How to Make the Most Out of Evolving Information: Function Identification Using Epi-Splines

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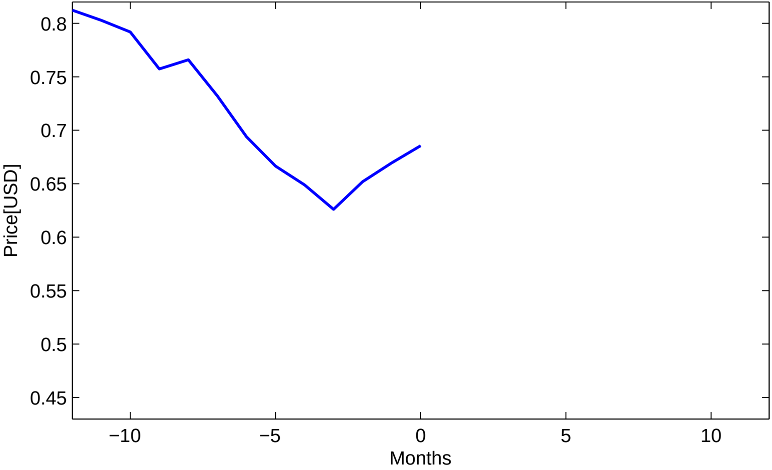
Goal: estimate, predict, forecast

Copper prices



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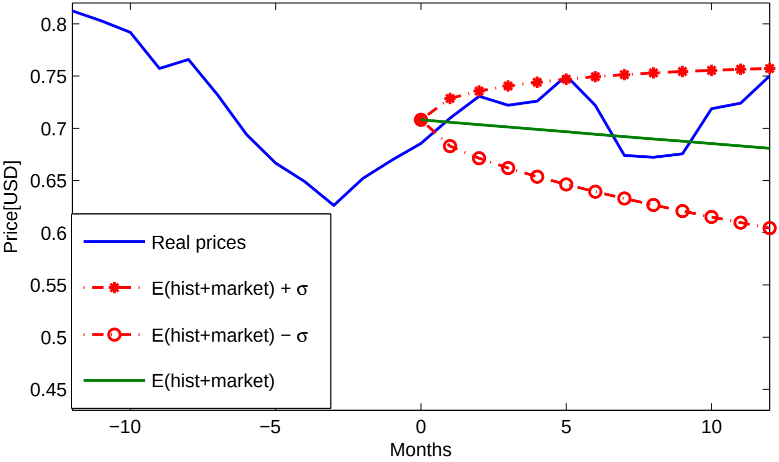
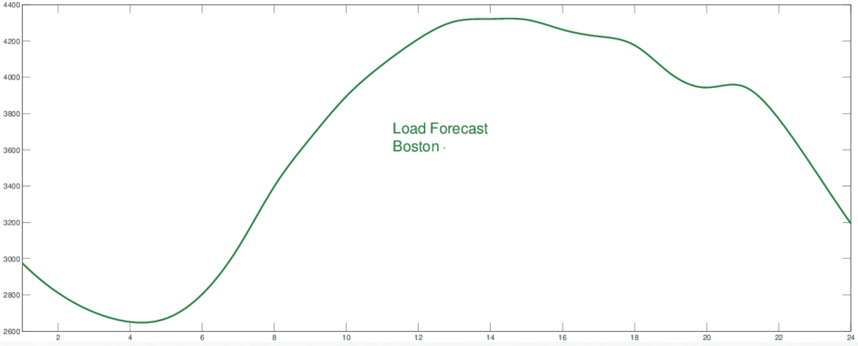
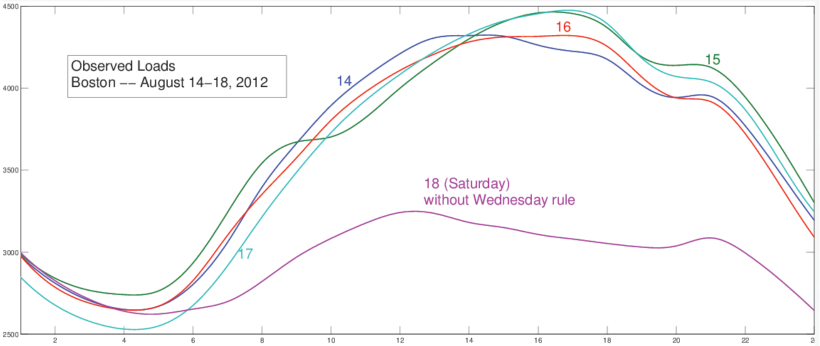


Hourly electricity loads



Hard information: data

- Observations
- Usually scarce, excessive, corrupted, uncertain



Soft information

- Indirect observations, external information
- Knowledge about
  - structures
  - established “laws”
  - physical restrictions
  - shapes, smoothness of curves

Evolving information

data acquisition, information growth

Function Identification Problem

Identify a function that

best represents all available information

Applications of function identification approach

- energy
- natural resources
- financial markets
- image reconstruction
- uncertain quantification
- variograms
- nonparametric regression
- deconvolution
- density estimation

Outline

- Illustrations and formulations (slides 12-33)
- Framework (slides 34-60)
- Epi-splines and approximations (slides 61-77)
- Implementations (slides 78-85)
- Examples (slides 86-124)

Illustrations and formulations

Identifying financial curves

Estimate discount factor curve given instruments $i = 1, 2, ..., I$ and

$$\text{payments} \quad p_0^i, p_1^i, ..., p_{N_i}^i \quad \text{at times} \quad 0 = t_0^i, t_1^i, ..., t_{N_i}^i$$

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Identify nonnegative, nonincreasing function f with $f(0) = 1$

$$\sum_{j=0}^{N_i} f(t_j^i) p_j^i \approx 0 \quad \text{for all } i$$

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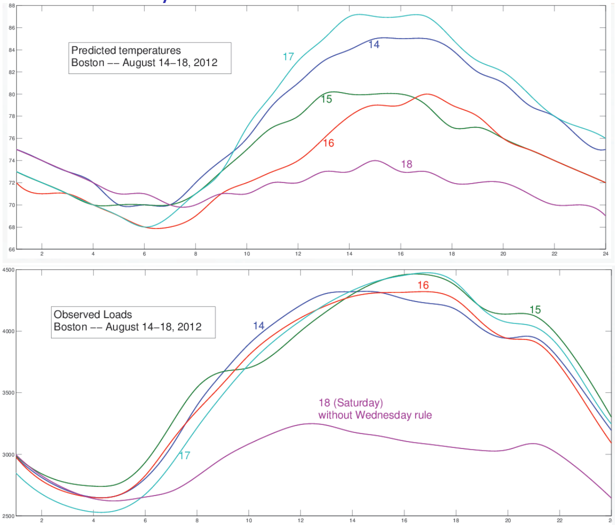
Constrained infinite-dimensional optimization problem:

$$\begin{aligned} &\text{minimize} \ \sum_{i=1}^{I} \left| \sum_{j=0}^{N_i} f(t_j^i) p_j^i \right| \\ &\text{such that} \ \ f(0) = 1, \ f \geq 0, \ f' \leq 0 \end{aligned}$$

Day-ahead electricity load forecast



Data: predicted temperature, dew point; observed load

Day-ahead load forecast problem

Using a relevant data and weather forecast for tomorrow, determine tomorrow's load curve and its uncertainty

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- $h = 1, 2, ..., 24$: hours of the day
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- d_h^j: predicted dew point in hour h of day j
- l_h^j: actual observed load in hour h of day j

Day-ahead load forecast problem (cont.)

"Functional" regression model:

$$l_h^j = f^{\mathrm{tmp}}(h) t_h^j + f^{\mathrm{dpt}}(h) d_h^j + e_h^j,$$

- regression "coefficients" f^{tmp} and f^{dpt} are functions of time
- e_h^j: error between the observed and predicted loads

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Regression problem: find functions f^{tmp} and f^{dpt} on $[0, 24]$ that

$$\text{minimize} \sum_{j=1}^{J} \sum_{h=1}^{24} \left| l_h^j - \left[f^{\mathrm{tmp}}(h)t_h^j + f^{\mathrm{dpt}}(h)d_h^j \right] \right|$$

subject to smoothness, curvature conditions

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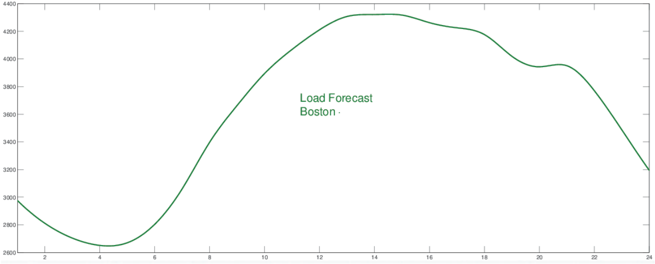
$$\text{load forecast becomes } f^{\mathrm{tmp}}(h)t(h) + f^{\mathrm{dpt}}(h)d(h), \quad h \in [0, 24]$$



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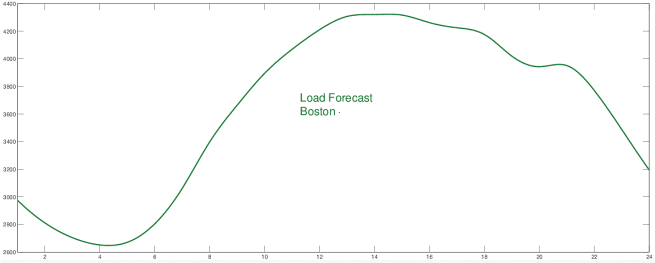


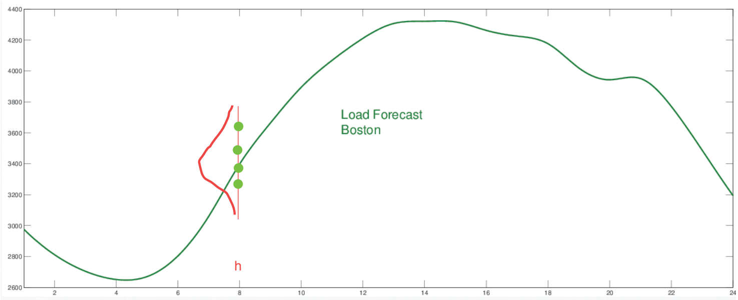
But, what about uncertainty?

