prob}\ \left[-cx + Q\left(\xi,.x\right) \leq v\right]$

Value-at-Risk(V@R) for $\alpha \in (0,1)$:

$$\text{V@R}\left(\alpha;x\right) = F^{-1}(\alpha;x) \quad \left(= \sup\left\{v \middle| F^{-1}(\alpha;x)\right\}\right)$$

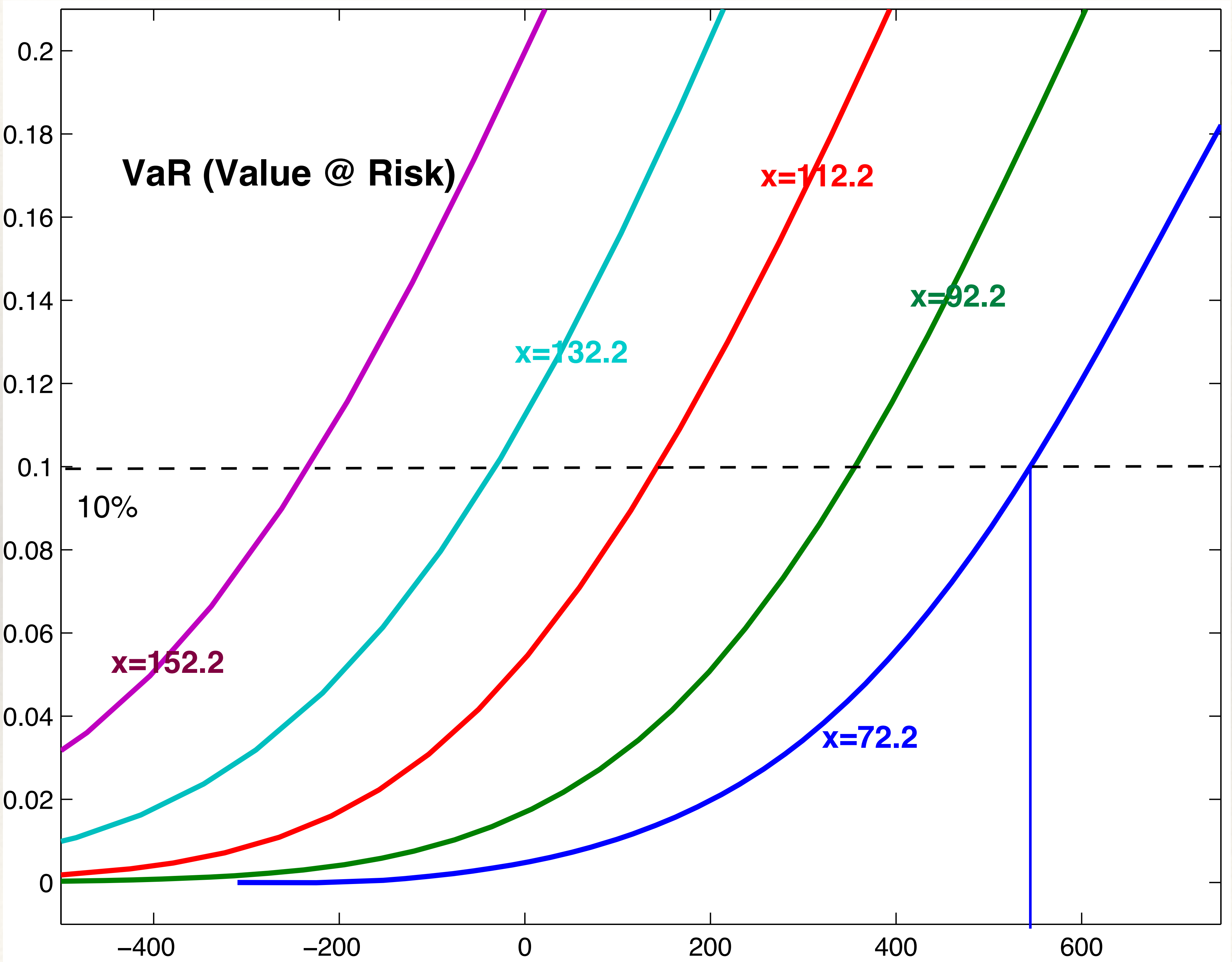
Objective: find x that maximizes $\text{V@R}\left(\alpha;x\right)$ given α

Challenge: $x \mapsto \text{V@R}\left(\alpha;x\right)$ isn't concave!

