Attenuation of the curse of dimensionality in optimal control by max-plus methods

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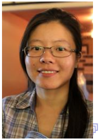
Survey of work with Akian, Lakhoua, McEneaney, Qu, Sridharan




Templates

Introduced in static analysis by Manna, Sankaranarayanan and Sipma (VMCAI'05)

A template is a collection of vectors $w_1, \dots, w_p \in \mathbb{R}^d$.
$\to$ sets parametrized by $\lambda \in \mathbb{R}^p$,

$$T(\lambda) = \{x \in \mathbb{R}^d,\ w_i \cdot x \leqslant \lambda_i, \quad 1 \leqslant i \leqslant p\}$$

Nonlinear templates, w_i are now non-linear functions (eg quadratic forms), Adjé, SG, Goubault (ESOP'10), Adjé's PhD

$$T(\lambda) = \{x \in \mathbb{R}^d,\ w_i(x) \leqslant \lambda_i, \quad 1 \leqslant i \leqslant p\}$$

Template methods developed/applied by several authors: Dang, Garoche, Gawlitza, Magron's PhD (formal proof) ...

A good idea often arises independently in different contexts

Max-plus basis methods

Introduced by Fleming and McEneaney 00, developed by McEneaney, Akian, Lakhoua, SG, Dower, Qu, Sridharan, ...

A function $f : \mathbb{R}^d \to \mathbb{R}$ is approximated by

$$f(x) \simeq F_\alpha(x) := \sup_{1 \leqslant i \leqslant p} (w_i(x) + \alpha_i)$$

maxplus basis functions $\xrightarrow{\text{level sets}}$ templates:

$$S(F_\alpha) := \{x \mid F_\alpha(x) \leqslant 0\} = \{x \mid w_i(x) \leqslant -\alpha_i\} = T(-\alpha) \ .$$

This talk:

survey of the maxplus basis methods in optimal control

curse of dimensionality attenuation $\leftrightarrow$ instance of dynamic templates

pruning of maxplus bases $\leftrightarrow$ compression of templates

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- $\text{“}2 + 3\text{”} =$

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- "$5/2$" $=$

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- “$\sqrt{-1}$” $=$

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$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

Lagrange problem / Lax-Oleinik semigroup

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s), \dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S_t)_{t\geqslant 0}$, $S_t\phi := v(t,\cdot)$.

Superposition principle: $\forall \lambda \in \mathbb{R}$, $\forall \phi, \psi$,

$$\begin{aligned} S_t(\sup(\phi,\psi)) &= \sup(S_t\phi, S_t\psi) \\ S_t(\lambda + \phi) &= \lambda + S_t\phi \end{aligned}$$

So S_t is max-plus linear.

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$$\begin{aligned} S_t(\text{“}\phi + \psi\text{”}) &= \text{“}S_t\phi + S_t\psi\text{”} \\ S_t(\text{“}\lambda\phi\text{”}) &= \text{“}\lambda S_t\phi\text{”} \end{aligned}$$

So S_t is max-plus linear.

The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0, \cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x, p) = \sup_u (L(x, u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \sup_{y \in \mathbb{R}^n} tL(\frac{x - y}{t}) + \phi(y) \ .$$

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Hopf formula, when $L = L(u)$ concave:

$$v(t,x) = \text{“}\int G(x-y)\phi(y)dy\text{”} \quad .$$

Classical	Maxplus
Expectation	sup
Brownian motion	$L(\dot{\mathbf{x}}(s)) = (\dot{\mathbf{x}}(s))^2/2$
Heat equation:	Hamilton-Jacobi equation:
$\frac{\partial v}{\partial t} = -\frac{1}{2}\Delta v$	$\frac{\partial v}{\partial t} = \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$
$\exp(-\frac{1}{2}\|x\|^2)$	$-\frac{1}{2}\|x\|^2$
Fourier transform:	Fenchel transform:
$\int \exp(i\langle x, y\rangle) f(x) dx$	$\sup_x \langle x, y\rangle - f(x)$
convolution	inf or sup-convolution

See Akian, Quadrat, Viot 97 Duality & Opt. ...

Max-plus basis / finite-element method

Fleming, McEneaney 00-;
Akian, Lakhoua, SG 04- Approximate the value function by a "linear comb." of "basis" functions with coeffs. $\lambda_i(t) \in \mathbb{R}$:

$$v(t,\cdot) \simeq \text{“}\sum_{i\in[p]} \lambda_i(t) w_i\text{”}$$

The w_i are given finite elements, to be chosen depending on the regularity of $v(t,\cdot)$

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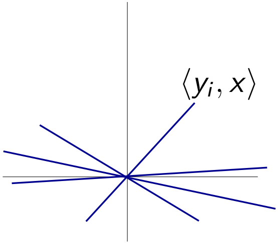
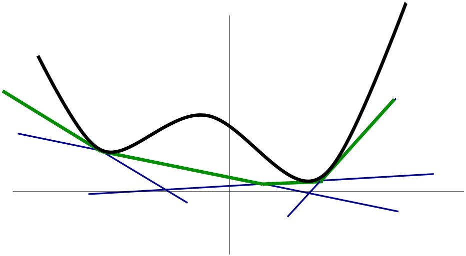
Best max-plus approximation

