Asset valuation via stochastic optimization

Teemu Pennanen
Department of Mathematics
King's College London

- Optimal investment and asset pricing are often treated as separate problems (Markovitz vs. Black–Scholes).
- In practice, valuations have been largely disconnected from investment and risk management. This lead to large losses during 2008 e.g. with credit derivatives.
- Building on convex stochastic optimization, we describe a unified approach to optimal investment, valuation and risk management.
- The resulting valuations
  - are based on hedging costs,
  - extend and unify financial and actuarial valuations,
  - reduce to “risk neutral valuations” for perfectly liquid securities.

- Pennanen, Optimal investment and contingent claim valuation in illiquid markets, Finance and Stochastics, 2014.
- Armstrong, Pennanen, Rakwongwan, Pricing and hedging of S&P500 options under illiquidity, manuscript.
- King, Koivu, Pennanen, Calibrated option bounds, Int. J. Theor. Appl. Finance, 2005.
- Nogueiras, Pennanen, Pricing and hedging EONIA swaps under illiquidity and credit risk, manuscript.
- Hilli, Koivu, Pennanen, Cash-flow based valuation of pension liabilities. European Actuarial Journal, 2011.
- Bonatto, Pennanen, Optimal hedging and valuation of oil refineries and supply contracts, manuscript.
- Pennanen, Perkkiö, Convex duality in optimal investment and contingent claim valuation in illiquid markets, manuscript.

Asset-Liability Management

Let $\mathcal{M}$ be the linear space of adapted sequences of cash-flows on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$.

- The financial market is described by a convex set $\mathcal{C} \subset \mathcal{M}$ of claims that can be superhedged without cost (i.e. each $c \in \mathcal{C}$ is freely available in the financial market).
- In models with a perfectly liquid cash-account,

$$\mathcal{C} = \{c \in \mathcal{M} \mid \sum_{t=0}^{T} c_t \in C\}$$

  where $C \subset L^0(\Omega, \mathcal{F}_T, P)$ are the claims at T that can be hedged without cost [Delbaen and Schachermayer, 2006].
- Conical $\mathcal{C}$: [Dermody and Rockafellar, 1991], [Jaschke and Küchler, 2001], [Jouini and Napp, 2001], [Madan, 2014].

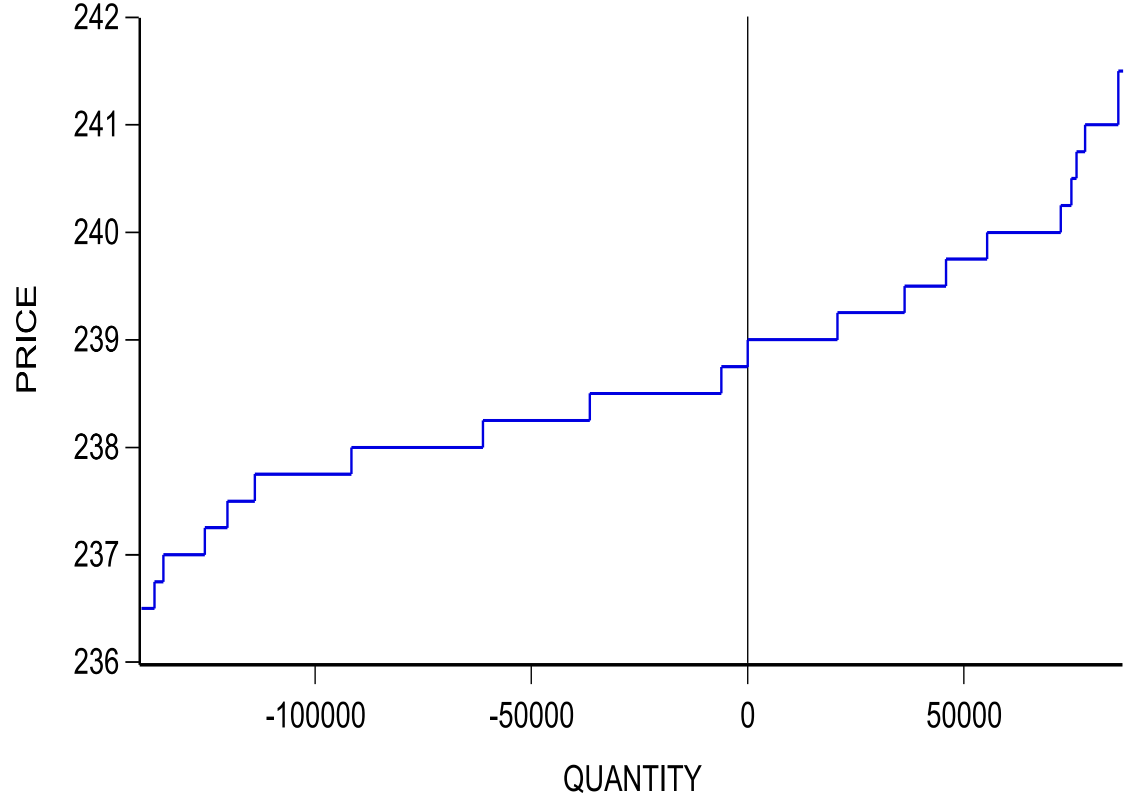
Example 1 (The classical model) *In the classical perfectly liquid market model with a cash-account*

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N} : \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\}$$

which is a convex cone. This set has been extensively studied in the literature; see e.g. [Föllmer and Schied, 2004] or [Delbaen and Schachermayer, 2006] and their references.