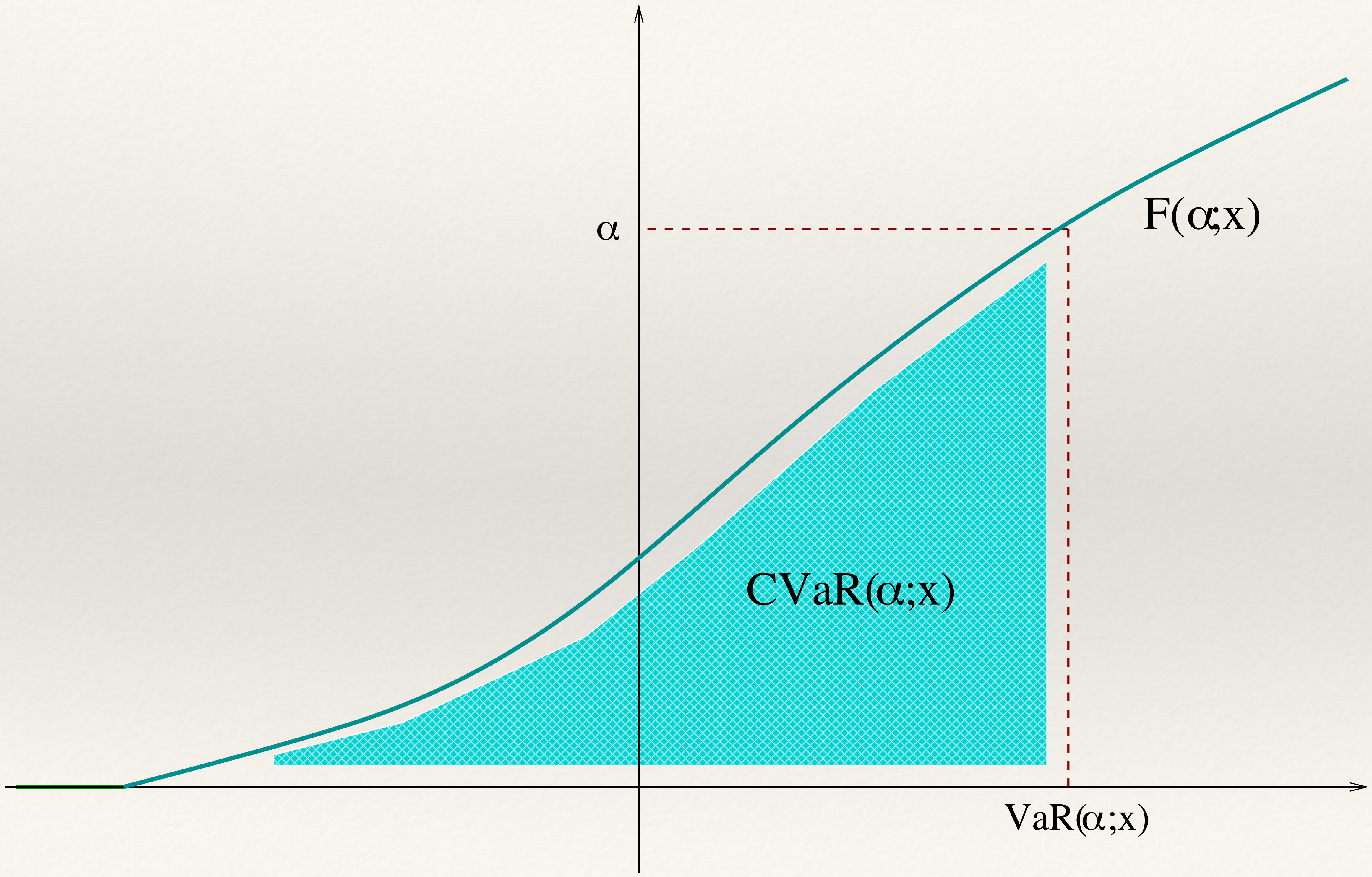
Heuristic: F is $\mathcal{N}\left(\mu(x),\sigma(x)^2\right)$ and

$$\boxed{\text{V@R}\left(\alpha;x\right) = \mathcal{N}\left(\alpha;\mu(x),\sigma(x)^2\right)}$$

V@E": the NewsVendor



CV@R: Conditional Value-at-Risk



CV@R: Conditional Value-at-Risk

$$G(v;x) = E\left\{-cx + Q\left(\xi, x\right) \middle| -cx + Q\left(\xi, x\right) \le v\right\}$$

Conditional Value-at-Risk(CV@R) for $\alpha \in (0,1)$:

$$CV@R\left(\alpha;x\right) = G^{-1}(\alpha;x) \quad \left(= \sup\left\{v \middle| G^{-1}(\alpha;x)\right\}\right)$$

$$= \min_r r + (1-\alpha)^{-1} E\left\{\left[-cx + Q(\xi,x) - r\right]_+\right\}$$

Objective: find x that maximizes $\mathrm{CV@R}\left(\alpha;x\right)$ given α

$x \mapsto CV@R(\alpha;x)$ is concave (convenient u)

Stochastic Programs with Recourse

.. with Simple Recourse

decision: $x \rightsquigarrow$ observation: $\xi \rightsquigarrow$ recourse cost evaluation.

cost evaluation 'simple' $\Rightarrow$ simple recourse, i.e.,

$$\min_{x \in S \subset \mathbb{R}^n} f_0(x) + \mathbb{E}\{Q(\xi, x)\} \quad Q \text{ 'simple'}$$

Product mix problem. With $\xi = (T, d)$,

$$f_0(x) = \langle c, x \rangle, \quad S = \mathbb{R}^4_+, \quad Q(\xi, x) = \sum_{i=c,f} \max\,[\,0, \gamma_i(\langle T_i, x \rangle - d_i)\,]$$

NewsVendor: cost: γ, sale price δ,

$\boldsymbol{\xi}$, demand distribution P, order x,

expected "loss": $\gamma x + \mathbb{E}\{Q(\boldsymbol{\xi}, x)\}$

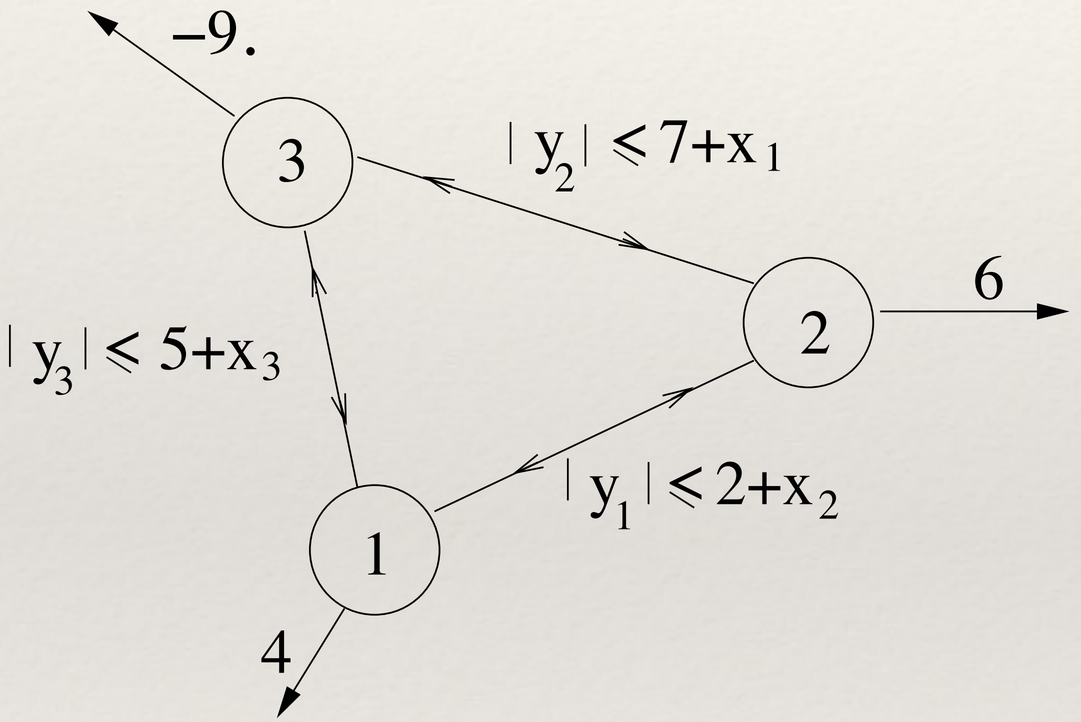
$Q(\xi, x) = -\delta \cdot \min\{x, \xi\}$