$$P(f) := \max\{g \leqslant f \mid g \text{ “linear comb.” of } w_i\}$$

linear forms $w_i : x \mapsto \langle y_i, x \rangle$



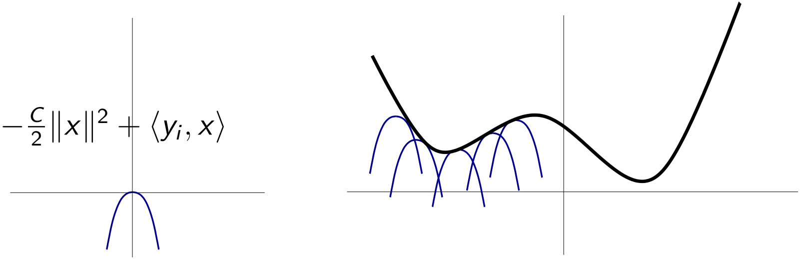


adapted if v is convex

Best max-plus approximation

$$P(f) := \max\{g \leqslant f \mid g \text{ “linear comb.” of } w_i\}$$

$w_i : x \mapsto -\frac{C}{2}\|x\|^2 + \langle y_i, x\rangle$



adapted if v is C-semi-convex, i.e. $v + C\|x\|^2/2$ convex

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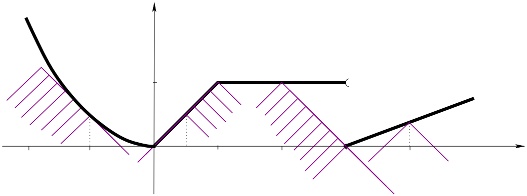
cone like functions $w_i : x \mapsto -C\|x - x_i\|$



adapted if v is C-Lip

Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

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&= \sup_i S_\tau[\lambda_i^t + \mathbf{w}_i] \\
&= \sup_i \lambda_i^t + S_\tau[\mathbf{w}_i] && \text{maxplus linearity} \\
&\simeq \sup_i \lambda_i^t + \tilde{S}_\tau[\mathbf{w}_i] && \text{semigroup approximation step} \\
&\simeq \sup_i \lambda_i^{t+\tau} + \mathbf{w}_i && \text{maxplus projection step}
\end{aligned}$$

In summary:

$$V_T(x) = \{S_\tau\}^N[V_0] \simeq \{\mathcal{P} \circ \tilde{S}_\tau\}^N[V_0] \ .$$

Max-plus basis methods

Several max-plus basis methods have been proposed:

- [Fleming,McEneaney 00]:
  A first development of max-plus basis method
- [Akian,SG,Lakhoua 06]:
  A finite element max-plus basis method
- [McEneaney 07]:
  A curse of dimensionality free method
- [McEneaney,Deshpande,SG 08],
  [Sridharan,James,McEneaney 10], [Dower,McEneaney 11], ...

Maxplus finite element projector

- Define the max-plus scalar product

$$\langle g|f\rangle := \sup_{x\in X} f(x)+g(x), \quad g\in\mathcal{G}, f\in\mathcal{F}.$$

- Two functions f and $\tilde{f}$

$$f\leqslant\tilde{f}\Leftrightarrow\langle g|f\rangle\leqslant\langle g|\tilde{f}\rangle,\ \forall g\in\mathcal{G}.$$

Maxplus finite element projector

- The Max-plus projector equals to

$$\mathscr{P}_{\mathcal{B}}[f] = \sup\{\tilde{f} \in \operatorname{span}\mathcal{B} : \tilde{f} \leqslant f\}$$
$$= \sup\{\tilde{f} \in \operatorname{span}\mathcal{B} : \langle g|\tilde{f}\,\rangle \leqslant \langle g|f\rangle, g \in \mathcal{G}\}.$$

- Analogous to Petrov-Galerkin Method
A finite set of basis functions $\mathcal{B} \subset \mathcal{F}$, a finite set of test functions $\mathcal{Z} \subset \mathcal{G}$

$$\Pi_{\mathcal{B}}^{\mathcal{Z}}[f] := \sup\{\tilde{f} \in \operatorname{Span}\mathcal{B} : \langle z|\tilde{f}\rangle \leqslant \langle z|f\rangle, \forall z \in \mathcal{Z}\}$$

Theorem ([Cohen,SG,Quadrat 96])

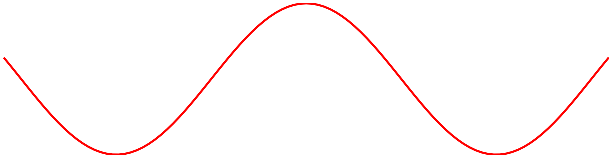
$$\Pi_{\mathcal{B}}^{\mathcal{Z}} = \mathscr{P}_{\mathcal{B}} \circ \mathscr{P}^{\mathcal{Z}}.$$

Example of max-plus finite element projector

$$\mathcal{B} = \{-|x|^2, -|x \pm 1.5|^2, -|x \pm 3|^2, -|x \pm 4|^2\}$$

$$-\mathcal{Z} = \{|x|, |x \pm \pi|, |x \pm \frac{\pi}{2}|\}$$

$$f(x) = \cos(x)$$



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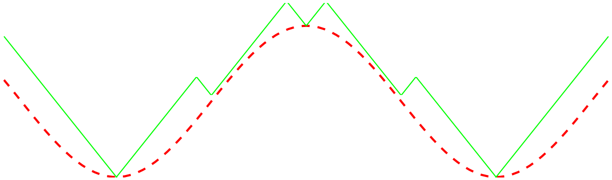
$\mathscr{P}^{\mathcal{Z}}[f]$:



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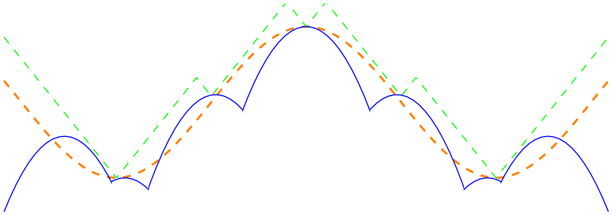
$\mathscr{P}_{\mathcal{B}} \circ \mathscr{P}^{\mathcal{Z}}[f]$:



Max-plus finite element method

$\mathcal{W}_h = \text{span}\{w_1, \ldots, w_p\}$ max-plus space of finite elements
$\mathcal{Z}_h = \text{span}\{z_1, \ldots, z_q\}$ max-plus space of test functions

Substitute $v^t \sim v_h^t := \sup_{1 \leqslant j \leqslant p} \lambda_j^t + w_j$ in $v^{t+\tau} = S^\tau v^t$, and take the scalar product with every z_i,

$$\sup_{1 \leqslant j \leqslant q} \lambda_j^{t+\tau} + \langle w_j, z_i \rangle = \sup_{1 \leqslant j \leqslant q} \lambda_j^t + \langle S^\tau w_j, z_i \rangle \qquad \forall 1 \leqslant i \leqslant q \ .$$

This is of the form $M\lambda^{t+\tau} = K\lambda^t$, M and K analogues of the mass and stiffness matrices, respectively.