Estimation of errors

For hour h:

$$\text{minimized errors } e_h^j = l_h^j - [f^{\mathrm{tmp}}(h)t_h^j + f^{\mathrm{dpt}}(h)d_h^j], \quad j = 1, \ldots, J$$

$=$ samples from actual error density



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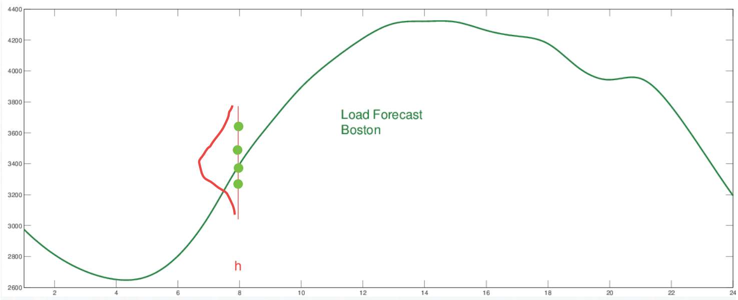

Probability density estimation problem with very small data set

Estimation of errors (cont.)

Probability density estimation problem:

Given iid sample $x^1, x^2, ..., x^\nu$,

find a function $h \in \mathcal{H}$ that maximizes $\prod_{i=1}^{\nu} h(x^i)$

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Constrained infinite-dimensional optimization problem

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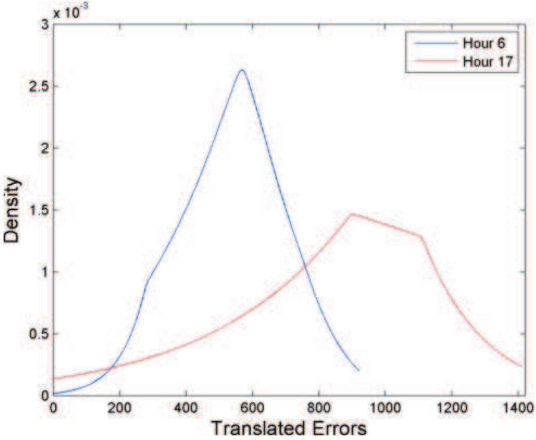

Generating load scenarios

Also need to consider conditioning:

- if load above regression function early, then likely above later
- error data for density estimation reduced further

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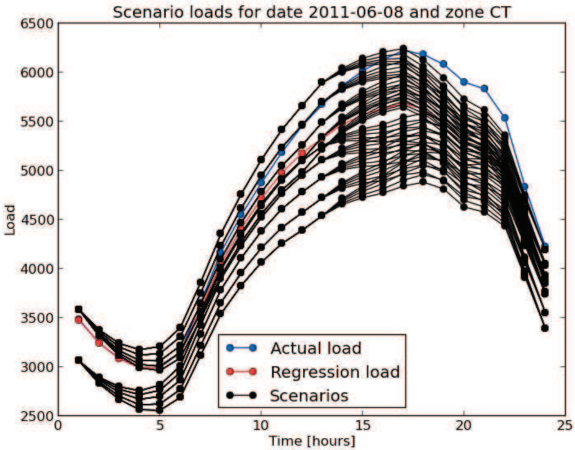


Forecast precision (Connecticut 2010-12)

Season	Mean % error
Fall	3.99
Spring	2.73
Summer	4.19
Winter	3.47

Uncertainty quantification (UQ)

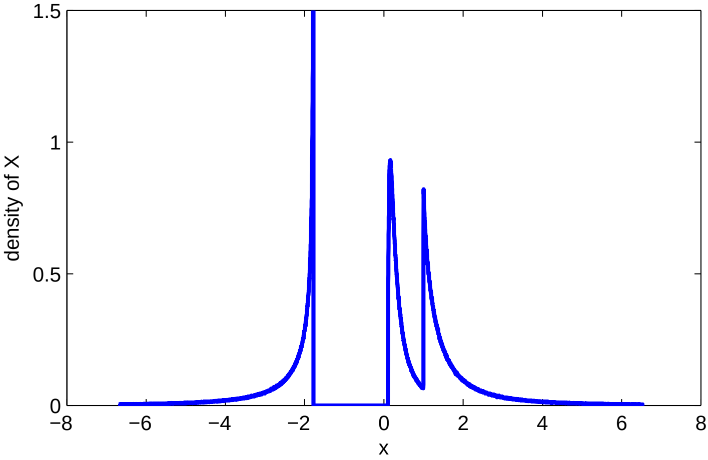
Engineering, biological, physical systems

- Input: random vector V ("known" distribution)
- System function g; implicitly defined e.g. by simulation
- Output: random variable

$$X = g(V)$$

Illustration of UQ challenges

Output amplitude of dynamical system:



How to estimate densities like this with a small sample?

Probability density estimation

Sample $X^1, \ldots, X^\nu$: maximize $\prod_{i=1}^{\nu} h(X^i)$ s.t. $h \in H^\nu \subset \mathcal{H}$

- Sample size might be growing
- Constraint set H^ν might be evolving

Evolving probability density estimation

maximize $\prod_{i=1}^{\nu}(h(X^i))^{1/\nu}$ s.t. $h \in H^\nu \subset \mathcal{H}$

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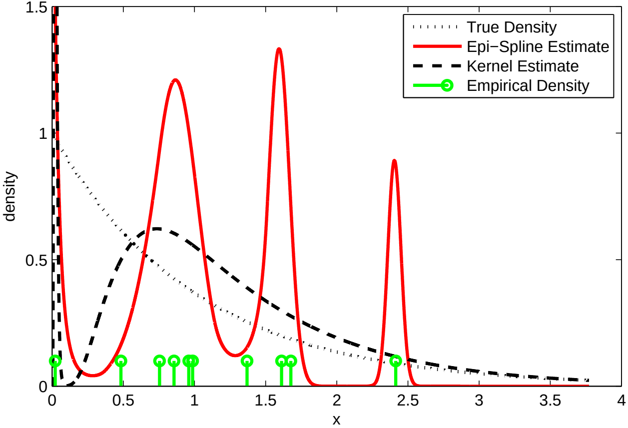
actual problem: maximize $E[\log h(X)]$ s.t. $h \in H \subset \mathcal{H}$

Evolving constraints

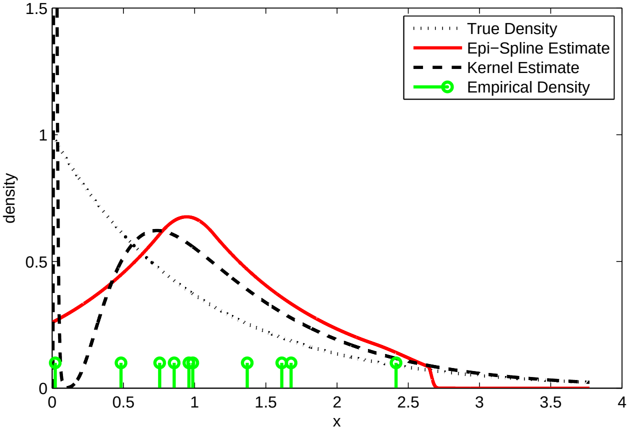
Constraints: nonnegative support, smooth, curvature bound



MSE $= 0.2765$ (epi-spline) and 0.3273 (kernel)

Evolving constraints (cont.)

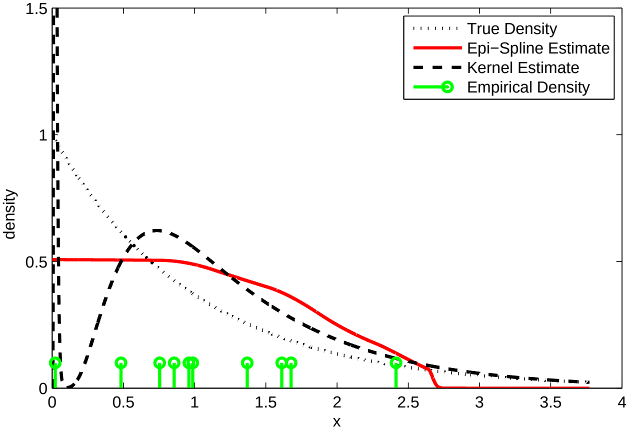
Also log-concave



MSE $= 0.1144$ (epi-spline) and 0.3273 (kernel)

Evolving constraints (cont.)

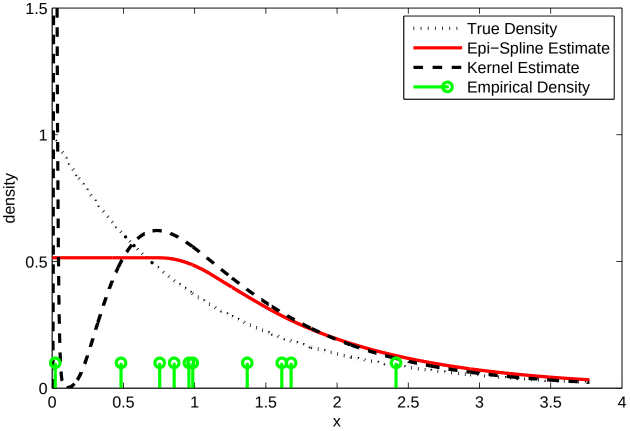
Also nonincreasing



MSE $= 0.0470$ (epi-spline) and 0.3273 (kernel)

Evolving constraints (cont.)

Also slope bounds



MSE $= 0.0416$ (epi-spline) and 0.3273 (kernel)

Take-aways so far

- Challenges in forecasting and estimation are function identification problems
- Day-ahead forecasting of electricity demand involves
  - functional regression to get trend
  - probability density estimation to get errors
- Soft information supplements hard information (data)

Questions?