$\Rightarrow$ explicit sol'n

Network capacity expansion

Deterministic Version:

$$\min \sum_{j=1}^{n} \psi_j(x_j), \text{ such that } 0 \le x_j \le v_j, \qquad j = 1, \dots, n$$
$$|y_j| \le \gamma_j + x_j, \; j = 1, \dots, n, \; \sum_{j \in \odot(i)} y_j \ge e_i, \; \imath = 1, \dots, m$$



Network capacity expansion

Deterministic Version:

$$\min \ \sum_{j=1}^{n} \psi_j(x_j), \ \text{such that } \ 0 \le x_j \le \upsilon_j, \qquad j = 1, \ldots, n$$

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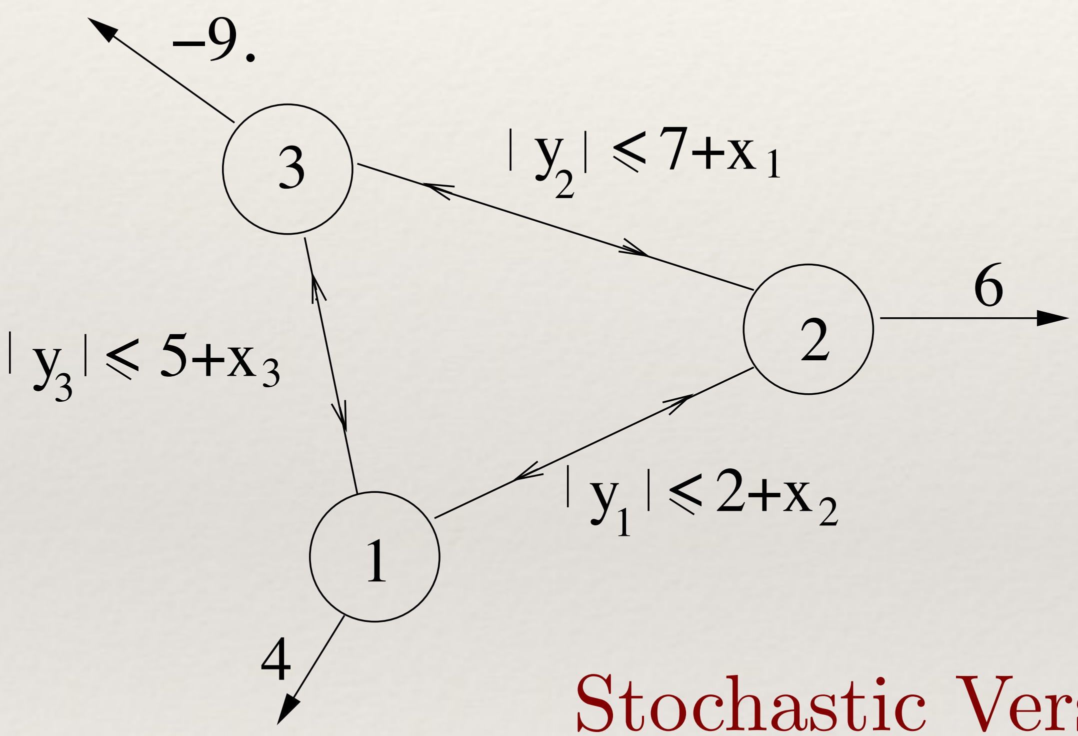

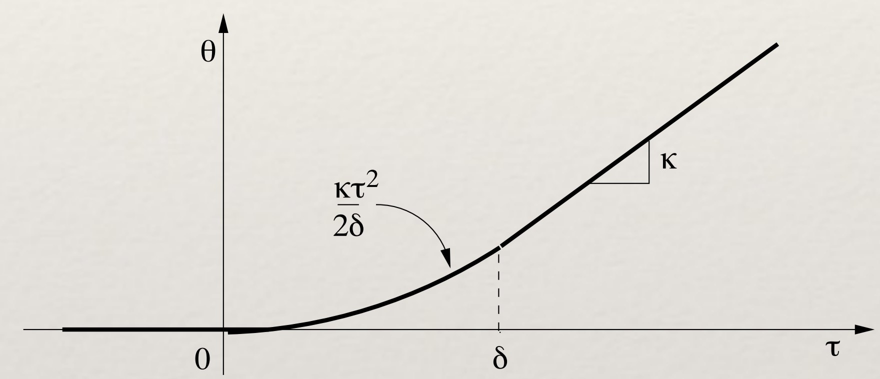

monitoring function

Stochastic Version: $\boldsymbol{\xi}^{\,l}, l = 1, \ldots, L,$

$$\min_{x, y^{\,l}} \sum_{l=1}^{L} \left[p_l \sum_{i=1}^{m} \theta_i \big(\xi_i^{\,l} - \sum_{j \in \odot(i)} \boldsymbol{y}_j^{\,l} \big) \right]$$

$$\text{s.t.} \ \ \sum_{j=1}^{n} \psi_j(x_j) \le \beta, \quad 0 \le x_j \le \upsilon_j, \ \ j = 1, \ldots, n,$$

$$|\boldsymbol{y}_j^{\,l}| - x_j \le \gamma_j, \ \ j = 1, \ldots, n, \ \ l = 1, \ldots, L$$

