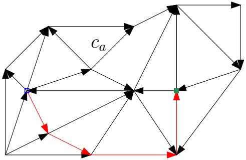
Minimizing risk Measures on paths in graphs

Axel Parmentier, Frédéric Meunier

CERMICS

SESO 2014 – 24/06/2014

Shortest path problem



- Directed graph $G = (V, A)$
- Origin o
- Destination d

Shortest path problem

- Cost $(c_a)_{a \in A}$

$$\min_{P \in \mathcal{P}_{o,d}} \sum_{a \in P} c_a$$

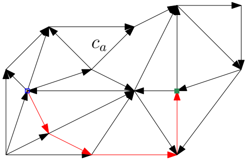
Stochastic shortest path problem

- Random variables $(X_a)_{a \in A}$

$$\min_{P \in \mathcal{P}_{o,d}} \Gamma \left(\sum_{a \in P} X_a \right)$$

$\Gamma(\cdot)$: risk measure.

Constrained shortest path



- Directed graph G
- Origin o
- Destination d
- Cost $(c_a)_{a\in A}$
- Resource constraint Γ_0

Resource constrained shortest path

- Weight w_a

$$\min_{P\in\mathcal{P}_{o,d}} \sum_{a\in P} c_a \quad s.t. \quad \sum_{a\in P} w_a \leq \Gamma_0$$

Stochastic resource constrained shortest path

- Random variables $X_a \geq_{st} 0$

$$\min_{P\in\mathcal{P}_{o,d}} \sum_{a\in P} c_a \quad s.t. \quad \Gamma\left(\sum_{a\in P} X_a\right) \leq \Gamma_0$$

$\Gamma(\cdot)$: risk measure.

Complexity

Problem	Complexity
Deterministic shortest path	$O(\|V\| \ln(\|V\|) + \|A\|)$ (Dijkstra)
Stochastic shortest path	$\mathcal{NP}$-complete for a wide range of $\Gamma(\cdot)$
Resource constrained shortest path	$\mathcal{NP}$-complete
Stochastic resource constrained shortest path	$\mathcal{NP}$-complete

Literature review

Gamma	Normal	Discrete
[4, 5]	[1, 6, 7, 8]	[9]

Table : Arc distributions in recent papers on stochastic shortest path. Stochastic resource shortest path has been little studied [3].

Concave	Convex	$\mathbb{P}(X \geq T_0)$	$\min\{t \mid \mathbb{P}(X \leq t) \geq \alpha\}$
Piecewise-linear [4, 5]	[6, 7, 8]	[2]	[1]

Table : Risk measure $\Gamma(\cdot)$

Our contribution: efficient algorithm for:

- STOCHASTIC SHORTEST PATH
- STOCHASTIC RESOURCE CONSTRAINED SHORTEST PATH

- Discrete distributions (allows fine discretization)
- Risk measures coherent with the usual stochastic order

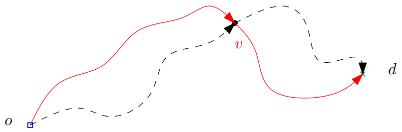
Table of contents

1. Key elements in deterministic shortest path algorithm
2. Stochastic dominance
3. Stochastic shortest path algorithm
4. Numerical results

Part 1

1. Key elements in deterministic shortest path algorithm
   - Unconstrained shortest path: subpaths
   - Unconstrained shortest path: lower bound on complete paths
   - Constrained shortest path: dominance
2. Stochastic dominance
3. Stochastic shortest path algorithm
4. Numerical results

Deterministic shortest path: subpath property



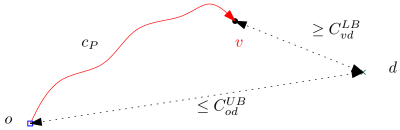
- Dijkstra algorithm
- Ford Bellman algorithm

Subpath property (deterministic short path)

Let $P_{od} = (o, \ldots, v, \ldots, d)$ be a path from o to d which goes through v.
P_{od} is a shortest path $\Leftrightarrow$ Subpaths $P_{ov} = (o, \ldots, v)$ and $P_{vd} = (v, \ldots, d)$ are shortest paths respectively from o to v and from v to d.