Outline

Framework

Actual problem

- Constrained infinite-dimensional optimization problem

$$\begin{aligned}&\min \psi(f)\\&\text{such that } f \in F \subseteq \mathcal{F} \subseteq \text{ some function space}\end{aligned}$$

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- criterion ψ (norms, error measures, etc)

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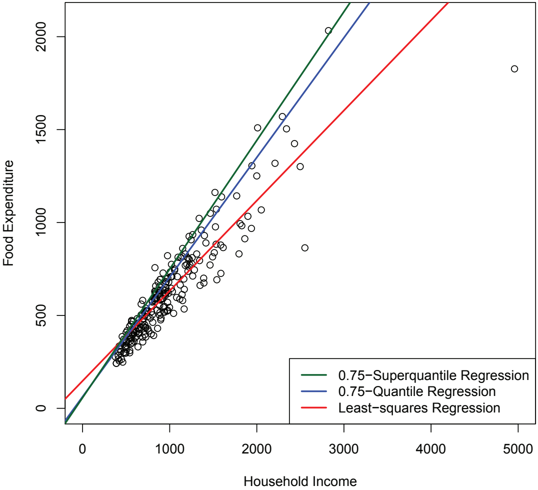
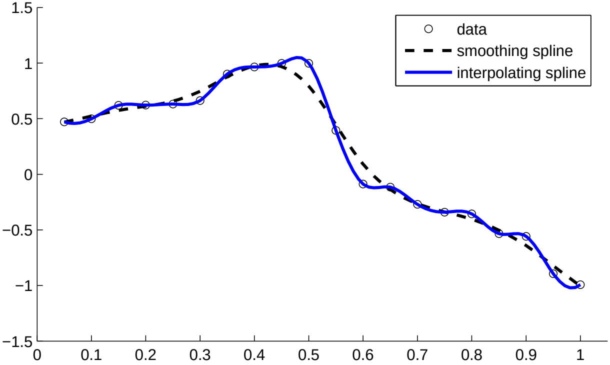
- criterion ψ (norms, error measures, etc)
- feasible set F (shape restrictions, external info)
- subspace of interest $\mathcal{F}$ (continuity, smoothness)

Special case: linear regression



find affine f that $\min \sum_j (y^j - f(x^j))^2$ or other error measures

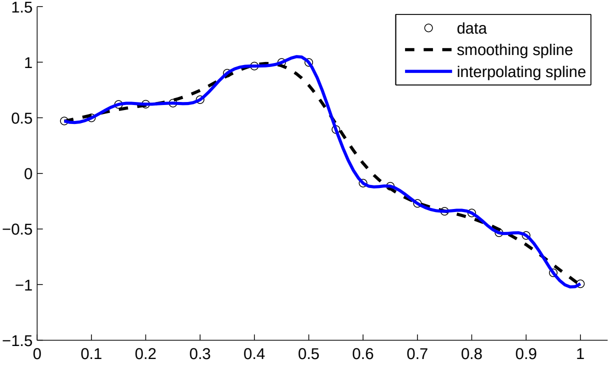
Special case: interpolation



$$\min \|f''\|^2$$

such that $f(x^j) = y^j$ for all j and $f \in$ Sobolev space

Special case: smoothing



$$\min \sum_j (y^j - f(x^j))^2 + \lambda \|f''\|^2$$

such that $f \in$ Sobolev space

Evolving problems

- Evolving infinite-dimensional optimization problems

$$\begin{aligned} &\min \psi^\nu(f) \\ &\text{such that } f \in F^\nu \subseteq \mathcal{F} \subseteq \text{ some function space} \end{aligned}$$

- evolving/approximating criterion ψ^ν
- evolving/approximating feasible set F^ν

Evolving problems

- Evolving infinite-dimensional optimization problems

$$\min \psi^\nu(f)$$
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Function space

extended real-valued lower semicontinuous functions on $\boldsymbol{R}^n$

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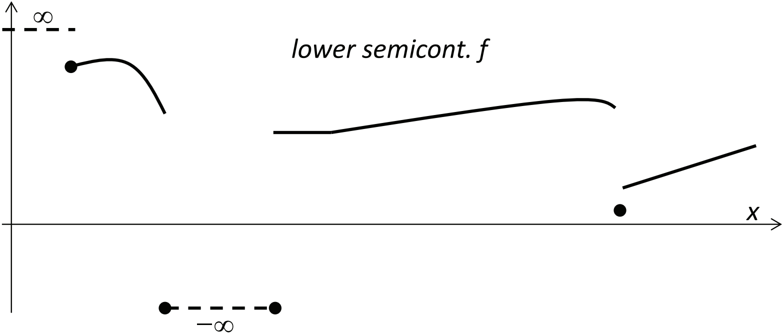

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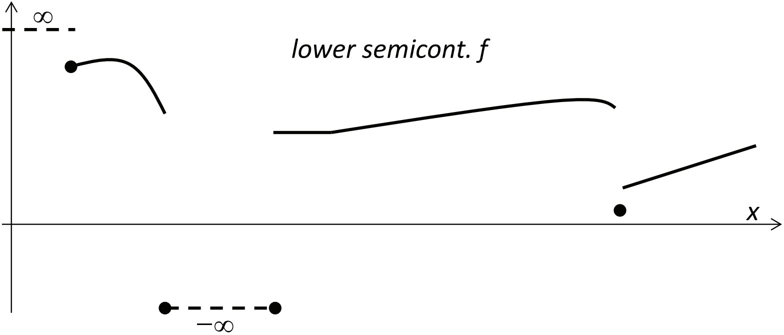
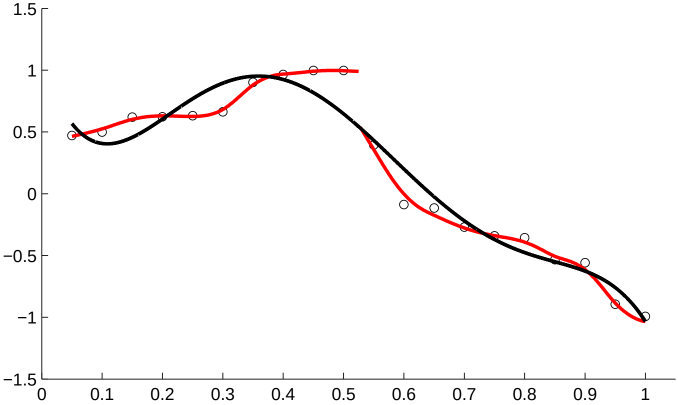


$(\text{lsc-fcns}(\boldsymbol{R}^n), dl)$: complete separable metric, dl = epi-distance

Modeling possibilities with lsc functions

- functions with jumps and high growth
- response surface building with implicit constraints
- system identification with subsequent minimization
- nonlinear transformations requiring $\pm\infty$
- functions with unbounded domain

Jumps



Fits by polynomial and lsc function

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Nonlinear transformations

Recall: probability density estimation with sample $X^1, ..., X^\nu$:

maximize $\frac{1}{\nu}\sum_{i=1}^{\nu} \log h(X^i)$ s.t. $h \in H^\nu \subset \mathcal{H}$ including $h \geq 0$

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Exponential transformation: $h(x) = \exp(-s(x))$

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minimize $\sum_{i=1}^{\nu} s(X^i)$ s.t. $s \in S^\nu \subset \mathcal{S}$ excluding nonnegativity

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But need s to take on ∞ for $h = \exp(-s(\cdot))$ to vanish

Identifying functions with unbounded domains

Complications for "standard" spaces

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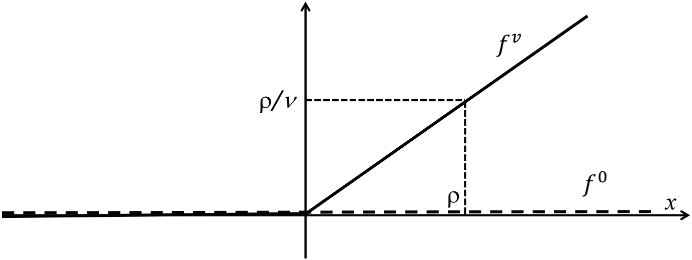
Is f^ν near f^0?



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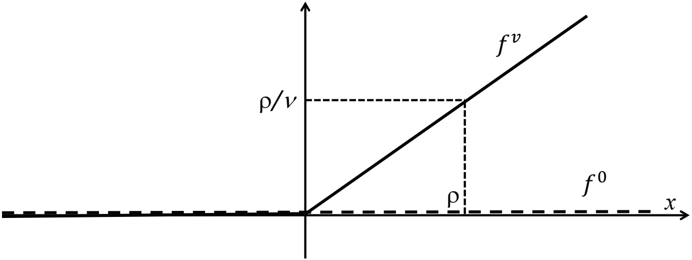

Not in $\mathcal{L}^p$-norm, but epi-distance $\leq 1/\nu$

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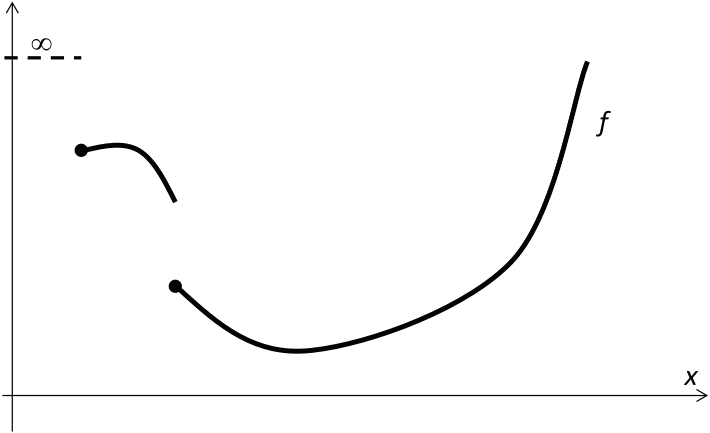
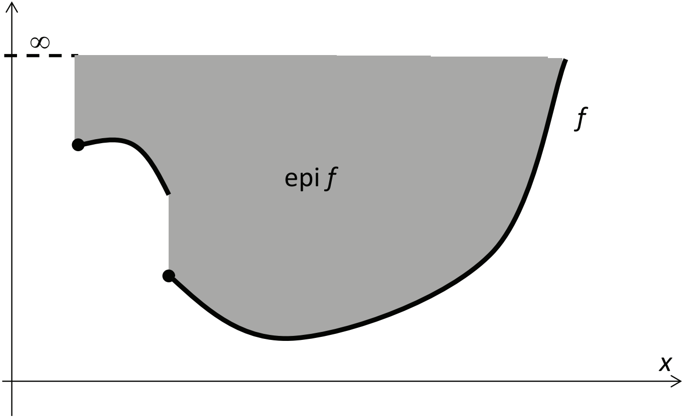
Epi-graph



Epi-graph



Geometric view of lsc functions

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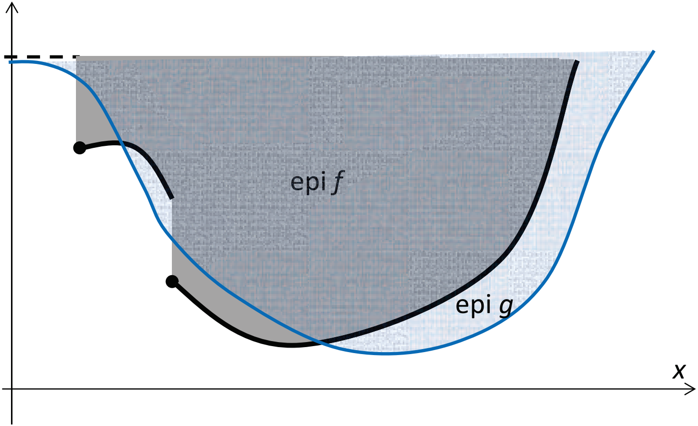
Notation:

- $\rho\boldsymbol{B} = \boldsymbol{B}(0,\rho) =$ origin-centered ball with radius ρ
- $d(y,S) = \inf_{y' \in S} \|y - y'\| =$ distance between y and S

Distances between epi-graphs



Distances between epi-graphs (cont.)

$$\rho\text{-}\textit{epi-distance} = dl_\rho(f,g) = \sup_{\bar{x}\in\rho\boldsymbol{B}} |d(\bar{x}, \mathrm{epi}\ f) - d(\bar{x}, \mathrm{epi}\ g)|, \quad \rho \geq 0$$

Distances between epi-graphs (cont.)