Extensive formulation

$$
\begin{array}{llllllll}
\min & \langle -c, x\rangle & +p_1\langle q, y^1\rangle & +p_2\langle q, y^2\rangle & \cdots + & p_L\langle q, y^L\rangle & & \\
\text{s.t.} & T^1 x & -y^1 & & & & \leq & d^1 \\
 & T^2 x & & -y^2 & & & \leq & d^2 \\
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 & T^L x & & & & -y^L & \leq & d^L \\
 & x \geq 0, & y^1 \geq 0, & y^2 \geq 0, & \cdots & y^L \geq 0. & &
\end{array}
$$

EXF

Deterministic Equivalent Problem

$$Q(\xi, x) = \min\left\{\langle q, y\rangle \,\middle|\, T_\xi x + y \geq d_\xi,\ y \geq 0\right\}$$

$$EQ(x) = \mathbb{E}\{Q(\boldsymbol{\xi}, x)\} = \sum_{\xi\in\Xi} p_\xi Q(\xi, x)$$

the equivalent deterministic program:

$$\min\langle -c, x\rangle + EQ(x) \quad \text{such that } x \in \mathbb{R}^n_+$$

DEP

product mix problem

RHS: random right-hand sides

deterministic version:

$$\min \langle c, x\rangle \quad \text{such that} \quad Ax = b,\ Tx = \hat{\xi},\ x \geq 0$$

stochastic program with simple recourse RHS:

$$\min_x \langle c, x\rangle + \mathbb{E}\{Q(\boldsymbol{\xi}, x)\} \quad \text{such that} \quad Ax = b,\ x \geq 0$$

Deterministic Equivalent Problem: (SPwSR)

$$\min_x \langle c, x\rangle + EQ(x) \quad \text{such that} \quad Ax = b,\ x \geq 0$$

- $f_0 = \langle c, x\rangle$ is linear;
- $S = \{x \in \mathbb{R}^n_+ \,|\, Ax = b\}$ is a polyhedral set, A is $m_1 \times n$;
- $Q(\xi, x) = q(\xi - Tx)$, T a non-random $m_2 \times n$ matrix;
- the *recourse cost function* $q\colon \mathbb{R}^{m_2} \to \mathbb{R}$ is convex;
- the *expectation functional* $EQ(x) = \int_\Xi q(\xi - Tx)\, P(d\xi)$.

Convex optim., linear constraints

$\mathcal{P}$: $\quad f_0 : \mathbb{R}^n \to \mathbb{R}$ convex, $\quad X$ polyhedral,

$$\begin{aligned} \min \quad & f_0(x), \qquad x \in X \subset \mathbb{R}^n \\ \text{such that} \quad & \langle A_i, x\rangle \geq b_i, \quad i = 1, \dots, s, \\ & \langle A_i, x\rangle = b_i, \quad i = s+1, \dots, m, \end{aligned}$$

x^* is an optimal solution of $(\mathcal{P})$ $\iff$ $\exists$, KKT-multipliers $y \in \mathbb{R}^m$:

(a) $\langle A_i, x^*\rangle \geq b_i,\ i = 1, \dots, s, \quad \langle A_i, x^*\rangle = b_i,\ i = s+1, \dots, m,$

(b) for $i = 1, \dots, s$: $\quad y_i \geq 0,\ \ y_i(\langle A_i, x^*\rangle - b_i) = 0,$

(c) $x^* \in \operatorname{argmin} \left\{ f_0(x) - \langle A^\top y, x\rangle \,\middle|\, x \in X \right\}.$

$\sim$ version of (c): $\exists\ -v \in N_X(x^*) = \left\{ u \,\middle|\, \langle u, x - x^*\rangle \leq 0, \forall\, x \in X \right\}$

Optimality: Simple Recourse

(SPwSR)

Suppose EQ finite-valued, x^* solves SP-simple recourse
$\iff$ one can find KKT-multipliers $u \in \mathbb{R}^{m_1}$
& summable KKT-multipliers $v : \Xi \to \mathbb{R}^{m_2}$:

1. $x^* \geq 0, \quad Ax^* = b;$
2. for all $\xi \in \Xi$: $\quad v(\xi) \in \partial q(\xi - Tx^*);$
3. $x^* \in \operatorname{argmin} \left\{ \langle c - A^\top u - T^\top \bar{v}, x \rangle \,\middle|\, x \in \mathbb{R}^n_+ \right\}$

where $\bar{v} = E\{v(\boldsymbol{\xi})\}$.