$M\lambda^{t+\tau} = K\lambda^t$

Need to compute $\lambda^{t+\tau}$ as a function of λ^t. The equation $M\mu = K\lambda^t$ may not have a solution, so we take the greatest solution of $M\mu \leqslant K\lambda^t$,

$$\lambda^{t+\tau} = M^\sharp K\lambda^t$$

where $M^\sharp$ is the adjoint (it is a min-plus linear operator):

$$(M^\sharp\nu)_j = \min_{1\leqslant i\leqslant q} -M_{ij} + \nu_i \enspace .$$

In summary: Approx. v^t by $v_h^t := \sup_{1\leqslant j\leqslant p} \lambda_h^t + w^t$ where the λ_j^0 are given, and

$$\lambda_j^{t+\tau} = \min_{1\leqslant i\leqslant q} -\langle w_j, z_i\rangle + \max_{1\leqslant k\leqslant p} \langle S^\tau w_k, z_i\rangle + \lambda_k^t \qquad t = \tau, 2\tau, \ldots$$

Zero-sum two player game !

To compute

$$\lambda_j^{t+\tau} = \min_{1\leqslant i\leqslant q} -\langle w_j, z_i\rangle + \max_{1\leqslant k\leqslant p} \langle S^\tau w_k, z_i\rangle + \lambda_k^t \quad t = \tau, 2\tau, \dots$$

we need to evaluate off line

$$M_{ij} = \langle w_j, z_i\rangle = \sup_{x\in X} w_j(x) + z_i(x)$$

$$K_{ik} = \langle z_i, S^\tau w_k\rangle = \sup_{x\in X, \mathbf{u}(\cdot)} z_i(x) + \int_0^\tau \ell(\mathbf{x}(s), \mathbf{u}(s))\, ds + w_k(x)$$

1. Typically, w_j or z_i are concave, say $x \mapsto p \cdot x - x^* Ax$ with A positive semidefinite, so computing M_{ij} is a convex programming problem (sometimes explicit formulæ).
2. Computing K_{ik} is an optimal control problem, horizon τ is small, final and terminal rewards w_j and z_i nice, so convexity propagates $\Rightarrow$ global optimum accessible by Pontryagin under suitable assumptions. Sometimes, Riccati!

Error estimates

- [Akian,SG,Lakhoua 06]

$$\|V_T - \tilde{V}_T\|_{\infty,X} \leqslant (1 + \frac{T}{\tau})\Big(\max_{i=1,\dots,p} \|S^\tau[\mathbf{w}_i] - \tilde{S}^\tau[\mathbf{w}_i]\|_{\infty,X}$$

$$+ \max_{t=0,\tau,\dots,T} \|V_t - \mathscr{P}_{\mathcal{B}}[V_t]\|_{\infty,X}$$

$$+ \max_{t=0,\tau,\dots,T} \|V_t - \mathscr{P}^{\mathcal{Z}}[V_t]\|_{\infty,X}\Big)$$

Error estimates

- [Akian,SG,Lakhoua 06]

$$\|V_T - \tilde V_T\|_{\infty,X} \leqslant (1+\frac{T}{\tau})\Big(\max_{i=1,\dots,p}\|S^\tau[\mathbf{w}_i]-\tilde S^\tau[\mathbf{w}_i]\|_{\infty,X}$$
$$+\max_{t=0,\tau,\dots,T}\|V_t-\mathscr{P}_{\mathcal{B}}[V_t]\|_{\infty,X}$$
$$+\max_{t=0,\tau,\dots,T}\|V_t-\mathscr{P}^{\mathcal{Z}}[V_t]\|_{\infty,X}\Big)$$

- This also applies to [Fleming,McEneaney 00]:

$$\|V_T - \tilde V_T\|_{\infty,X} \leqslant (1+\frac{T}{\tau})\Big(\max_{i=1,\dots,p}\|S^\tau[\mathbf{w}_i]-\tilde S^\tau[\mathbf{w}_i]\|_{\infty,X}$$
$$+\max_{i=1,\dots,p}\|\mathscr{P}_{\mathcal{B}}\circ\tilde S^\tau[\mathbf{w}_i]-\tilde S^\tau[\mathbf{w}_i]\|_{\infty,X}\Big)$$

Error estimates

- Approximation of the semigroup (Euler scheme)

$$S^\tau[\mathbf{w}_i] \simeq \tilde{S}^\tau[\mathbf{w}_i] = \mathbf{w}_i + \tau H(x, \nabla \mathbf{w}_i).$$

When X is bounded, under some technical assumptions,

$$\max_{i=1,\dots,p} \|S^\tau[\mathbf{w}_i] - \tilde{S}^\tau[\mathbf{w}_i]\|_{\infty,X} \sim O(\tau^2).$$

- Maxplus projection error of a C-semiconvex function f

$$\|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X}$$

where $\mathcal{B} = \{-\frac{1}{2}(x - x_i)^\top C(x - x_i)\}_{i=1,\dots,p}$.

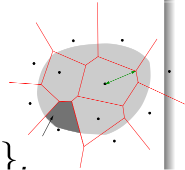
Maxplus projection error: an upper bound

Theorem ([Akian,SG,Lakhoua 06])

Let X be a compact convex subset of $\mathbb{R}^d$ and f be a C-semiconvex function and Lipschitz continuous of Lipschitz constant L, then

$$\|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \leqslant |C|\rho(\hat{X}; x_1, \dots, x_p)\,\mathrm{diam}\, X$$

where $\hat{X} = X + B(0, \frac{L}{|C|})$ and $\rho(\hat{X}; x_1, \dots, x_p)$ is the maximal radius of the Voronoi cells of the space $\hat{X}$ divided by the points $\{x_1, \dots, x_p\}$.