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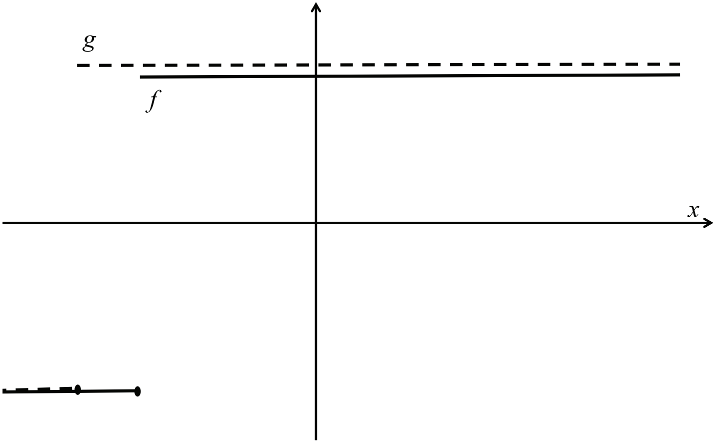

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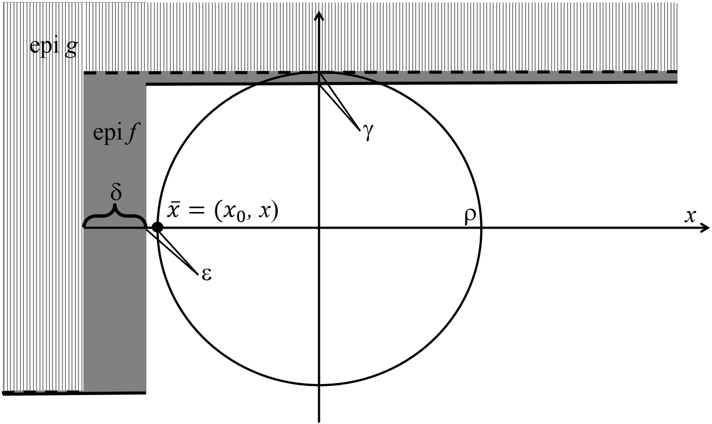
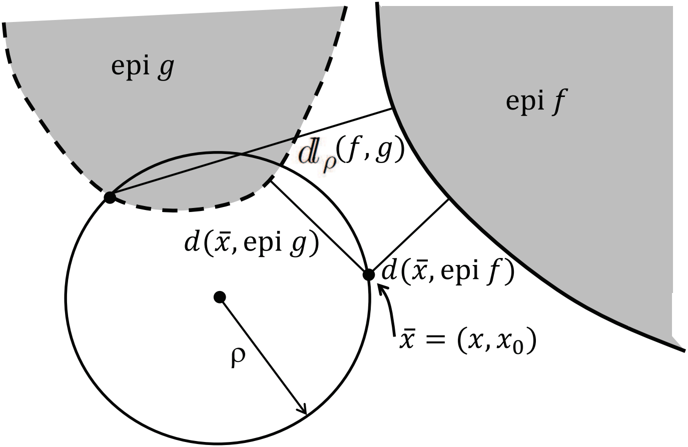


$$dl_\rho(f,g) = \delta$$

Distances between epi-graphs (cont.)

$$\textit{ρ-epi-distance} \ = dl_\rho(f,g) = \sup_{\bar{x}\in\rho\boldsymbol{B}} |d(\bar{x}, \mathrm{epi}\ f) - d(\bar{x}, \mathrm{epi}\ g)|, \quad \rho \geq 0$$



Metric on the lsc functions

ρ-epi-distance dl_ρ is a pseudo-metric on $\mathrm{lsc\text{-}fcns}(\boldsymbol{R}^n)$

$$\text{epi-distance} = dl(f,g) = \int_0^\infty dl_\rho(f,g)e^{-\rho}d\rho$$

dl is a metric on $\mathrm{lsc\text{-}fcns}(\boldsymbol{R}^n)$

The epi-distance induces the epi-topology on $\mathrm{lsc\text{-}fcns}(\boldsymbol{R}^n)$
$(\mathrm{lsc\text{-}fcns}(\boldsymbol{R}^n), dl)$ is a complete separable metric space (Polish)

Epi-convergence

$f^\nu \in \text{lsc-fcns}(\boldsymbol{R}^n)$ epi-converges to f if $dl(f^\nu, f) \to 0$

Actual problem

- Constrained infinite-dimensional optimization problem

$$\begin{aligned} &\min \psi(f) \\ &\text{such that } f \in F \subseteq \mathcal{F} \subseteq \mathrm{lsc\text{-}fcns}(\boldsymbol{R}^n) \end{aligned}$$

- criterion ψ (norms, error measures, etc)
- feasible set F (shape restrictions, external info)
- subspace of interest $\mathcal{F}$ (continuity, smoothness)

Take-aways in this section

- Actual problem:
  - find best extended real-valued lower semicontinuous function
  - captures regression, curve fitting, interpolation, density estimation etc:
  - allows functions on $\boldsymbol{R}^n$, jumps, transformation requiring $\pm\infty$
- Evolving problem due to changes in information and approximations
- Theory about lsc functions:
  - metric space under epi-distance
  - epi-distance $=$ distance between epi-graphs

Questions?

Outline

- Illustrations and formulations (slides 12-33)
- Framework (slides 34-60)
- Epi-splines and approximations (slides 61-77)
- Implementations (slides 78-85)
- Examples (slides 86-124)

Epi-splines and approximations

Epi-splines = piecewise polynomial + lower semicont.

Piecewise polynomial not automatic; must be constructed

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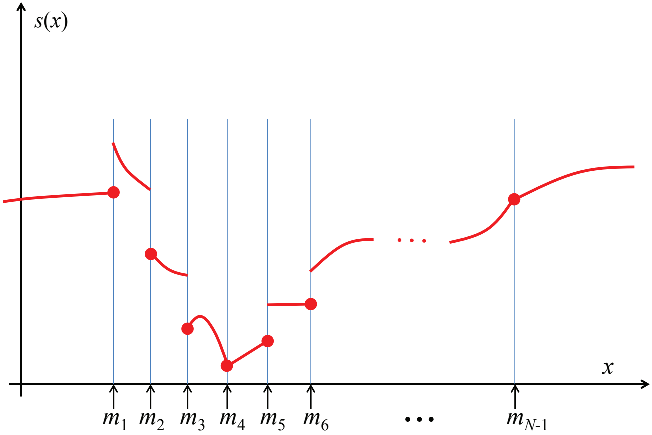

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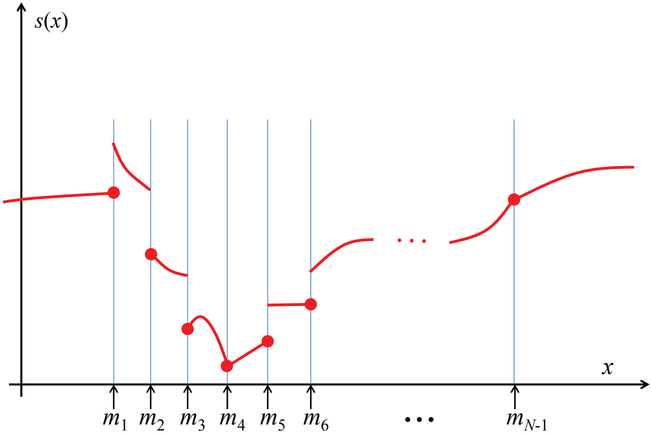
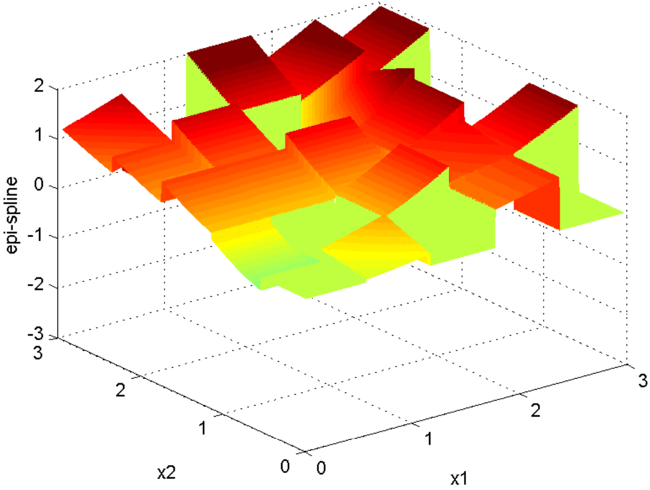
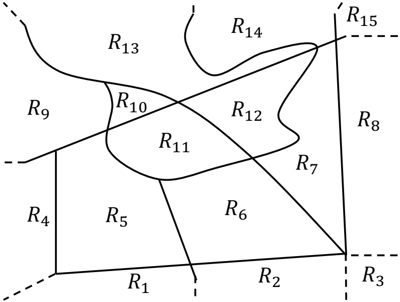


Optimization over new class of piecewise polynomial functions

n-dim epi-splines



n-dim epi-splines (cont.)

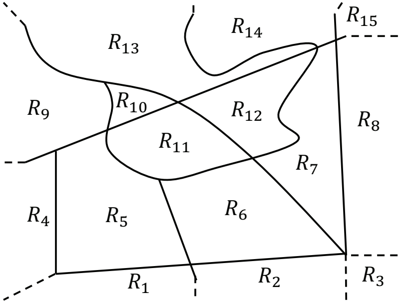


Epi-spline s of order p defined on $\boldsymbol{R}^n$ with partition $\mathcal{R} = \{R_k\}_{k=1}^{N}$ is a real-valued function that

on each R_k, $k = 1, ..., N$, is polynomial of total degree p and

for every $x \in \boldsymbol{R}^n$, has $s(x) = \liminf_{x' \to x} s(x')$

n-dim epi-splines (cont.)