	Constrained SP	*Stochastic SP*	*Stochastic Res. Constr. SP*
$\Rightarrow$	False	False	False
$\Leftarrow$	True if $P_{ov}P_{vd}$ is feasible	False	True if $P_{ov}P_{vd}$ is feasible

Deterministic shortest path: bounds on complete paths



- $C_{od}^{UB} \geq \min_{P \in \mathcal{P}_{o,d}} c_P$
- $C_{vd}^{LB} \leq c_P, \forall P \in \mathcal{P}_{vd}$

Restricted search area

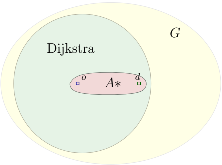
Path $P \in \mathcal{P}_{ov}$ such that $c_P + C_{vd}^{LB} > C_{od}^{UB}$ is not a subpath of a shortest path

A* algorithm (unconstr.):

- Generate all non dominated path satisfying

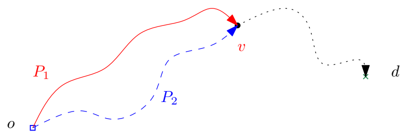
$$c_P + C_{vd}^{LB} \leq C_{od}^{UB}$$

- Update C_{od}^{UB}



Constrained shortest path version: bounds on cost *and on weights*.

Constrained shortest path: dominance



$P_1 \in \mathcal{P}_{ov}$ *dominates* $P_2 \in \mathcal{P}_{ov}$ if

$$c_{P_1} \leq c_{P_2} \text{ and } w_{P_1} \leq w_{P_2}$$

Path dominance

If path $P_1 \in \mathcal{P}_{ov}$ dominates $P_2 \in \mathcal{P}_{ov}$, then for any path P starting in v

1. $P_2 + P$ feasible $\Rightarrow P_1 + P$ is feasible
2. $c_{P_1+P} \leq c_{P_2+P}$

Only non-dominated paths need to be generated when searching a shortest path

Label setting / pruning algorithm: Generate all non dominated paths

Algorithm *convergence speed* is strongly influenced by *enumeration order*.

Part 2

1. Key elements in deterministic shortest path algorithm
2. Stochastic dominance
   - Stochastic order
   - Paths dominance
3. Stochastic shortest path algorithm
4. Numerical results

Usual stochastic order

The *usual stochastic order* $\leq_{st}$ is defined as:

$$X \leq_{st} Y \Leftrightarrow F_X(t) \geq F_Y(t), \forall t$$

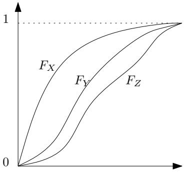


Figure : $X \leq_{st} Y \leq_{st} Z$

Properties

1. $$\left.\begin{array}{c} X_1 \leq_{st} X_2 \\ \text{and} \\ Y_1 \leq_{st} Y_2 \end{array}\right\} \Rightarrow X_1 + X_2 \leq_{st} Y_1 + Y_2$$
   for independent random variables.
2. $X \leq_{st} Y \Rightarrow \Gamma(X) \leq \Gamma(Y)$ for a wide range of Risk Measures $\Gamma(\cdot)$ — Examples: $Var(\cdot)$, $CVar(\cdot)$, $\mathbb{E}(f(\cdot))$ with f non decreasing, $\mathbb{P}(\cdot > T_0)$.