Maxplus projection error: an upper bound

Theorem ([Akian,SG,Lakhoua 06])

Let X be a compact convex subset of $\mathbb{R}^d$ and f be a $(C-\alpha)$-semiconvex function and Lipschitz continuous of Lipschitz constant L, then

$$\|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \leqslant \frac{\rho(\hat{X}; x_1, \ldots, x_p)^2}{\alpha} \operatorname{diam} X$$

Maxplus projection error: an upper bound

We have

$$\|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \leqslant O(\rho(\hat{X}; x_1, \ldots, x_p)^2)$$

It is known (covering surface with discs [Hlawka 49, Rogers 64]) that:

$$\min_{x_1,\ldots,x_p} \rho(\hat{X}; x_1, \ldots, x_p) \sim O(\frac{1}{p^{\frac{1}{d}}}), \quad \text{as } p \to +\infty$$

Therefore, the minimal number of basis functions $p(\epsilon)$ needed to obtain an error of order $O(\epsilon)$ is bounded by

$$p(\epsilon) \leqslant O(\frac{1}{\epsilon^{\frac{d}{2}}})$$

.

Minimal number of basis functions $p(\epsilon)$ needed to obtain an error of order $O(\epsilon)$ bounded by

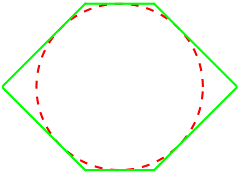
$$p(\epsilon) \leqslant O(\frac{1}{\epsilon^{\frac{d}{2}}})$$

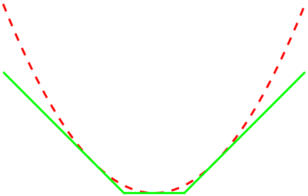
First generation maxplus methods (Fleming-McEneaney, Akian, SG, Lakhoua):

$\rightarrow$ same complexity as Falcone's grid based semilagrangean schemes (state of the art competitor) + theory of projection errors (explicit bounds).

Maxplus projection error: an asymptotic estimates

Strong analogy between





Covering a convex body by a circumscribed polytope with at most p faces

Asymptotic estimates for best approximation of convex bodies by P.M.Gruber .

Projecting a convex function into a max-plus linear subspace generated by at most p linear basis functions

Asymptotic estimates for best max-plus projection

Analogous result of [Gruber 93, Gruber 07]

Theorem ([SG,McEneaney,Qu 11])

Let X be a compact convex set and $f \in \mathcal{C}^2(\mathbb{R}^d : \mathbb{R})$ be a convex function such that $f''(x) > 0$, $\forall x \in \mathbb{R}^d$. Then,

$$\min_{x_1,\dots,x_p} \|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \sim O(\frac{1}{p^{\frac{2}{d}}}), \qquad \text{as } p \to \infty$$

$$\min_{x_1,\dots,x_p} \|\mathscr{P}_{\mathcal{B}}[f] - f\|_{1,X} \sim O(\frac{1}{p^{\frac{2}{d}}}), \qquad \text{as } p \to \infty$$

A negative result

Corollary

Minimal number of linear forms $p(\epsilon)$ to reach an approximation of order $O(\epsilon)$ is of order:

$$p(\epsilon) \sim O(\frac{1}{\epsilon^{\frac{d}{2}}})$$

A negative result for the max-plus basis method

A negative result

Corollary

Minimal number of linear forms $p(\epsilon)$ to reach an approximation of order $O(\epsilon)$ is of order:

$$p(\epsilon) \sim O(\frac{1}{\epsilon^{\frac{d}{2}}})$$

A negative result for the max-plus basis method and for the dual dynamic programming method. (if the value function is regular and strictly convex)

Asymptotic estimates for best max-plus projection

Analogous result of [Gruber 93, Gruber 07]

Theorem ([SG,McEneaney,Qu 11])

Under the same assumptions, as $p \to \infty$,

$$\min_{x_1,\dots,x_p} \|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \sim \frac{C_1}{p^{\frac{2}{d}}}, \min_{x_1,\dots,x_p} \|\mathscr{P}_{\mathcal{B}}[f] - f\|_{1,X} \sim \frac{C_2}{p^{\frac{2}{d}}},$$

where

$$C_1 = \alpha_1 \Big(\int_X (\det(f''(x)))^{\frac{1}{d+2}} \, dx \Big)^{\frac{d+2}{d}}$$

$$C_2 = \alpha_2 \Big(\int_X (\det(f''(x)))^{\frac{1}{2}} dx \Big)^{\frac{2}{d}}$$

Second generation maxplus methods

McEneaney's curse of dimensionality attenuation

Switched optimal control problem

- Infinite horizon switched optimal control problem [McEneaney 07]:

$$V(x) = \sup_{\mu} \sup_{\mathbf{u}} \int_0^{\infty} \frac{1}{2}\mathbf{x}(t)' D^{\mu(t)}\mathbf{x}(t) - \frac{\gamma^2}{2}|\mathbf{u}(t)|^2 dt,$$

where

$$\mathcal{D}_\infty \doteq \{\mu : [0,\infty) \to \{1,\dots,M\} : \mathrm{measurable}\} \ ,$$
$$W \doteq L_2^{\mathrm{loc}}([0,\infty);\mathbb{R}^k) \ ,$$

and $\mathbf{x}(\cdot)$ satisfies:

$$\dot{\mathbf{x}}(t) = A^{\mu(t)}\mathbf{x}(t) + \sigma^{\mu(t)}\mathbf{u}(t), \ \ \mathbf{x}(0) = x \in \mathbb{R}^d \ ,$$

arising from H_∞ robust control, nonconvex $(D^1,\dots,D^M \succcurlyeq 0)$.