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$\text{e-spl}_n^p(\mathcal{R})$ = epi-splines of order p on $\boldsymbol{R}^n$ with partition $\mathcal{R}$

Evolving and approximating problems

$$\min \psi^\nu(s)$$
$$\text{such that } s \in S^\nu = F^\nu \cap \mathcal{S}^\nu$$

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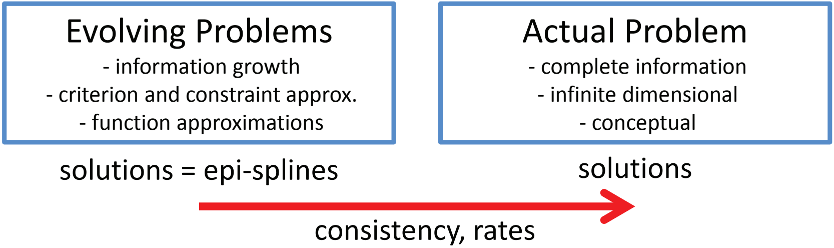
- evolving/approximate criterion ψ^ν

Function identification framework



Convergence of evolving problems

Epi-convergence of functionals on a metric space:

Convergence of evolving problems

Epi-convergence of functionals on a metric space:

Definition: $\{\psi^\nu : S^\nu \to \boldsymbol{R}\}_{\nu=1}^{\infty}$ epi-converge to $\psi : F \to \boldsymbol{R}$ if and only if

(i) for every $s^\nu \to f \in \mathrm{lsc\text{-}fcns}(\boldsymbol{R}^n)$, with $s^\nu \in S^\nu$, we have $\liminf \psi^\nu(s^\nu) \geq \psi(f)$ if $f \in F$ and $\psi^\nu(s^\nu) \to \infty$ otherwise;

(ii) for every $f \in F$, there exists $\{s^\nu\}_{\nu=1}^{\infty}$, with $s^\nu \in S^\nu$, such that $s^\nu \to f$ and $\limsup \psi^\nu(s^\nu) \leq \psi(f)$.

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Equivalent to previous definition

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Equivalent to previous definition

Also say that the evolving optimization problems epi-converge to the actual problem.

Consequence of epi-convergence

If evolving problems epi-converge to the actual problem, then

$$\limsup \left(\inf_{s \in S^\nu} \psi^\nu(s) \right) \leq \inf_{f \in F} \psi(f).$$

Consequence of epi-convergence

If evolving problems epi-converge to the actual problem, then

$$\limsup \left(\inf_{s \in S^\nu} \psi^\nu(s) \right) \leq \inf_{f \in F} \psi(f).$$

Moreover, if s^k are optimal for the evolving problems $\min_{s \in S^{\nu_k}} \psi^{\nu_k}(s)$ and $s^k \to f^0$, then f^0 is optimal for the actual problem and

$$\lim_{k \to \infty} \inf_{s \in S^{\nu_k}} \psi^{\nu_k}(s) = \inf_{f \in F} \psi(f) = \psi(f^0).$$

When will evolving problems epi-converge to actual?

$$\min \psi(f) \quad \text{such that } f \in F \subseteq \mathcal{F} \subseteq \text{lsc-fcns}(\boldsymbol{R}^n)$$

$$\min \psi^\nu(s) \quad \text{such that } s \in S^\nu = F^\nu \cap \mathcal{S}^\nu$$

Theorem: A sufficient condition for epi-convergence is

- ψ^ν converges continuously to ψ relative to $\mathcal{F}$
- Epi-splines dense in lsc functions
- F^ν set-converges to F
- F is solid

Epi-splines dense in lsc functions

$\{\mathcal{R}^\nu\}_{\nu=1}^\infty$, with $\mathcal{R}^\nu = \{R_1^\nu, ..., R_{N^\nu}^\nu\}$, is an infinite refinement if

for every $x \in \boldsymbol{R}^n$ and $\epsilon > 0$, there exists a positive integer $\bar{\nu}$ s.t. $R_k^\nu \subset \boldsymbol{B}(x, \epsilon)$ for every $\nu \geq \bar{\nu}$ and k satisfying $x \in \mathrm{cl}\, R_k^\nu$.

Epi-splines dense in lsc functions

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$R_k^\nu \subset \boldsymbol{B}(x, \epsilon)$ for every $\nu \geq \bar{\nu}$ and k satisfying $x \in \mathrm{cl}\, R_k^\nu$.

Theorem: For any $p = 0, 1, 2, ...,$ and infinite refinement $\{\mathcal{R}^\nu\}_{\nu=1}^\infty$,

$$\bigcup_{\nu=1}^{\infty} \text{e-spl}_n^p(\mathcal{R}^\nu) \text{ is dense in } \text{lsc-fcns}(\boldsymbol{R}^n)$$

Dense in continuous functions under simplex partitioning

Definition: A simplex S in $\boldsymbol{R}^n$ is the convex hull of $n+1$ points $x^0, x^1, ..., x^n \in \boldsymbol{R}^n$, with $x^1 - x^0$, $x^2 - x^0$, ..., $x^n - x^0$ linearly independent.

Dense in continuous functions under simplex partitioning

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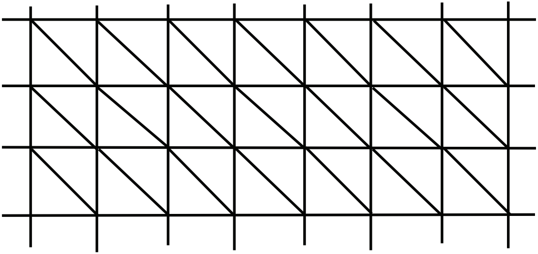
Definition: A partition $R_1, R_2, ..., R_N$ of $\boldsymbol{R}^n$ is a simplex partition of $\boldsymbol{R}^n$ if $\mathrm{cl}\, R_1$, ..., $\mathrm{cl}\, R_N$ are "mostly" simplexes.



Dense in cnts fcns under simplex partitioning (cont.)

Theorem: For any $p = 1, 2, ..,$ and $\{\mathcal{R}^\nu\}_{\nu=1}^{\infty}$, an infinite refinement of $\boldsymbol{R}^n$ consisting of simplex partitions of $\boldsymbol{R}^n$,

$$\bigcup_{\nu=1}^{\infty} \text{e-spl}_n^p(\mathcal{R}^\nu) \cap \mathcal{C}^0(\boldsymbol{R}^n) \text{ is dense in } \mathcal{C}^0(\boldsymbol{R}^n).$$

Rates in univariate probability density estimation

For convex and finite-dimensional estimation problem:

Theorem: For any "correct" soft information,

$$\nu^{1/2} d_{KL}(h^0 || h_\epsilon^\nu) = O_p(1) \text{ for some } h_\epsilon^\nu = \text{ near-optimal solution}$$

Decomposition

Theorem: For every $s \in \text{e-spl}^p_n(\mathcal{R})$, with $n > p \geq 1$ and $\mathcal{R} = \{R_k\}_{k=1}^N$, there exist $q_{k,i} \in \text{poly}^p(\boldsymbol{R}^{n-1})$, $i = 1, 2, ..., n$ and $k = 1, 2, ..., N$, such that

$$s(x) = \sum_{i=1}^{n} q_{k,i}(x_{-i}), \text{ for all } x \in R_k.$$

an n-dim epi-spline is the sum of n, $(n-1)$-dim epi-splines

Take-aways in this section

- Epi-splines are piecewise polynomials defined on arbitrary partition of $\boldsymbol{R}^n$
- Only lower-semicontinuity required (not smoothness)
- Dense in space of extended real-valued lower-semicontinuous functions under epi-distance (i.e., epi-splines can approximate every lsc function to an arbitrary accuracy)
- Solutions of evolving problem (in terms of epi-splines) are approximate solutions of actual problem

Questions?

Outline

Implementations

Implementation considerations

- Selection of epi-spline order and composition $h = \exp(-s)$
- Selection of partition: go fine!
- Implementation of criterion functional and soft info

Implementation considerations

- Selection of epi-spline order and composition $h = \exp(-s)$
- Selection of partition: go fine!
- Implementation of criterion functional and soft info

Provide details for one-dimensional epi-splines of order 1

Focusing on criteria and constraints for probability densities

Implementation of criteria functionals

Probability densities $h \in \mathrm{e\text{-}spl}^1(m)$, with mesh $m = \{m_0, m_1, ..., m_N\}$,

$-\infty < m_0 < m_1 < ... < m_N < \infty$

Implementation of criteria functionals

Probability densities $h \in \text{e-spl}^1(m)$, with mesh $m = \{m_0, m_1, ..., m_N\}$,

$-\infty < m_0 < m_1 < ... < m_N < \infty$

First-order epi-splines (piecewise linear):
$h(x) = a_0^k + a^k x$ for $x \in (m_{k-1}, m_k)$

Implementation of criteria functionals (cont.)

Log-likelihood for observations $x^1, ..., x^\nu$:

$$\log \prod_{i=1}^{\nu} h(x^i) = \sum_{i=1}^{\nu} \log(a_0^{k_i} + a^{k_i} x^i), \text{ where } k_i \text{ such that } x^i \in (m_{k_i - 1}, m_{k_i})$$

Implementation of criteria functionals (cont.)

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Entropy:

$$-\int h(x) \log h(x) dx = -\sum_{k=1}^{N} \int_{m_{k-1}}^{m_k} (a_0^k + a^k x) \log(a_0^k + a^k x)$$

Implementation of criteria functionals (cont.)

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Both concave in epi-parameters a_0^k and a^k, $k = 1, ..., N$

Soft information

Nonnegativity: $h \geq 0$ if $a_0^k + a^k m_{k-1} \geq 0$ and $a_0^k + a^k m_k \geq 0$, $k = 1, ..., N$

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Integrate to one:

$$\int h(x)dx = \sum_{k=1}^{N} a_0^k (m_k - m_{k-1}) + \sum_{k=1}^{N} \frac{a^k}{2} (m_k^2 - m_{k-1}^2) = 1$$

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Continuity: $a_0^k + a^k m_k = a_0^{k+1} + a^{k+1} m_k$ for $k = 1, ..., N-1$.

Soft information

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Continuity: $a_0^k + a^k m_k = a_0^{k+1} + a^{k+1} m_k$ for $k = 1, ..., N-1$.

Log-concavity, convexity, monotonicity

Soft information (cont.)