Approximation

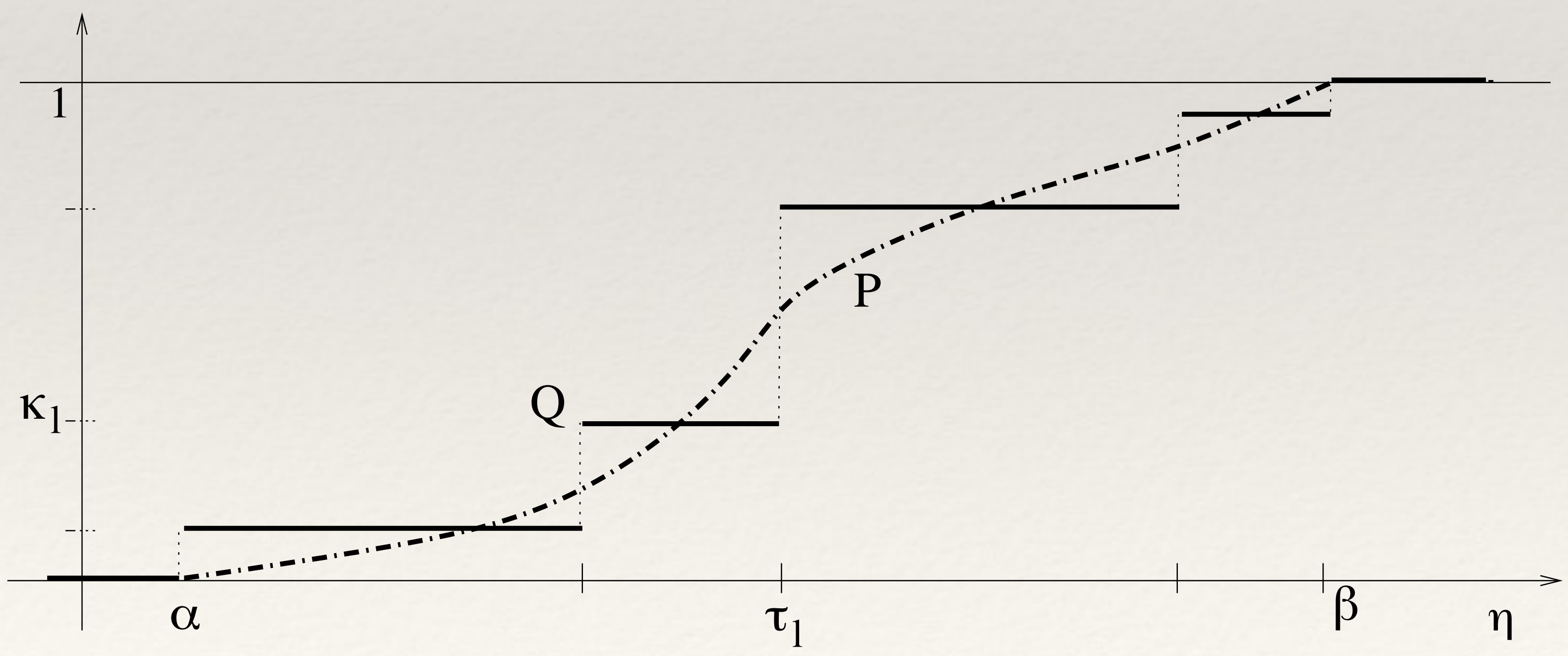
Let $P : \mathbb{R} \to [0,1]$ continuous, increasing on interval Ξ,

$$E\psi(\chi) = \delta(\bar{\xi} - \chi) + (\delta - \gamma)\big[\,\chi P(\chi) - \int_{-\infty}^{\chi} \zeta\, P(d\zeta)\,\big];$$

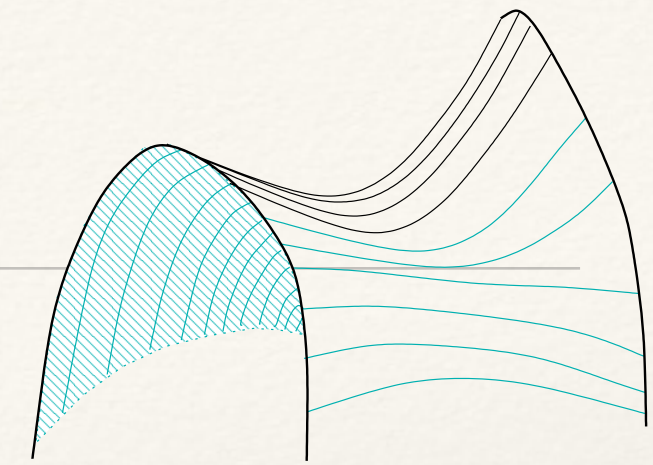
Hence,

$$P(\chi^*) = \frac{\delta - \bar{v}}{\delta - \gamma} =: \kappa, \qquad \boxed{\chi^* = P^{-1}(\kappa).}$$

When Q has the same κ-quantile, $Q^{-1}(\kappa) = P^{-1}(\kappa)$, <u>same</u> optimal sol'n.
$\Longrightarrow$ choose Q is 'quantile close' to P



Lagrangian Duality



min $\; f_0(x),\; x \in X \subset \mathbb{R}^n$ polyhedral

$\langle A_i, x\rangle \geq b_i,\; i = 1, \ldots, s, \quad \langle A_i, x\rangle = b_i,\; i = s+1, \ldots, m$

The Lagrangian:

$L(x, y) = f_0(x) + \langle y, b - Ax\rangle$ on $X \times Y,\; Y = \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$.

x^*optimal $\iff \exists$ a pair (x^*, y^*) that satisfies:

$x^* \in \text{argmin}_{x\in\mathbb{R}^n} L(x, y^*), \qquad y^* \in \text{argmax}_{y\in Y} L(x^*, y)$.

primal & dual problems:

linear programs: $\min\langle c, x\rangle,\; Ax = b,\; x \geq 0$ & $\max\langle b, y\rangle,\; A^\top y \leq 0$

quadratic programs: $\min\langle c, x\rangle + \frac{1}{2}\langle x, Qx\rangle,\; Ax \geq b,\; x \in \mathbb{R}^n$

$\max \alpha + \langle d, y\rangle - \frac{1}{2}\langle y, Py\rangle, y \in \mathbb{R}^m_+$

$\alpha = -\frac{1}{2}\langle c, Q^{-1}c\rangle,\; d = b + AQ^{-1}c$ and $P = AQ^{-1}A^\top$.