McEneaney's curse of dimensionality free method

- Semigroup approximation:

$$S_\tau \simeq \tilde{S}_\tau = \sup_m S_\tau^m$$

- S_t^m is the semigroup associated to the control problem by letting the switching control μ equal to $m \in \{1, \dots, M\}$:

$$S_t^m[\phi](x) = \sup_{\mathbf{u}} \int_0^t \frac{1}{2}\mathbf{x}(t)' D^m \mathbf{x}(t) - \frac{\gamma^2}{2}|\mathbf{u}(t)|^2 dt + \phi(\mathbf{x}(t)).$$

$$\dot{\mathbf{x}}(s) = A^m \mathbf{x}(s) + \sigma^m \mathbf{u}(s); \;\; \mathbf{x}(0) = x \in \mathbb{R}^d \quad .$$

- $S_t^m[\phi]$ is a quadratic function if ϕ is. (Riccati)

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \quad .$$

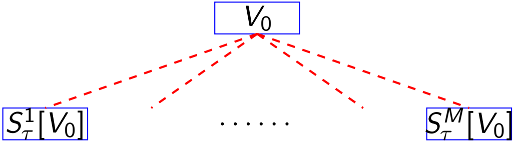
Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\ldots,i_1} S_\tau^{i_N} \circ \ldots S_\tau^{i_1}[V_0] \ .$$

V_0

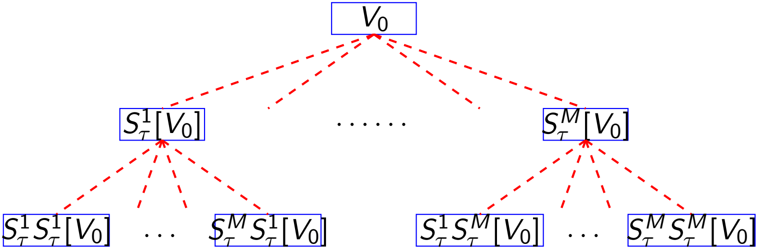
Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \ .$$



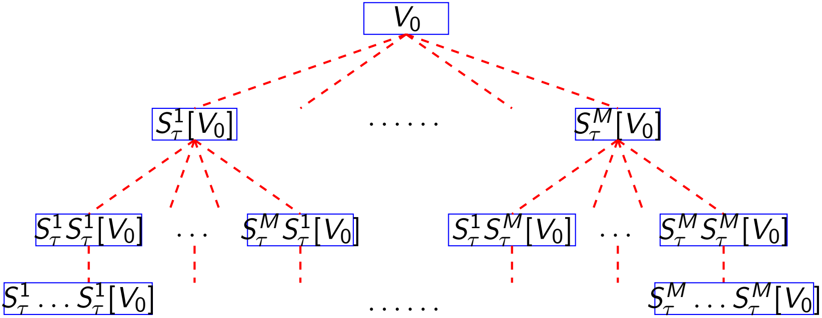
Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \ .$$



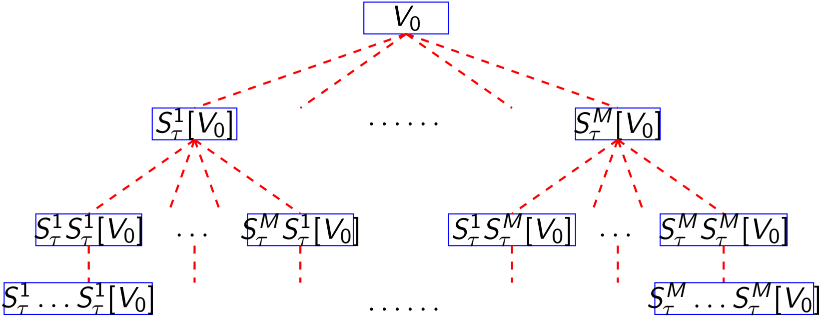
Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \ .$$



Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \; .$$



Comput. complexity: $O(M^N d^3)$ $\Rightarrow$curse of dimensionality free

Tree approximation + pruning

Computational complexity: $O(M^N d^3)$

The method has been applied to solve approximately problems of

- dimension $d = 4$, number of switches $M = 3$, in [McEneaney 07]
- dimension $d = 6$, number of switches $M = 6$, in [McEneaney,Deshpande,SG 08] (with a semidefinite programming pruning technique)
- dimension $d = 15$, number of switches $M = 6$, in [Sridharan,James,McEneaney 10] (quantum optimal gate synthesis, $SU(4)$)

Switched infinite horizon optimal control problem

Static HJ equation:

$$H(x, \nabla V) = 0, \ \forall x \in \mathbb{R}^d; V(0) = 0 \ .$$

where $H(x,p) = \sup_{m \in \{1,\dots,M\}} \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p + (A^m x)' p.$

Assumption (existence)

$$0 \prec D^m \preccurlyeq c_D I_d, \ 0 \prec \Sigma^m \preccurlyeq c_\Sigma I_d, \forall m$$
$$x' A^m x \leqslant -c_A |x|^2, \ \forall x \in \mathbb{R}^d, \forall m. \ c_A^2 > c_D c_\Sigma.$$

Assumption Σ:

$$\Sigma^m = \Sigma, \ \ m = 1, \dots, M \ .$$

Assumption contraction:

$$D^m \geqslant m_D I_d, \ m_D c_\Sigma > (c_A - \sqrt{c_A^2 - c_D c_\Sigma})^2.$$

Error bound

Theorem (Zheng Qu, PhD 2013, to appear in SICON)

Under Assumption existence and Assumption contraction, the computational complexity to reach an error of order ϵ is

$$O(M^{-\log(\epsilon)/\epsilon}d^3) \ .$$

Compare with $O(1/\epsilon^{d/r})$ for a grid scheme with an error of order $(\Delta x)^r$.

Take home message :

invariant Finsler metrics on the cone of positive definite matrices

the standard Riccati flow is a contraction in these metrics

$$\dot{P} = A'P + PA + D - P\Sigma P \ , \qquad D, \Sigma > 0 \ .$$

Thompson's part metric

C closed convex pointed cone in a Banach space C,
$x \leqslant y$ iff $y - x \in C$

$$d_T(x,y) = \inf\{\log\alpha \mid \alpha^{-1}x \leqslant y \leqslant \alpha x\}$$

$C = \mathbb{R}_+^n$, $d_T(x,y) = \|\log x - \log y\|_\infty$.