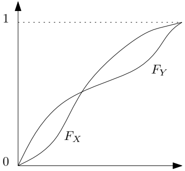
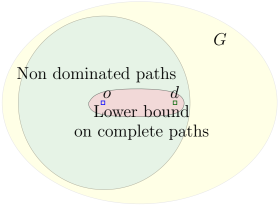


Figure : $X \nleq_{st} Y$ and $Y \nleq_{st} X$

Dominance: stochastic shortest path

Path $P_1 \in \mathcal{P}_{uv}$ *dominates* a path $P_2 \in \mathcal{P}_{uv}$ if $X_{P_1} \leq_{st} X_{P_2}$



Propagation

If $P_1 \in \mathcal{P}_{uv}$ dominates $P_2 \in \mathcal{P}_{uv}$, then $P_1 + P$ dominates $P_2 + P$

Dominance

A shortest path is non-dominated, and its subpaths are non-dominated.

Only non-dominated paths need to be generated.

Complete path test

Let Γ_{od}^{UB} and X_{ov}^{LB} be such that:

- $\exists P \in \mathcal{P}_{od} : \Gamma\left(X_P\right) \leq \Gamma_{od}^{UB}$
- $\forall P \in \mathcal{P}_{ov}, X_P \geq_{st} X_{ov}^{LB}$

If $P \in \mathcal{P}_{vd}$ is the subpath of a shortest path, then $\Gamma\left(X_P + X_{od}^{LB}\right) \leq \Gamma_{od}^{UB}$

Part 3

1 Key elements in deterministic shortest path algorithm

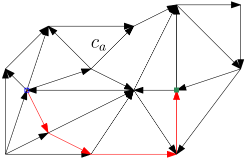
2 Stochastic dominance

3 Stochastic shortest path algorithm
- Exact algorithm
- Forward lower bound algorithm
- Stochastic resource constrained shortest path

4 Numerical results

Stochastic shortest path problem

We now give an algorithm to solve the stochastic shortest path problem

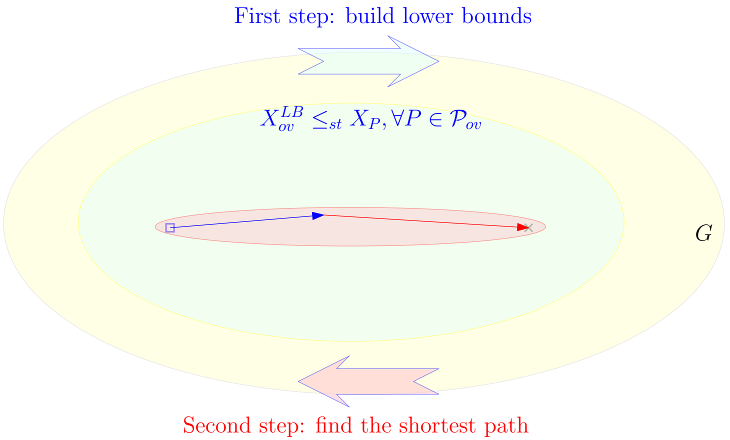


Stochastic shortest path problem

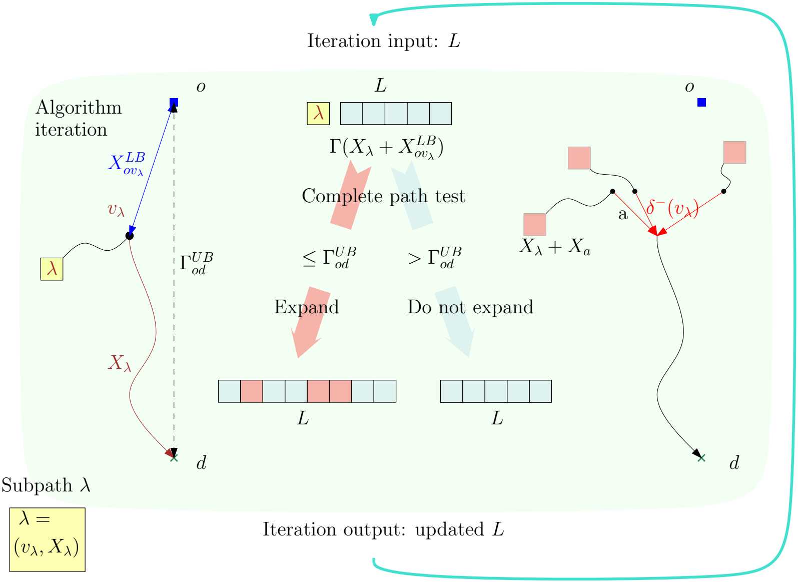
- A directed graph G
- An origin o
- A destination d
- Random variables X_a