Monitoring Functions

penalty substitutes

A *monitoring function* $\theta_{P,Q}(x) = \sup_{v \in \mathbb{R}^n} \left\{ \langle x, v \rangle - \frac{1}{2}\langle v, Qv \rangle \,\middle|\, v \in P \right\}$,
convex *linear-quadratic function* for P polyhedral, Q psd. On $\mathbb{R}$

$$\theta_{r,\beta} = \theta_{I,\beta}(x) = \sup \left\{ xv - \frac{\beta}{2}v^2 \,\middle|\, v \in I \subset \mathbb{R} \right\}, \quad \beta \geq 0,\ P = [0, r]\ (r \leq \infty)$$

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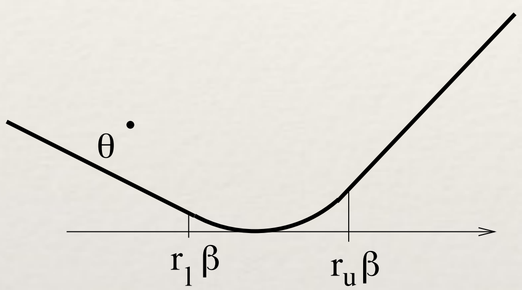

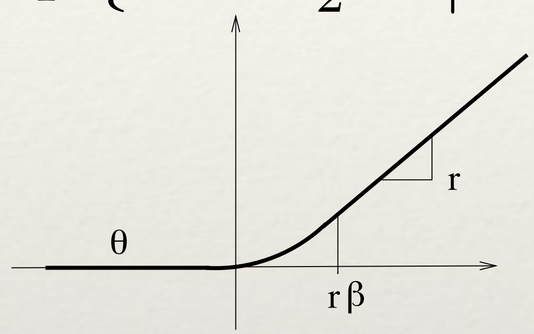

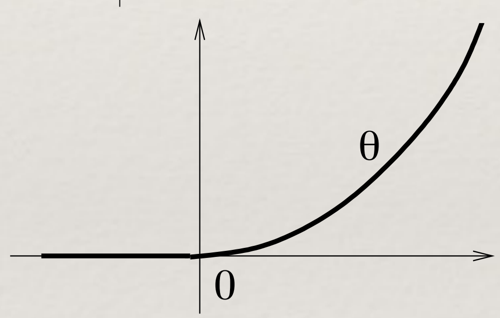

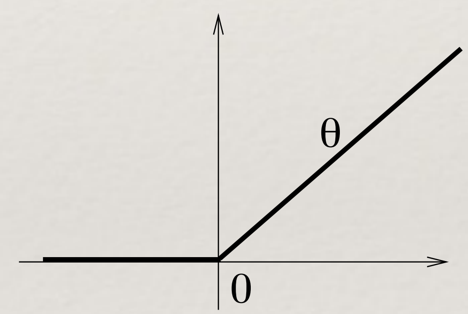

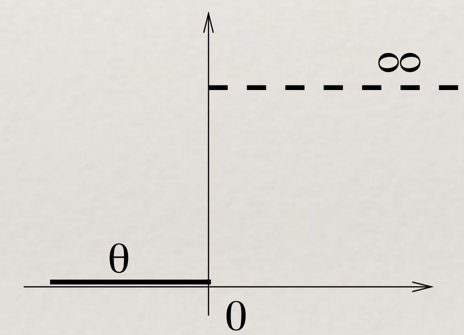

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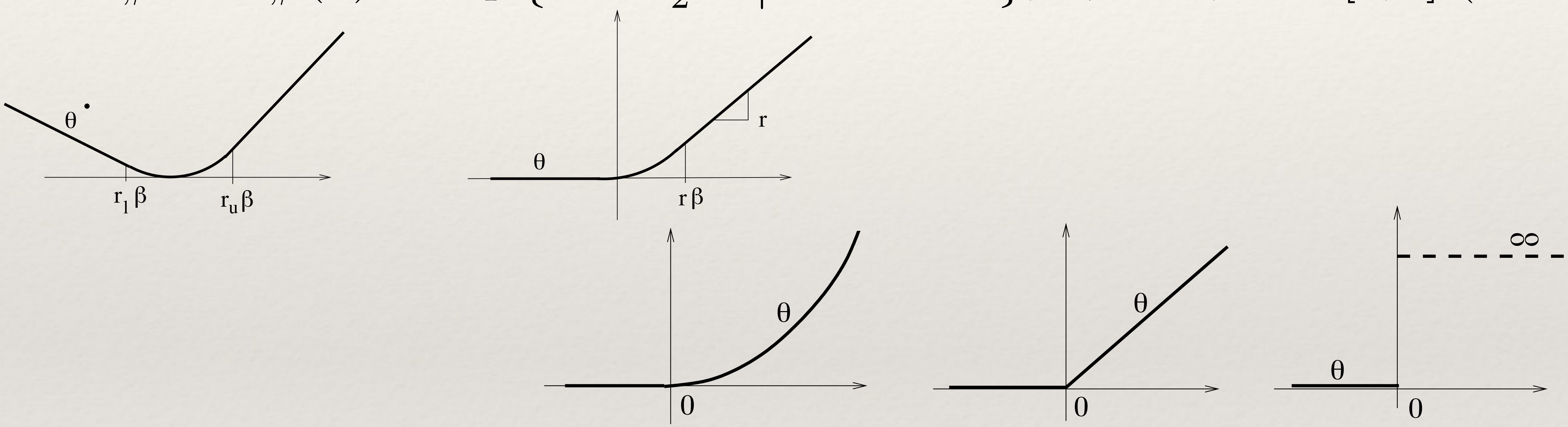
$$\theta_{r,\beta} = \theta_{I,\beta}(x) = \sup \left\{ xv - \frac{\beta}{2}v^2 \,\middle|\, v \in I \subset \mathbb{R} \right\}, \quad \beta \geq 0,\ P = [0, r]\ (r \leq \infty)$$