$C = S_n^+$, $d_T(A,B) = \|\operatorname{spec}\log A^{-1/2}BA^{-1/2}\|_\infty = \|\log\operatorname{spec} A^{-1}B\|_\infty$.

Invariant metrics on the cone of positive matrices

- Thompson's part metric:

$$d_T(A,B) = \|\log \operatorname{spec} A^{-1}B\|_\infty, \quad A, B \succ 0$$

$$d_T(UAU', UBU') = d_T(A,B) \ , U \in GL(n)$$

$$d_T(A,B) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\gamma(t)^{-1}\|_\infty \, dt.$$

- Riemannian metric:

$$d_2(A,B) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\gamma(t)^{-1}\|_2 dt$$

$$= \|\log \operatorname{spec} A^{-1}B\|_2$$

- Invariant Finsler metric, ν convex positively homogeneous

$$d_\nu(A,B) = \inf_\gamma \int_0^1 \nu(\dot\gamma(t)\gamma(t)^{-1}) dt = \nu(\log \operatorname{spec} A^{-1}B)$$

Theorem ([Bougerol 93])

The standard Riccati operator (flow) is a strict contraction mapping in the invariant Riemannian metric.

Theorem ([Liverani and Wojtkowski.94, Lawson and Lim 07])

The standard Riccati operator (flow) is a strict contraction mapping in Thompson's part metric.

Theorem ([Lee and Lim 07])

The standard Riccati operator (flow) is a strict contraction mapping in any invariant Finsler metric.

The proof relies on the symplectic structure of the standard Riccati flow

Ingredient: contraction property of Riccati flow

For all $m \in \{1, \ldots, M\}$, the semigroup $\{S_t^m\}_t$ corresponds to the flow of an indefinite Riccati equation:

$$\dot{P} = (A^m)'P + PA^m + D^m + P\Sigma^m P \quad . \qquad (1)$$

sign changed, $-P\Sigma^m P$ in Bougerol, Liverani, Wojtwoski. . . !

Theorem ([SG, Qu JDE14])

Under Assumption existence and Assumption contraction, there is $P_0 \succ 0$ and $\alpha > 0$ such that for all solutions $P_1(\cdot), P_2(\cdot) : [0, T] \to (0, P_0)$ of the indefinite Riccati flow (1) we have:

$$d_T(P_1(t), P_2(t)) \leqslant e^{-\alpha t} d_T(P_1(0), P_2(0)), \;\; \forall t \in [0, T] \;\; .$$

Contraction rate in Thompson's part metric

Denote by $M_s^t(\cdot)$ the flow associated to the ODE

$$\dot{x}(t) = \phi(t, x(t))$$

ϕ is a continuous function, $\mathscr{C}^1$ in the second variable with bounded differential. Let $\mathcal{U} \subseteq \operatorname{int} C$ be an open set satisfying $\lambda\mathcal{U} \subseteq \mathcal{U}$ for all $\lambda \in (0, 1]$.

Theorem ([SG, Qu JDE14])

If the flow is order-preserving, then the best constant α such that

$$d_T(M_s^t(x_1), M_s^t(x_2)) \leqslant e^{-\alpha(t-s)} d_T(x_1, x_2),\ s \leqslant t < t_{\mathcal{U}}(s, x_i)$$

holds for all $x_i \in \mathcal{U}$, $i = 1, 2$, is given by

$$\alpha := -\sup_{s \in J,\ x \in \mathcal{U}} \lambda_{\max}\Big(\big(D\phi_s(x)x - \phi(s,x)\big)x^{-1}\Big) \ .$$

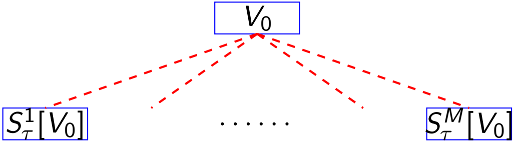
Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \ .$$

V_0

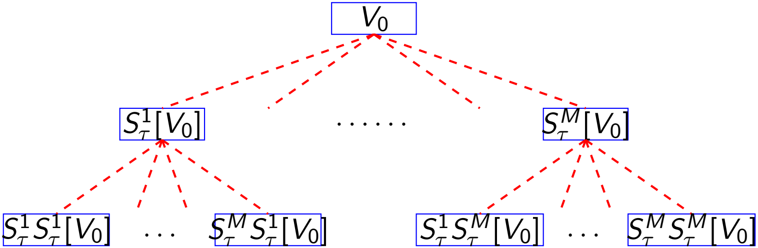
Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\ldots,i_1} S_\tau^{i_N} \circ \ldots S_\tau^{i_1}[V_0] \ .$$



Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \ .$$



Arborescent propagation

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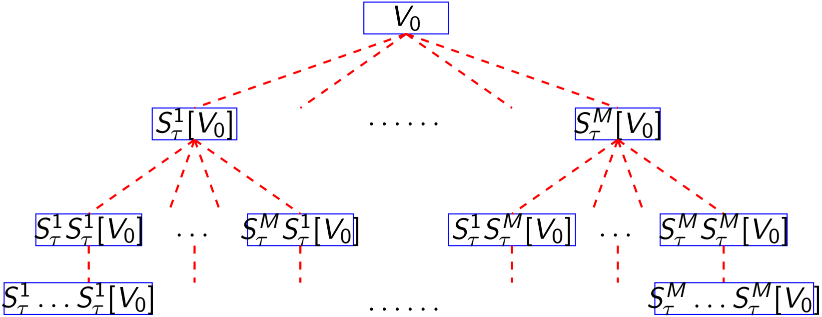
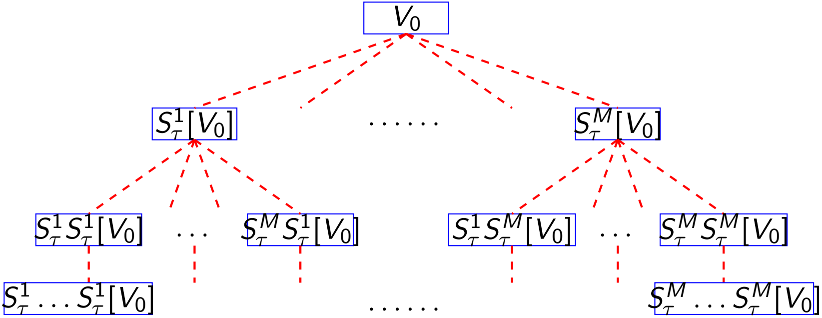


Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \ .$$



Comput. complexity: $O(M^N d^3)$ $\Rightarrow$curse of dimensionality free

The Pruning Problem

Given

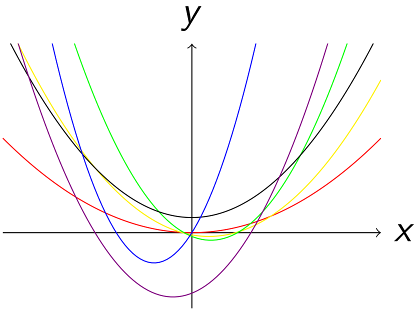
$$f = \sup_{i \in [p]} \phi_i, \qquad \phi_i \text{ quadratic } \mathbb{R}^d \to \mathbb{R}$$

and $k \ll p$, find $I \subset [p]$, $|I| = k$, with a best approximation of f by

$$\sup_{i \in I} \phi_i \quad .$$

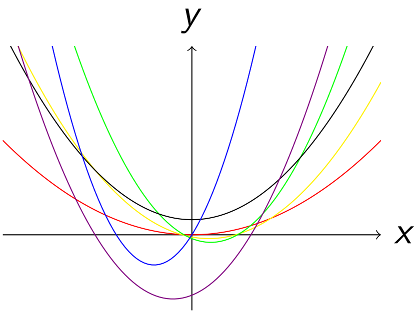
Equivalent to template compression

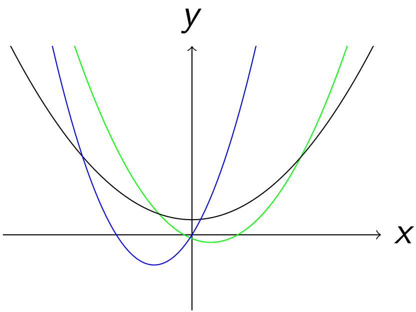
Pruning operation:



$$\phi = \sup(\phi_{green}, \phi_{red}, \phi_{violet}, \phi_{yellow}, \phi_{black}, \phi_{blue})$$

Pruning operation:





$$\phi = \sup(\phi_{green}, \phi_{red}, \phi_{violet}, \phi_{yellow}, \phi_{black}, \phi_{blue})$$

$$\phi = \sup(\phi_{green}, \phi_{black}, \phi_{blue})$$

Pruning algorithms

Let $Q_1, \ldots, Q_n$ be $(d+1)\times(d+1)$ symmetric matrices such that the quadratic functions are given by

$$\phi_i(x) = (x^T \ 1) Q_i \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad i = 1, \ldots, n.$$

1. Pairwise pruning

$$\phi_i \geqslant \phi_j \Leftrightarrow Q_i \geqslant Q_j \Rightarrow \text{ Remove the function } \phi_j$$

2. Global pruning (nonconvex quadratic programming, NP hard)

$$\sup_{i\neq j} \phi_i \geqslant \phi_j \Rightarrow \text{ Remove the function } \phi_j$$

Pruning algorithms

- Importance metric

$$\sup_{i\neq j} \phi_i \geqslant \phi_j \Leftrightarrow \nu_j := \sup_x \frac{\phi_j(x) - \sup_{i\neq j} \phi_i(x)}{1 + |x|^2} \leqslant 0$$

ν_j is called the *importance metric* of the function ϕ_j.

- Optimisation form

$$\nu_j = \max \quad \nu$$
$$\nu \leqslant z^\top (Q_j - Q_i) z, \quad \forall i \neq j$$
$$z^T z = 1.$$

Pruning algorithms

Optimisation problem

$$\nu_j = \max \ \nu$$
$$\nu \leqslant z^\top (Q_j - Q_i) z$$
$$z^T z = 1.$$

SDP relaxation

$$\overline{\nu}_j = \max \ \nu$$
$$\nu \leqslant \mathrm{tr}((Q_j - Q_i) Z)$$
$$Z \geqslant 0, \ \ \mathrm{tr}(Z) = 1.$$

- Conservative pruning[McEneaney,Deshpande,SG 08]:

$$\overline{\nu}_j \leqslant 0 \Rightarrow \nu_j \leqslant 0 \Rightarrow: \ \text{Remove the function } \phi_j.$$

- Over-pruning[McEneaney,Deshpande,SG 08]:

sort $\overline{\nu}_j$, keep at most k functions and prune the rest.

Pruning = facility location for Bregman dist.

- Discrete points $\tilde{X} = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$. The *loss* at point x_k if we remove the function ϕ_j is:

$$c(j,k) := \sup_i \phi_i(x_k) - \sup_{i \neq j} \phi_i(x_k) \geqslant 0$$

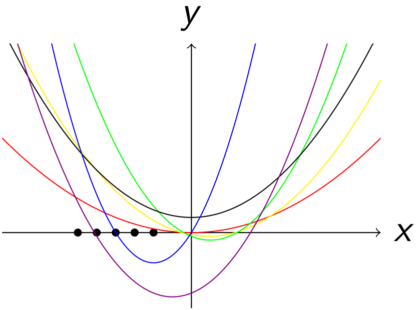
- Combinatorial optimisation problem
  - Minimizing the average lost → discrete k-median problem:

$$\min_{|S|=k} \sum_{k=1}^{N} [\min_{j \in S} c(j,k)] \ .$$

  - Minimizing the maximal lost → discrete k-center problem:

$$\min_{|S|=k} \max_{k=1,\dots,N} [\min_{j \in S} c(j,k)] \ .$$

Pruning algorithms: distribution of witness points



Pruning algorithms: generation of witness points

[SG,McEneaney,Qu 11]