Asset-Liability Management

The limit order book of TDC A/S in Copenhagen Stock Exchange on January 12, 2005 at 13:58:19.43.



Asset-Liability Management

- Consider an agent with liabilities $c \in \mathcal{M}$, access to $\mathcal{C}$ and a loss function $\mathcal{V} : \mathcal{M} \to \overline{\mathbb{R}}$ that measures disutility/regret/risk/…of delivering $c \in \mathcal{M}$. For example,

$$\mathcal{V}(c) = E \sum_{t=0}^{T} -u_t(-c_t).$$

- The agent's ALM problem can be written as

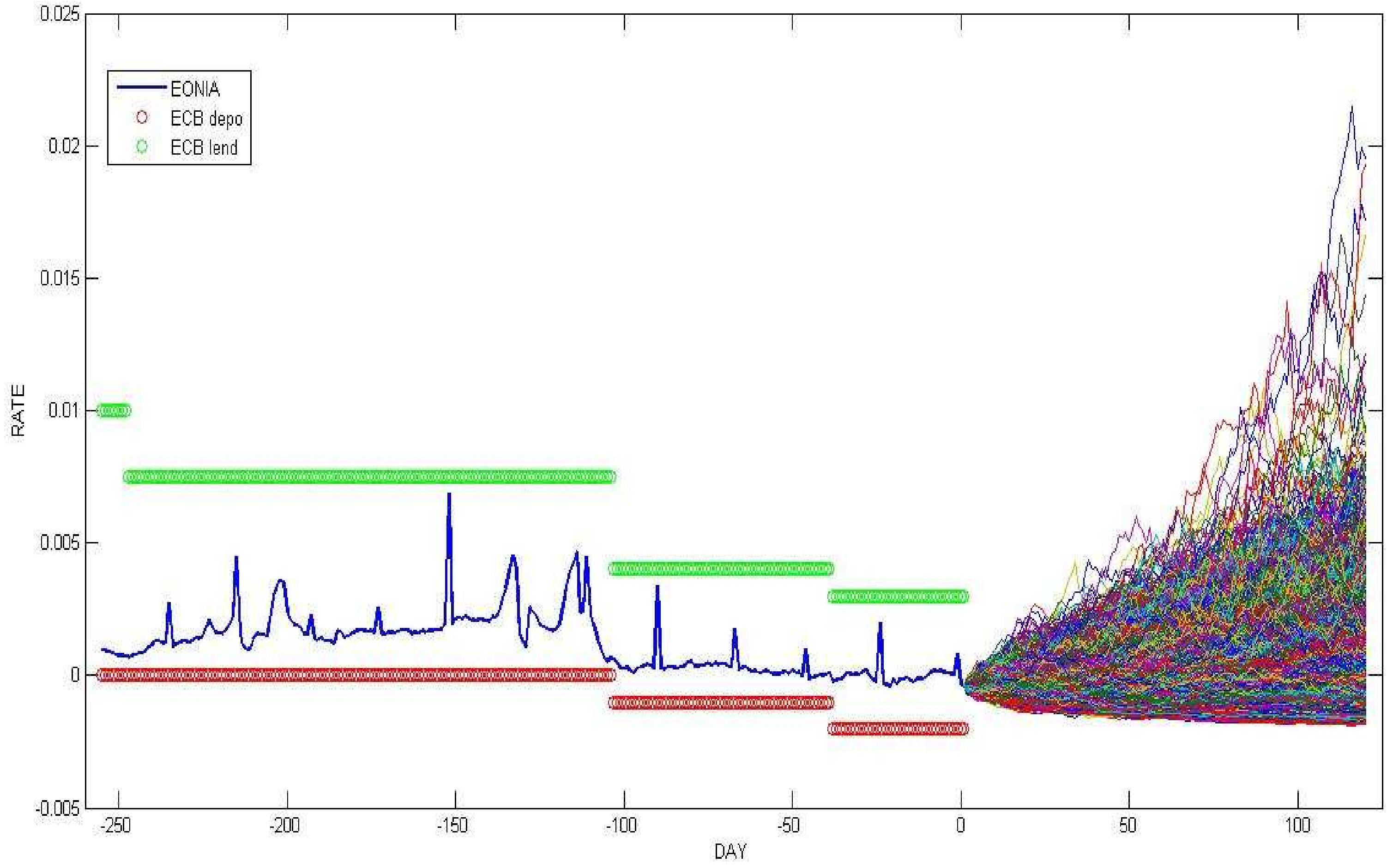
$$\varphi(c) = \inf_{d \in \mathcal{C}} \mathcal{V}(c - d)$$

- We assume that $\mathcal{V}$ is convex and nondecreasing with $\mathcal{V}(0) = 0$.