Find $\arg\min_{P \in \mathcal{P}_{o,d}} \Gamma\left(\sum_{a \in P} X_a\right)$

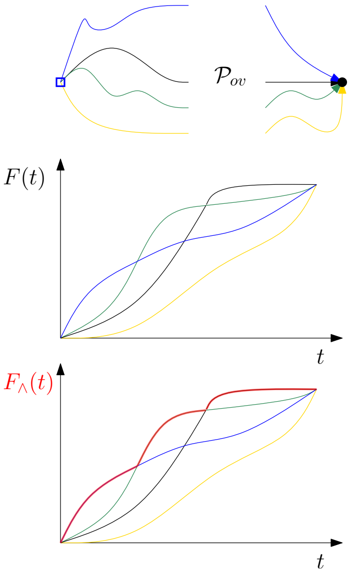
Algorithm principle



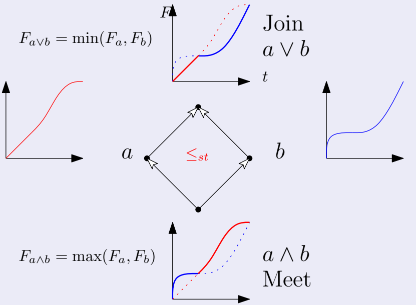
Second step: exact algorithm using bounds as input



Lower bounds and lattice structure



Lattice structure of $(\{X\}, \leq_{st})$



$$X_{ov}^{LB} \leq_{st} X_P, \ \forall P \in \mathcal{P}_{ov} \quad \Leftrightarrow X_{ov}^{LB} \leq_{st} \bigwedge_{\mathcal{P}_{ov}} X_P$$

Lower bound algorithm main ideas

- Computing $\wedge_{\mathcal{P}_{ov}} X_P$ is difficult
- We can simply compute $X \wedge Y$

Lemma:

A solution of system (1) satisfies

$$Z_v \leq_{st} \bigwedge_{\mathcal{P}_{ov}} X_P$$

$$\begin{cases} Z_o = 0, \\ Z_v = \bigwedge_{u \in N^-(v)} \left(Z_u + X_{(u,v)} \right) \quad \text{for all } v \in V \setminus \{o\}. \end{cases} \tag{1}$$

Stochastic on Time Arrival problem

System (1) admits a unique solution which can be computed efficiently [9].

Tests for stochastic resource constrained shortest path

Stochastic resource constrained shortest path

- Directed graph G
- Origin o
- Destination d
- Cost $(c_a)_{a\in A}$
- Resource constraint Γ_0

- Random variables $X_a \geq_{st} 0$

$$\min_{P\in\mathcal{P}_{o,d}} \sum_{a\in P} c_a$$
$$s.t.\quad \Gamma\left(\sum_{a\in P} X_a\right) \leq \Gamma_0$$

Given C_0 and $(X_{ov}^{LB})_v$ such that:

- $\exists P \in \mathcal{P}_{od} : C_0 = \sum_{a\in P} c_a$ and $\Gamma\left(\sum_{a\in P} X_a\right) \leq \Gamma_0$.
- $\forall P \in \mathcal{P}_{ov}, X_{ov}^{LB} \leq_{st} X_P$
- $\forall P \in \mathcal{P}_{ov}, \Pi_{ov} \leq C_P$