Deconvolution $Y = X + W$: Given densities h_W and h_Y,

$$h_Y(y) = \int h(x) h_W(y-x)dx = \sum_{k=1}^{N} a_0^k \int_{m_{k-1}}^{m_k} h_W(y-x)dx + \sum_{k=1}^{N} a^k \int_{m_{k-1}}^{m_k} x h_W(y-x)dx \text{ for all } y$$

Soft information (cont.)

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$$+ \sum_{k=1}^{N} a^k \int_{m_{k-1}}^{m_k} x h_W(y-x)dx \text{ for all } y$$

Inverse problem $Y = g(X)$: Given $v_q = \int y^q h_Y(y)dy$, $q = 1, 2, \ldots$

$$v_q = \int [g(x)]^q h(x)dx \approx \sum_{k=1}^{N} \sum_{j=1}^{M^k} w^{jk} [g(x^{jk})]^q (a_0^k + a^k x^{jk})$$

Soft information (cont.)

Deconvolution $Y = X + W$: Given densities h_W and h_Y,

$$h_Y(y) = \int h(x) h_W(y-x)dx = \sum_{k=1}^{N} a_0^k \int_{m_{k-1}}^{m_k} h_W(y-x)dx + \sum_{k=1}^{N} a^k \int_{m_{k-1}}^{m_k} x h_W(y-x)dx \text{ for all } y$$

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Convex optimization problems

Software, references

- Papers and tutorials: http://faculty.nps.edu/joroyset
- Software for univariate probability density estimation
  - Matlab toolbox http://faculty.nps.edu/joroyset/XSPL.html
  - R toolbox (S. Buttrey) http://faculty.nps.edu/sebuttre/home/Software/expepi/index.html

Outline

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Examples: forecasting and fitting

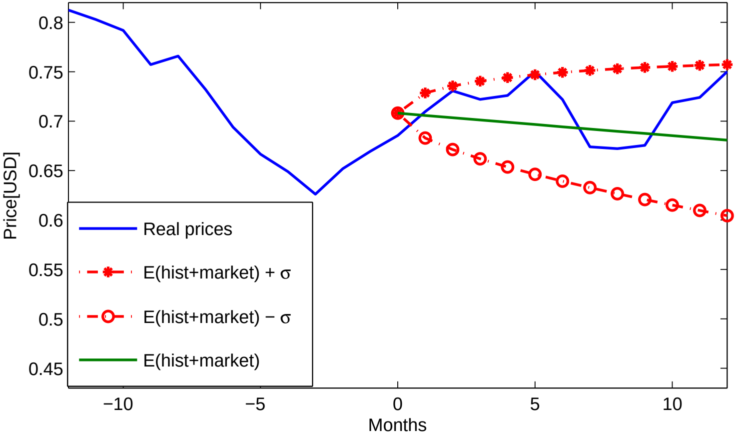
Copper price forecast

Stochastic differential equation: estimate drift, volatility

Steps:

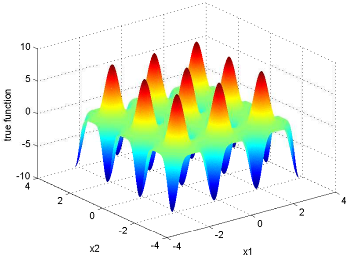
- identify discount factor curve using epi-splines and futures market information $\Longrightarrow$ future spot prices
- identify drift using epi-splines, historical data, future spot prices, etc
- identify volatility using epi-splines and observed/estimated errors

Copper price forecast (cont.)



Surface reconstruction

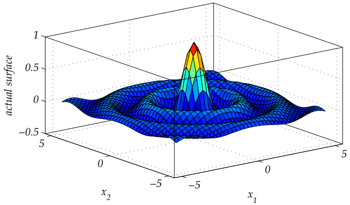
$f(x) = (\cos(\pi x_1) + \cos(\pi x_2))^3$ on $[-3,3]^2$
continuity; second-order epi-splines; $N = 400$; 900 uniform points

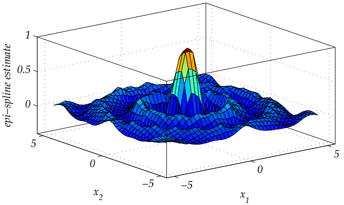


Actual function and epi-spline approximation

Surface reconstruction

$f(x) = \sin(\pi\|x\|)/(\pi\|x\|)$ for $x \in [-5,5]^2 \setminus \{0\}$, $f(0) = 1$
cont. diff.; second-order epi-splines; $N = 225$; 600 random points





Actual function and epi-spline approximation

Examples: density estimation

Examples of probability density estimation

Find a density h that maximizes log-likelihood of observations and satisfies soft information

Examples: earthquake losses, queuing, robustness, deconvolution, UQ, mixture, bivariate densities

Earthquake losses

Earthquake losses, Vancouver Region*

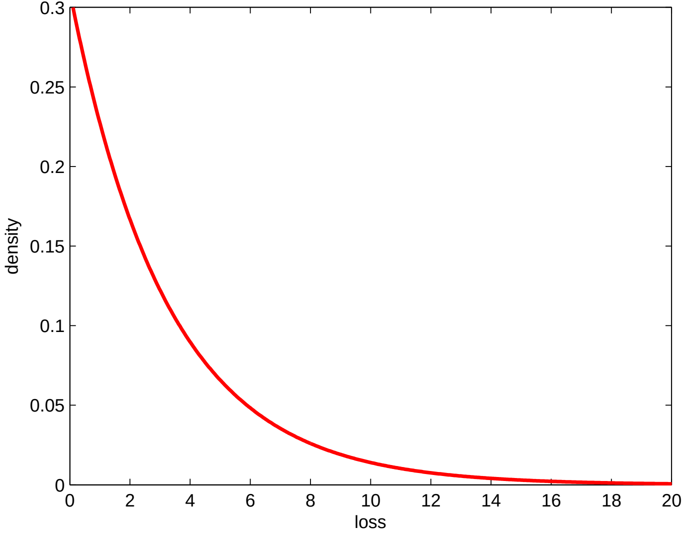
Comprehensive damage model (284 random variables)

100,000 simulations of 50-year loss

*Data from Mahsuli, 2012; see also Mahsuli & Haukaas, 2013

Earthquake losses (cont.)

Probability density of 50-year loss (billion CAD)



Use fewer simulations?

Using 100 simulations only and kernel estimator

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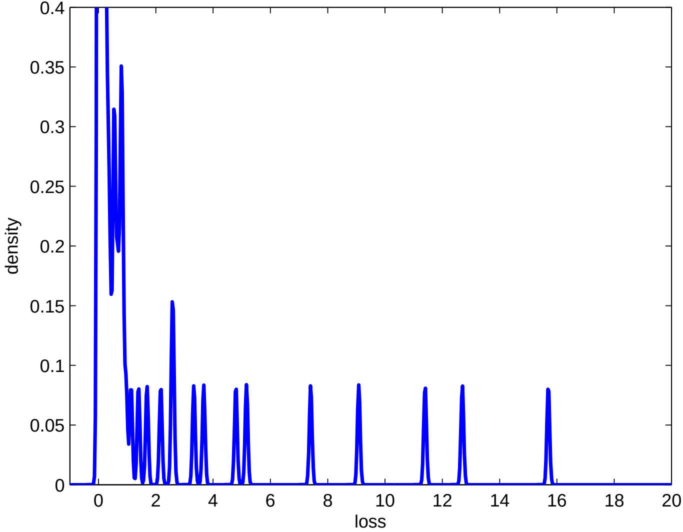

Using function identification approach

Max likelihood of 100 simulations; second-order exp. epi-splines

Eng. knowledge: nonincreasing, smooth, nonnegative support

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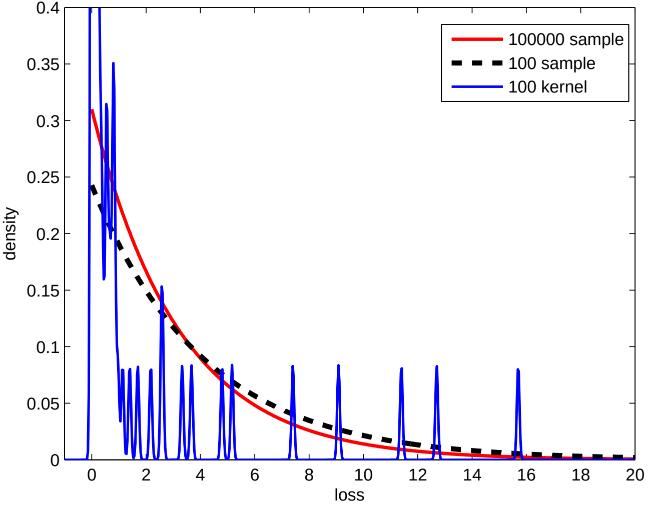
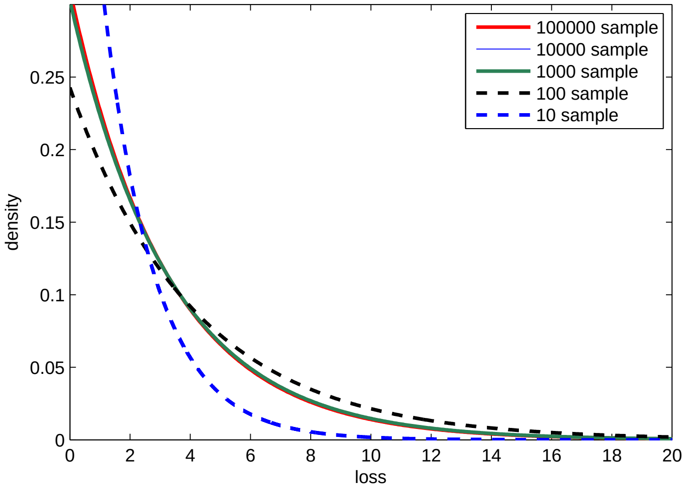


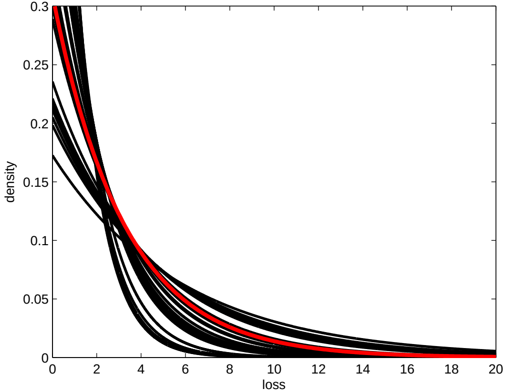
Using function identification approach (cont.)

Varying number of simulations; same soft information



Using function identification approach (cont.)