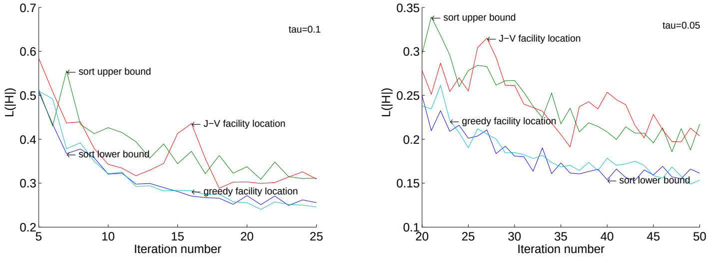
Optimisation problem

$$\begin{aligned} \nu_j = \max \ & \nu \\ & \nu \leqslant z^\top (Q_j - Q_i) z \\ & z^T z = 1. \end{aligned}$$

SDP relaxation

$$\begin{aligned} \overline{\nu}_j = \max \ & \nu \\ & \nu \leqslant \mathrm{tr}((Q_j - Q_i) Z) \\ & Z \geqslant 0, \ \ \mathrm{tr}(Z) = 1. \end{aligned}$$

Randomization technique [Aspremont,Boyd 03]: For each function j, let Z_j be the optimal solution of the SDP, generate random points of distribution $\mathcal{N}(0, Z_j)$.



(a) $\tau = 0.1$ (b) $\tau = 0.05$

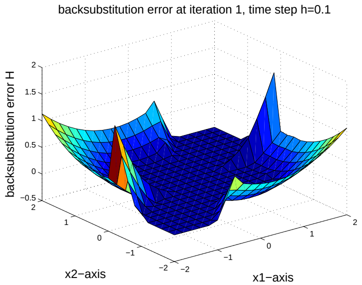
Figure: Comparison of the four pruning techniques by the evolution of the discrete L_1 norm of backsubstitution error on the rectangle $[-2,2] \times [-2,2]$ of the $x_1 - x_2$ plane, with respect to the number of iterations.

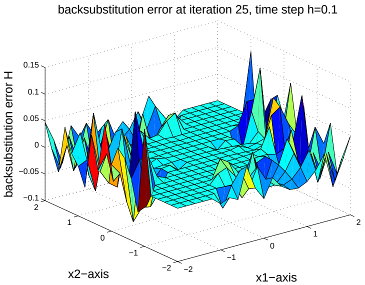
Table: CPU time

τ=0.2, K=25	Total time	Propagation	SDP	Pruning
sort lower	1.04h	1.85%	98.15%	0.00%
sort upper	1.34h	1.52%	98.43%	0.05%
J-V p-d	1.38h	1.45%	89.47%	9.08%
greedy	1.43h	1.63%	97.84%	0.53%

Experimental results

Instance : $d = 6$, $M = 6$ Backsubstitution error at point x : $H(x, \nabla V(x))$.





The method has been applied to a quantum optimal control of dimension 15 where the state space is the unitary group $SU(4)$, dim 15, see [Sridharan,James,McEneaney 10].

$$H(U,p) = \sup_{|v|=1} \langle p, -i\{\sum_{k=1}^{M} v_k H_k\} U\rangle - \sqrt{v^T R v} \ ,$$

for all $p, U \in M_n(\mathbb{C})$. Here $\{H_1,\ldots,H_M\} \subset M_n(\mathbb{C})$ generate the Lie algebra of the special unitary group $SU(n)$.

$$w_i(X) = \langle P_i, X\rangle + c_i, \ \forall X \in M_n(\mathbb{C}) \ , \ i = 0,\ldots,m \ ,$$

$P_0, P_1, \ldots, P_m \subset U(n)$ unitary, $c_0, c_1, \ldots, c_m \in \mathbb{R}$.

SDP relaxation is exact

$$\bar{\delta}_0 = \max_{X \in U(n)} \min_{1 \leqslant i \leqslant m} \langle P_i - P_0, X \rangle + c_i - c_0 \ . \qquad (2)$$

The following is a SDP relaxation of the importance metric

$$\bar{\delta}_1 = \max_{X \in B(n)} \min_{1 \leqslant i \leqslant m} \langle P_i - P_0, X \rangle + c_i - c_0 \ . \qquad (3)$$

Theorem ([SG,Qu,Sridharan MTNS14])

$\bar{\delta}_0 = \bar{\delta}_1$.

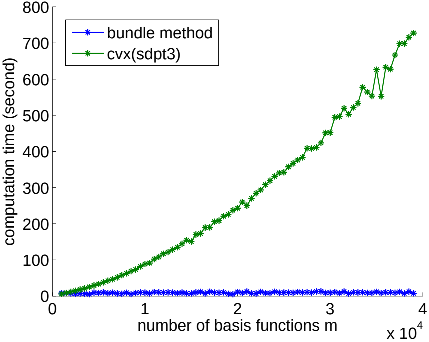


Figure: Computation time in seconds V.S. the number of basis functions m, obtained by the bundle method (blue curve) and by the interior point algorithm (green curve) for solving (3)

Recent maxplus randomized method by Zh. Qu, almost sure convergence, experimentally faster than McEneaney's curse of dim scheme (10^3 speedup on quantum gate synthesis problem), but no cod estimate/guarantee.

Concluding remarks

- Max plus basis method = nonlinear templates
- Reason of success: easy propagation of basis functions (eg Riccati)
- Generalizes to non-switched systems (approximate a general Hamiltonian by sup of LQ)
- Max-plus linearity of the Lax-Oleinik semigroup is used, extensions to second order PDE is Isaacs/game PDE are less practicable.
- Max-plus approx. $\rightarrow$ coarse but sometimes cod free estimates.
- Current work by McEneaney and Kaise, stochastic extension.

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