Complete path test

If $P \in \mathcal{P}_{vd}$ is the subpath of a shortest path, then

- $\Gamma\left(X_P + X_{od}^{LB}\right) \leq \Gamma_0$
- and $C_P + \Pi_{ov} \leq C_0$

Part 4

1 Key elements in deterministic shortest path algorithm

2 Stochastic dominance

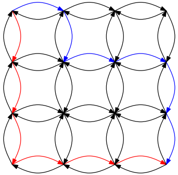
3 Stochastic shortest path algorithm

4 Numerical results
- Stochastic shortest path

Instances

	Distribution type	Largest value
R	Random	≤ 100
L	Log Normal (random variance)	≤ 100

Risk Measure $\Gamma(\cdot)$	
CVaR	$CVaR(\cdot)$



Name	$\lvert V \rvert$	$\lvert A \rvert$	Opt.
g40L	1600	6240	1100.43
g100L	10000	39600	2538.18

Name	$\lvert V \rvert$	$\lvert A \rvert$	Opt.
g10R	100	360	465.6
g40R	1600	6240	1499.6
g100R	10000	39600	3329.6
g200R	40000	159200	6650.7
g300R	90000	358800	10848.2

Algorithms are implemented in C++. Experiments on a MacBook Pro with a 2,5 GHz Intel Core i5 processor and 4 Go 1600 MHz DDR3 ram.

Numerical results

Instance	CPU time	LB	UB	λ treat.	λ exp.	CPU time	Sol. Value	Tot. CPU time
g40L	1.94278	1100.42	1100.43	307	79	0.041655	1100.43	2.0007
g100L	147.567	2537.36	2538.35	50962	12756	10.766	2538.18	158.447
g10R	0.009537	464.525	465.672	70	19	0.001925	465.672	0.012638
g40R	0.285869	1498.9	1499.75	607	156	0.031716	1499.66	0.332614
g100R	2.8309	3325.55	3337.34	16600	4479	1.56209	3329.66	4.50626
g200R	48.4813	6649.26	6650.94	7187	1806	1.10682	6650.76	50.1919
g300R	175.299	10846.3	10850.8	74418	19085	25.8252	10848.2	202.705

Table : Stochastic shortest path – Experimental results on grids – CPU times are in seconds

Instance	CPU time	λ treat.	λ exp.	CPU time	Sol. Value	Tot. CPU time
g20R	0.012	154	41	0.012	344.3	0.054
g40R	0.107	308	79	0.027	611.7	0.151
g100R	0.330	815	205	0.092	1656.7	0.497
g300R	2.781	977	245	0.229	2019.88	3.53

Table : Stochastic resource constrained shortest path (minimum cost, probability of on time arrival greater than a threshold fixed to forbid the determinist optimal solution) – Experimental results on grids – CPU times are in seconds

Bi Yu Chen, William HK Lam, Agachai Sumalee, Qingquan Li, Hu Shao, and Zhixiang Fang.
Finding reliable shortest paths in road networks under uncertainty.
Networks and spatial economics, 13(2):123–148, 2013.

H Frank.
Shortest paths in probabilistic graphs.
Operations Research, 17(4):583–599, 1969.

Stefanie Kosuch and Abdel Lisser.
Stochastic shortest path problem with delay excess penalty.
Electronic Notes in Discrete Mathematics, 36:511–518, 2010.

Ishwar Murthy and Sumit Sarkar.
A relaxation-based pruning technique for a class of stochastic shortest path problems.
Transportation Science, 30(3):220–236, 1996.

Ishwar Murthy and Sumit Sarkar.

Stochastic shortest path problems with piecewise-linear concave utility functions.
Management Science, 44(11-part-2):S125–S136, 1998.

Evdokia Nikolova.
High-performance heuristics for optimization in stochastic traffic engineering problems.
In *Large-Scale Scientific Computing*, pages 352–360. Springer, 2010.