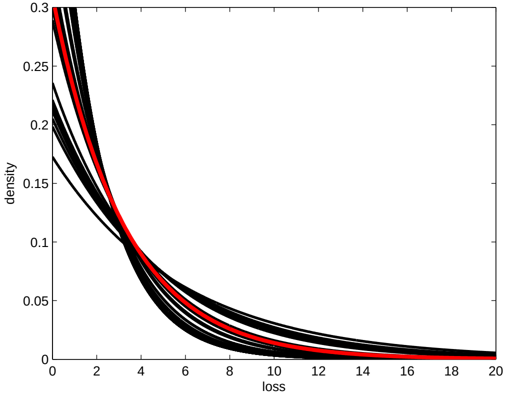
30 meta-replications of 100 simulations



	100,000 sim.	100 sim.
mean	3.2	3.1($\pm$1.3)
average 10% highest	10.6	10.3($\pm$4.4)

Using function identification approach (cont.)

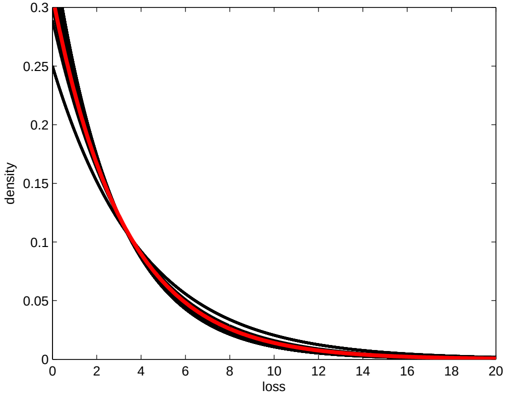
Also pointwise Fisher info.: $h'(y)/h(y) \in [-0.5, -0.1]$



	100,000 sim.	100 sim.
mean	3.2	3.2($\pm$1.2)
average 10% highest	10.6	10.6($\pm$4.0)

Using function identification approach (cont.)

Also pointwise Fisher info.: $h'(x)/h(x) \in [-0.35, -0.25]$



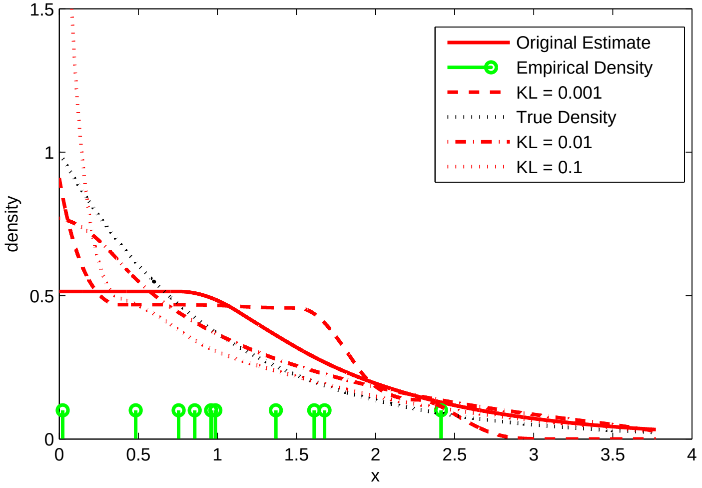
	100,000 sim.	100 sim.
mean	3.2	$3.3(\pm 0.5)$
average 10% highest	10.6	$10.9(\pm 1.7)$

Building robustness

Diversity of estimates using KL-divergence

Returning to exponential density

Continuously diff., nonincreasing, nonnegative support

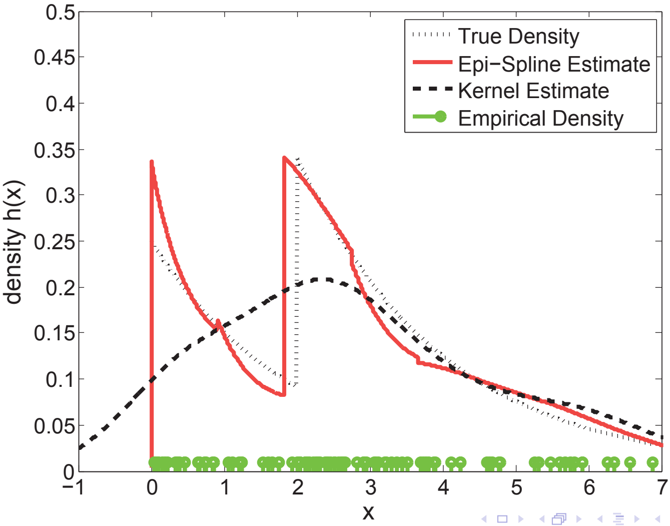


Queuing

M/M/1; 50% of customers delayed for fixed time

X = customer time-in-service; 100 obs.; exp. epi-spline

Soft info: lsc, $X \geq 0$, pointwise Fisher, unimodal upper tail



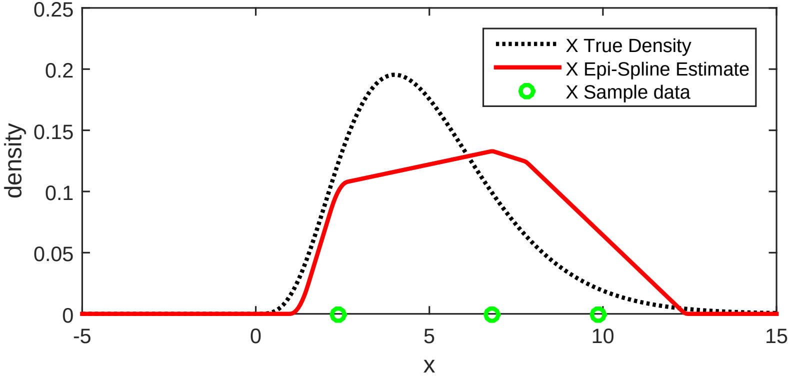
Deconvolution

True density Gamma(5,1)

First-order epi-spline on $[0, 23]$, $N = 1000$

Three sample points

Contin., unimodal, convex tails, bounds on gradient jumps



Also noisy observations

5000 observations of $Y = X + W$

W independent normal noise; mean 0, stdev 3.2

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$|h_Y(y) - \int h(x) h_W(y-x) dx| \leq 0.005$ for 101 y points

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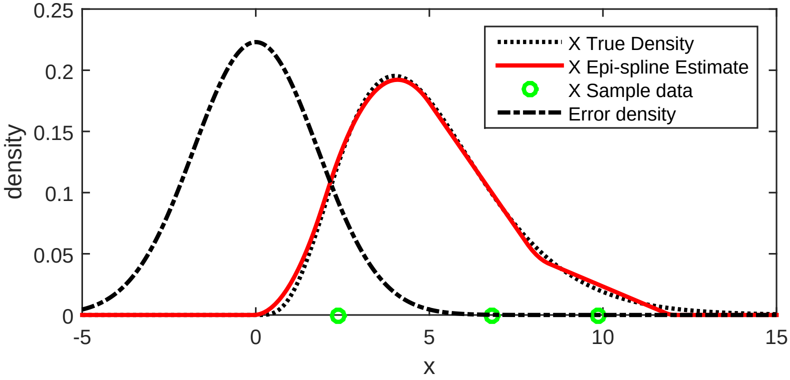
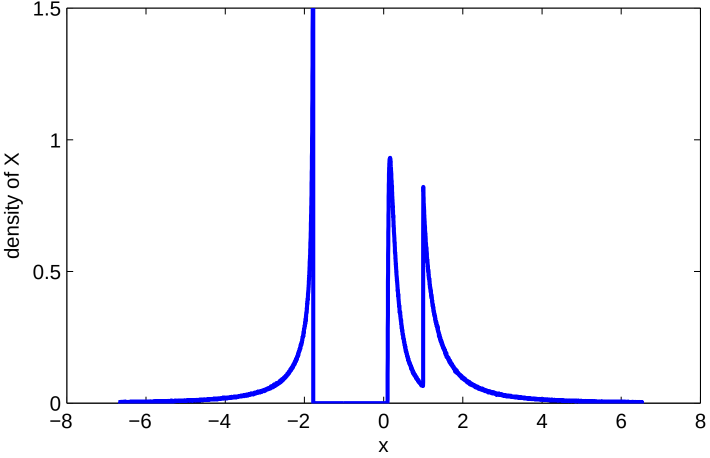


Uncertainty quantification

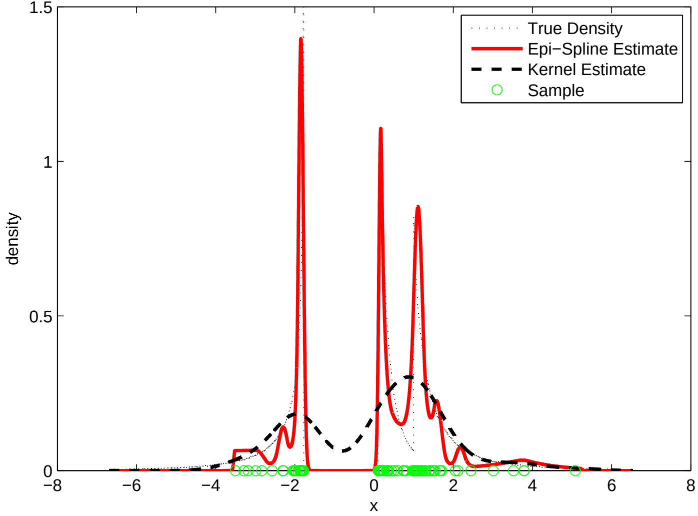
Dynamical system

Recall the true density of the amplitude



Density of amplitude

Sample size 100; cont. diff.; unimodal tails, exp. epi-splines



Gradient information

Gradient information for bijective $g : \boldsymbol{R} \to \boldsymbol{R}$
Recall: If $X = g(V)$, then

$$h(x) = h_V(g^{-1}(x))/|g'(g^{-1}(x))|$$

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Recall: If $X = g(V)$, then

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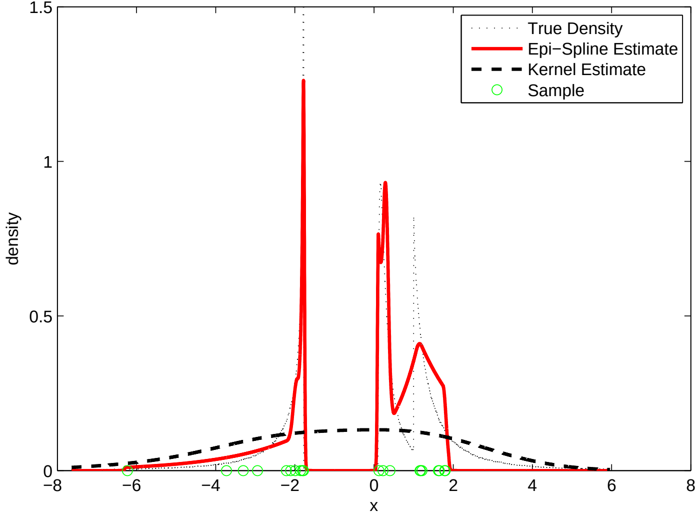
Present context *without* a bijection and data $x^i = g(v^i), g'(v^i)$:

$$h(x^i) \geq \frac{h_V(v^i)}{|g'(v^i)|}$$

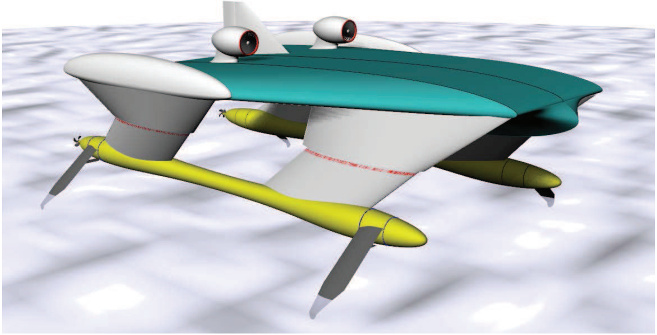
Value of pdf bounded from below at x^i

Gradient information (cont.)