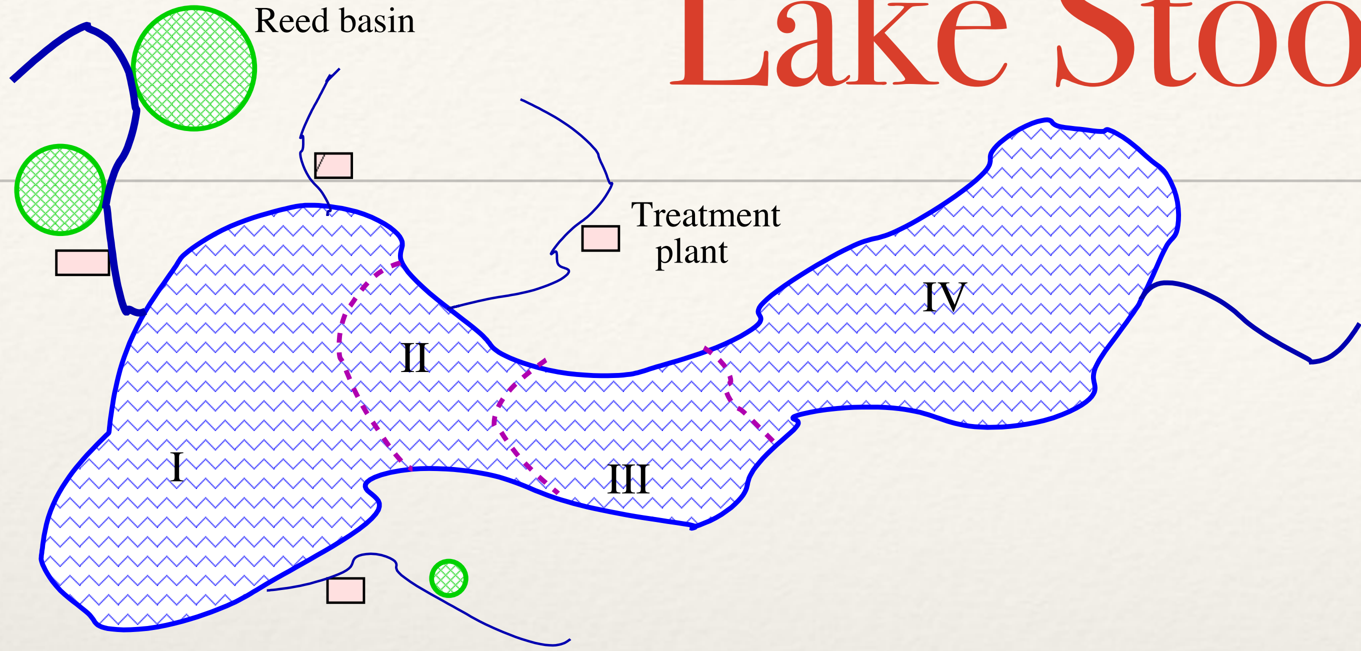
$$\min_{0 \leq x \leq s}\ \langle c, x \rangle + \frac{1}{2}\sum_{j=1}^{n} \delta_j x_j^2 + \sum_{i=1}^{m} \theta_{r_i,\beta_i}(d_i - \langle T_i, x \rangle)$$

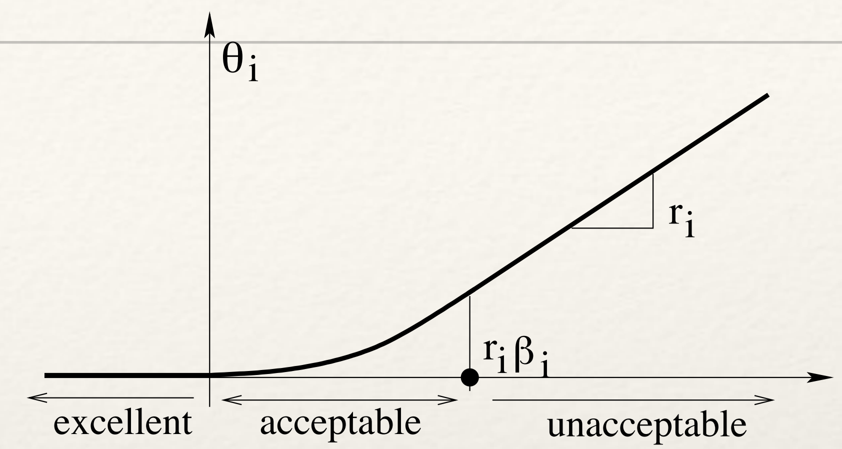
and

$$\max_{0 \leq v \leq r}\ \langle d, v \rangle - \frac{1}{2}\sum_{i=1}^{m} \beta_i v_i^2 - \sum_{j=1}^{n} \theta_{s_j,\delta_j}(\langle T^j, v \rangle - c_j)$$

are a dual pair when $\delta_j, r_j, \beta_i, s_i$ strictly positive.

Lake Stoöpt





Water quality: $\boldsymbol{z}_i = T_i(\boldsymbol{\xi})x - d(\boldsymbol{\xi})$, $i = I, \ldots, IV$, technical, ...: $Ax \le b$

Monitoring deviations from the desired quality: for $i = I, \ldots, , IV$,

$$\theta_{r_i,\beta_i}(\tau) = \begin{cases} 0 & \text{if } \tau < 0, \quad \text{(excellent)} \\ \tau^2/2\beta_i & \text{if } \tau \in [0, r_i\beta_i] \quad \text{(acceptable)} \\ r_i\tau - r_i^2\beta_i/2 & \text{if } \tau > r_i\beta_i \quad \text{(unacceptable)} \end{cases}$$

+ direct costs with building treatment plants and reed basins,

$\sum_{j=1}^n c_j x_j + \frac{1}{2}\sum_{j=1}^n \delta_j x_j^2$, $D = \mathrm{diag}(\delta_1, \ldots, \delta_n)$: leads to

$$\min\langle c, x\rangle + \tfrac{1}{2}\langle x, Dx\rangle + \mathbb{E}\Big\{\sum_{i=I}^{IV} \theta_{r_i,\beta_i}(d_i(\boldsymbol{\xi}) - \langle T_i(\boldsymbol{\xi}), x\rangle)\Big\},\ Ax \le b, 0 \le x \le$$

with random rhs & technology T-matrix (hydro-dynamics, atmospheric).