Evdokia Nikolova, Matthew Brand, and David R Karger.
Optimal route planning under uncertainty.
In *ICAPS*, volume 6, pages 131–141, 2006.

Evdokia Nikolova, Jonathan A Kelner, Matthew Brand, and Michael Mitzenmacher.
Stochastic shortest paths via quasi-convex maximization.
In *Algorithms–ESA 2006*, pages 552–563. Springer, 2006.

Samitha Samaranayake, Sebastien Blandin, and A Bayen.

A tractable class of algorithms for reliable routing in stochastic networks.
Transportation Research Part C: Emerging Technologies, 20(1):199–217, 2012.

Appendix

5. Appendix
   - Risk Measures and stochastic dominance
   - Detailed algorithm

Stochastic order and risk measures

A risk measure $\Gamma(\cdot)$ is

- *Law invariant* if $P_X = P_Y \Rightarrow \Gamma(X) = \Gamma(Y)$
- *Monotone* if $X \leq Y \Rightarrow \Gamma(X) = \Gamma(Y)$ where $X \leq Y$ mean $X(\omega) \leq Y(\omega)$ for any event ω.

Coherent risk measures

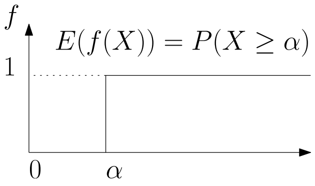
$X \leq_{st} Y \Rightarrow \Gamma(X) \leq \Gamma(Y)$ for any law invariant and monotone risk measure $\Gamma(\cdot)$.

Composition

$X \leq_{st} Y \Rightarrow f(X) \leq_{st} f(Y)$ for all increasing function f

Examples:

- $CVaR$
- $CVaR \circ f$ with f non-decreasing
- $\alpha CVaR + \beta \mathbb{E}$



Exact algorithm

Algorithm 1 Exact algorithm for stochastic shortest path

Require: $G, o, d, X_a, X_{ov}^{LB}, \Gamma_{od}^{UB}, (Z_v^{LB})_v$

Ensure: λ_o defines a shortest path

$L \leftarrow (d, 0)$ with key $\Gamma\left(X_{od}^{LB}\right)$

while L is not empty **do**

  $\lambda \leftarrow$ head of L

  Remove λ form L

  if $\Gamma\left(Y_\lambda^R + X_{ov_\lambda}^{LB}\right) \leq \Gamma_{od}^{UB}$ **then**

    Expand λ:

    for $(v, v_\lambda) \in \delta^-(v_\lambda)$ **do**

      Add $(v,\, Y_\lambda^R + X_{(v,v_\lambda)})$ to L with key $\Gamma\left(Y_\lambda^R + X_{ov_\lambda}^{LB}\right)$

    end for

    if $v_{=}o$ **then**

      $\lambda_o = \lambda$

    end if

  end if

end while

return λ_o

Lower bound algorithm

Algorithm 2 Lower bound for stochastic shortest path

Require: A graph $G = (V, A)$, arc distributions $X_a \geq_{st} 0$, and an origin o.

$Z_v \leftarrow \infty, \forall v \in V$.
$L \leftarrow [(o, 0)]$.
while L is not empty **do**
  $\lambda \leftarrow$ head of L
  Remove λ from L
  if $Y_\lambda^{LB} \not\geq_{st} Z_{v_\lambda}$ **then**
    $Z_{v_\lambda} \leftarrow Y_\lambda^{LB} \wedge_{st} Z_{v_\lambda}$
    for $(v_\lambda, v) \in \delta^+(v_\lambda)$ **do**
      Insert $\left(v, Z_{v_\lambda} + X_{(v_\lambda, v)}\right)$ in L with key $\Gamma\left(Z_{v_\lambda} + X_{(v_\lambda, v)}\right)$
    end for
  end if
end while
return (Z_v)