Sample size 20



Fluid dynamics



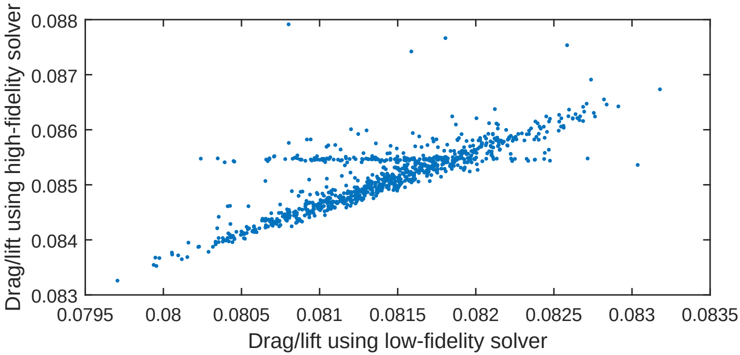
Drag/lift estimates:

High-fidelity RANSE solves (each 4 hours on 8 cores)

Low-fidelity potential flow solves (each 5 sec on 1 core)

Predicting high-fidelity performance

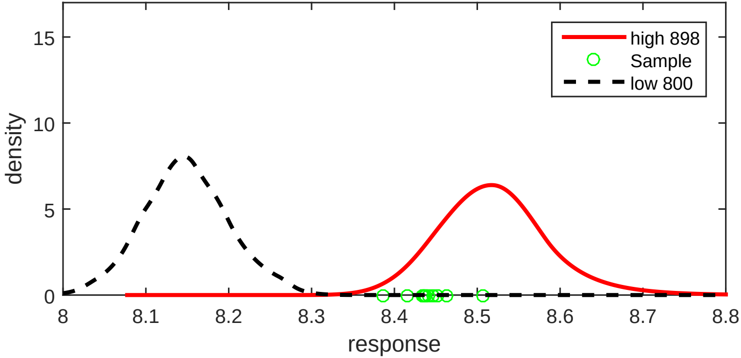
898 high- and low-fidelity solves



Learn from low-fidelity solves and avoid (many) high-fidelity solves

Density using high or low solves

Exponential epi-splines of second order; mesh $N = 50$
Soft info: log-concavity and bounds on second-order derivatives



Estimating conditional error

X =high-fidelity; Y =low-fidelity

$h_X(x) = \int h_{X|Y}(x|y)h_Y(y)dy$

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Normal linear least-squares regression model on training data of size 50 $\rightarrow$ $h_{X|Y}(x|y)$

Estimating conditional error

X =high-fidelity; Y =low-fidelity

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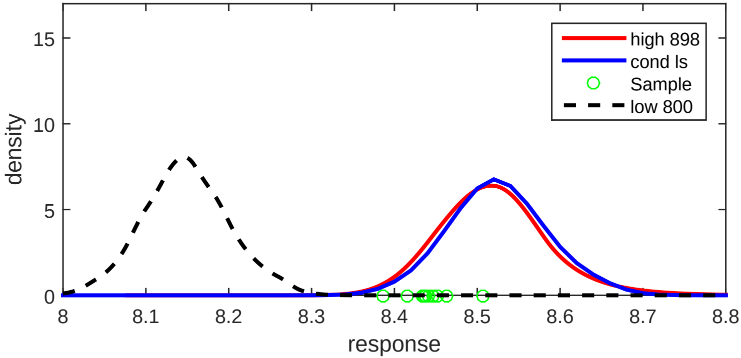
Normal linear least-squares regression model on training data of size 50 $\rightarrow \quad h_{X|Y}(x|y)$



Information fusion

Max likelihood using 10 high-fidelity simulations

$0.5 h_X(x) \le \int h_{X|Y}(x|y) h_Y(y) dy \le 1.5 h_X(x)$

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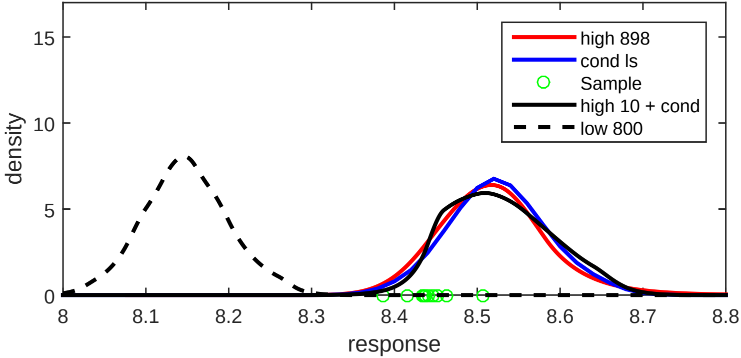
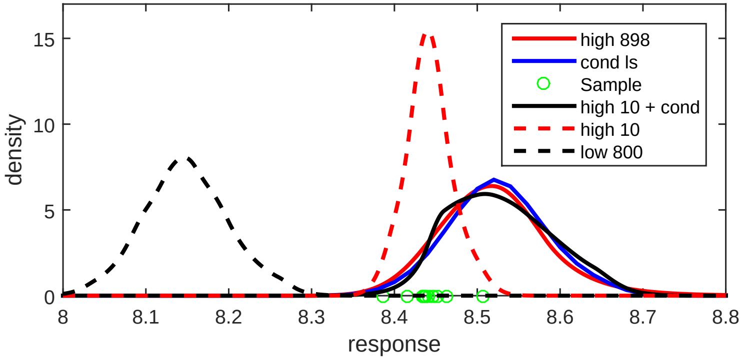


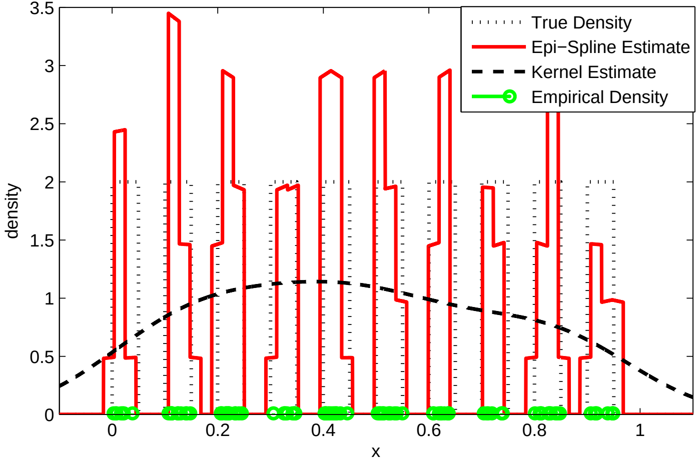
Only 10 high-fidelity



Uniform mixture density

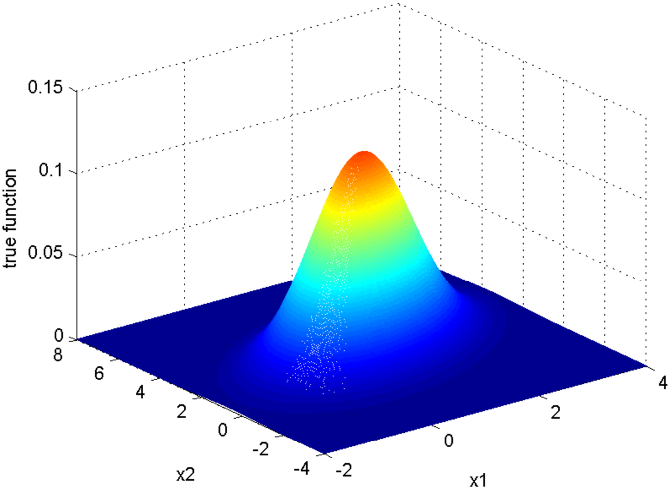
Uniform mixture density

Sample size 100; lsc, slope constraints; exp. epi-splines

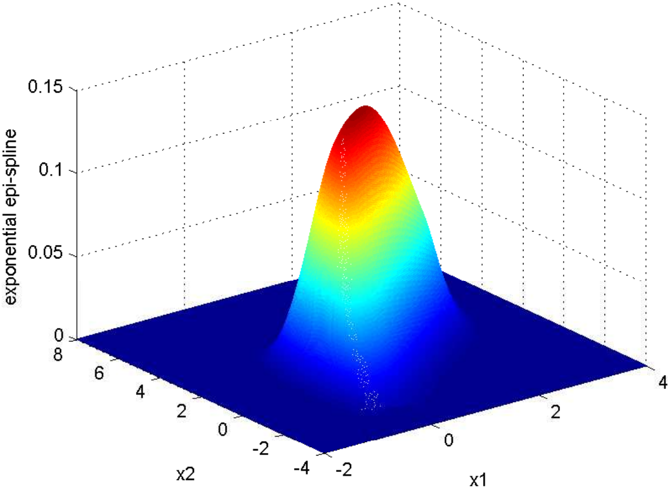


Bivariate normal density

Bivariate normal probability density



Bivariate normal probability density



Curvature, log-concave, 25 sample points, exp. epi-spline

Conclusions

- Function identification problems: rich class

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Conclusions

- Function identification problems: rich class
- lsc functions provide modeling flexibility
- Epi-convergence allows evolution of info./approx.
- Epi-splines central approximation tools

References

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