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Exploiting duality

$$\min \langle c, x\rangle + \tfrac{1}{2}\langle x, Dx\rangle + \sum_{i=1}^{m_2} \mathbb{E}^i\{\theta_{r_i,\beta_i}(\boldsymbol{w}_i)\},\ Ax \geq b, \quad \boldsymbol{w} = \boldsymbol{d} - \boldsymbol{T}x,\ 0 \leq x \leq s$$

for all i, $\boldsymbol{\xi}_i = (\boldsymbol{r}_i, \boldsymbol{\beta}_i, \boldsymbol{d}_i, \boldsymbol{t}_{i1}, \ldots, \boldsymbol{t}_{in})$, $\boldsymbol{w}$ and, later, $\boldsymbol{v}$ as *very* long vectors.
The dual takes on the form, with $u \geq 0, \quad 0 \leq \boldsymbol{v}_i \leq \boldsymbol{r}_i, \ \forall\, i$

$$\max \ \langle b, u\rangle - \sum_{j=1}^{n} \theta_{s_j,\delta_j}(z_j) + \sum_{i=1}^{m_2} E^i\{\boldsymbol{d}_i \boldsymbol{v}_i - \tfrac{1}{2}\boldsymbol{\beta}_i \boldsymbol{v}_i^2\}$$

$$\text{such that } \ z_j = \langle A^j, u\rangle + \sum_{i=1}^{m_2} E^i\{\boldsymbol{t}_{ij}\boldsymbol{v}_i\} - c_j, \ \ j = 1, \ldots, n,$$

Only "simple" stochastic box constraints: $0 \leq \boldsymbol{v}_i \leq \boldsymbol{r}_i, \quad i = 1, \ldots, m_2$.
Lake Stoöpt: r and β are non-random.

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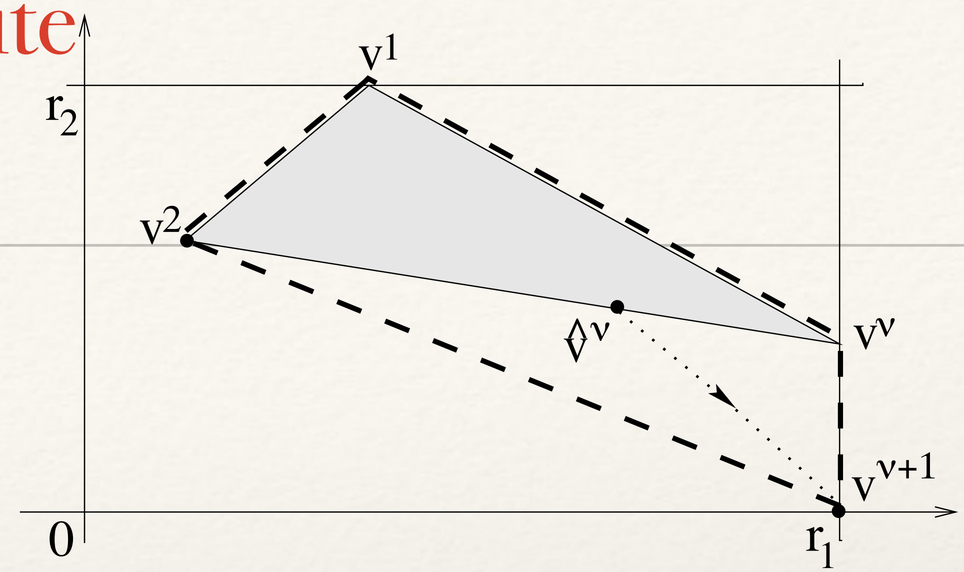
Lagrangian Finite Generation



Step 0. $\boldsymbol{V}_\nu = [\boldsymbol{v}^1, \dots, \boldsymbol{v}^\nu]$, $\boldsymbol{v}_i^k : \Xi_i \to [0, \boldsymbol{r}_i]$, $\forall i$

Step 1. Compute $d^\nu = E\{\boldsymbol{V}_\nu^\top \boldsymbol{d}\}$, $\quad B^\nu = E\{\boldsymbol{V}_\nu^\top B \boldsymbol{V}_\nu\}$, $\quad T_\nu^\top = E\{\boldsymbol{T}^\top \boldsymbol{V}_\nu\}$

Step 2. Solve the (deterministic) approximating dual program:

$$\max \langle b, u\rangle + \langle d^\nu, \lambda\rangle - \tfrac{1}{2}\langle \lambda, B^\nu \lambda\rangle - \textstyle\sum_{j=1}^n \theta_{s_j,\delta_j}(z_j)$$

$$\text{s.t } A^\top u + T_\nu^\top \lambda - z = c, \quad \textstyle\sum_{k=1}^\nu \lambda_k = 1, \quad u \geq 0, \quad \lambda \geq 0$$

$(u^\nu, \lambda^\nu, z^\nu)$ optimal, and x^ν KKT-multipliers of equality constraints.

$$\text{set } \hat{\boldsymbol{v}}^\nu = \boldsymbol{V}_\nu \lambda^\nu, \quad \boldsymbol{w}_i^\nu = \boldsymbol{d}_i - \langle \boldsymbol{T}_i, x^\nu\rangle,\ i = 1, \dots, m_2$$

Step 3 (*saddle point check*) Stop, if for each i,

$$\hat{\boldsymbol{v}}_i^\nu \in \text{argmax}_{\boldsymbol{v}_i \in [0, r_i]} \langle \boldsymbol{w}_i^\nu, \boldsymbol{v}_i\rangle - \tfrac{1}{2}\beta_i \boldsymbol{v}_i^2,$$

otherwise, for $i = 1, \dots, m_2$ and every $\zeta \in \Xi_i$, define

$$v_i^{\nu+1}(\zeta) \in \text{argmax}_{\,0 \leq v \leq r_i} [w_i^\nu(\zeta) v - \tfrac{1}{2}\beta_i v^2] \text{ with value } \theta_{r_i,\beta_i}(w_i^\nu(\zeta))$$

Augment $\boldsymbol{V}_{\nu+1} = [\boldsymbol{V} \ \ \boldsymbol{v}^{\nu+1}]$, set $\nu \leftarrow \nu + 1$, return to Step 1.