Example: EONIA swaps

- EONIA (Euro Over Night Index Average) is the average overnight interest rate on agreed interbank lending.
- We study indifference swap rates of EONIA swaps (Overnight Index Swaps).
- The hedging instruments consist of EONIA and other EONIA swaps.

Example: EONIA swaps



Example: EONIA swaps

Table 1: Swap data:

OIS Maturity	OIS Rate
1W	$-0.2730E-3$
2W	$-0.0500E-3$
3W	$-0.0300E-3$
1M	$-0.0100E-3$
2M	$-0.0700E-3$
3M	$-0.1400E-3$
6M	$-0.1300E-3$

Example: EONIA swaps

We have

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_0, z \in \mathbb{R}^K\colon\ x_t + c_t \leq (1+r_t)x_{t-1} + \sum_{k\in K} z_k c_t^k\}$$

where

- x_t amount of overnight deposits,
- r_t EONIA rate,
- c_t agent's cash-flows to be hedged,
- $c_{k,t}$ net cash-flows of the kth swap,
- z_k position in the kth swap (to be optimized).

Example: EONIA swaps

- We describe risk preferences by

$$\mathcal{V}(c) = \begin{cases} E\exp[\gamma c_T] & \text{if } c_t \leq 0 \text{ for } t < T, \\ +\infty & \text{otherwise.} \end{cases}$$

  where $\gamma > 0$ describes the risk aversion of the agent.

- The ALM-problem can then be written as

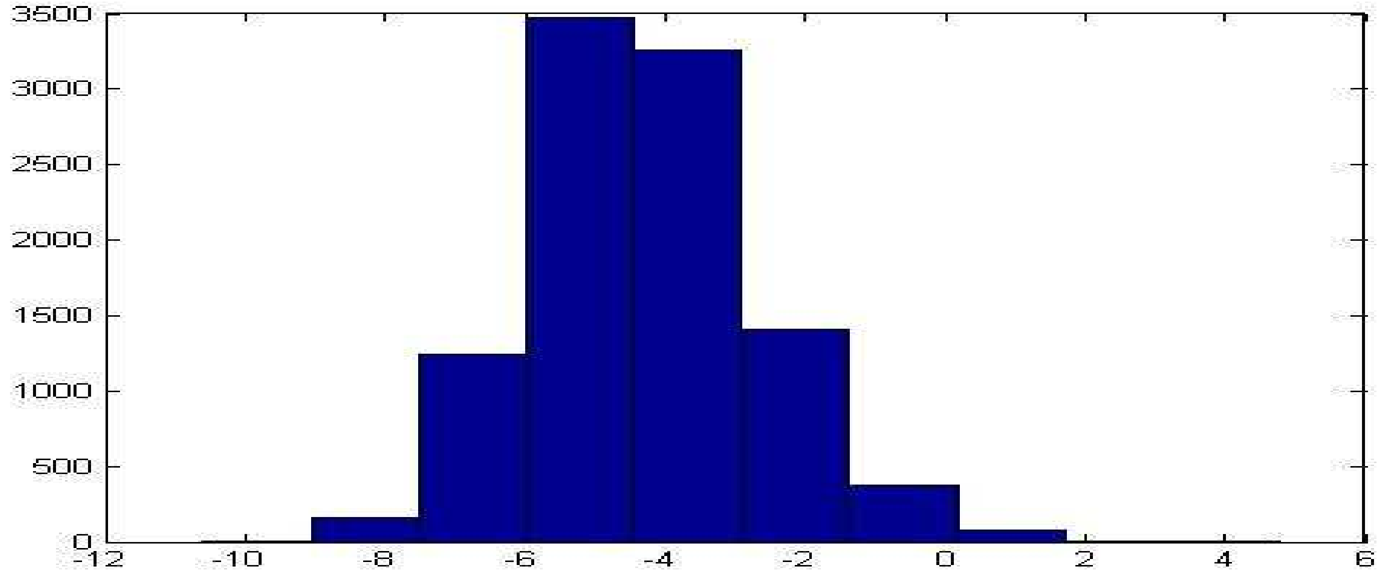
$$\text{minimize} \quad E\exp(-\gamma x_T) \quad \text{over} \quad z \in \mathbb{R}^K,$$

  where x_T is given by the recursion

$$x_t = (1 + r_{t-1})x_{t-1} + \sum_{k \in K} z_k c_{k,t} - c_t.$$

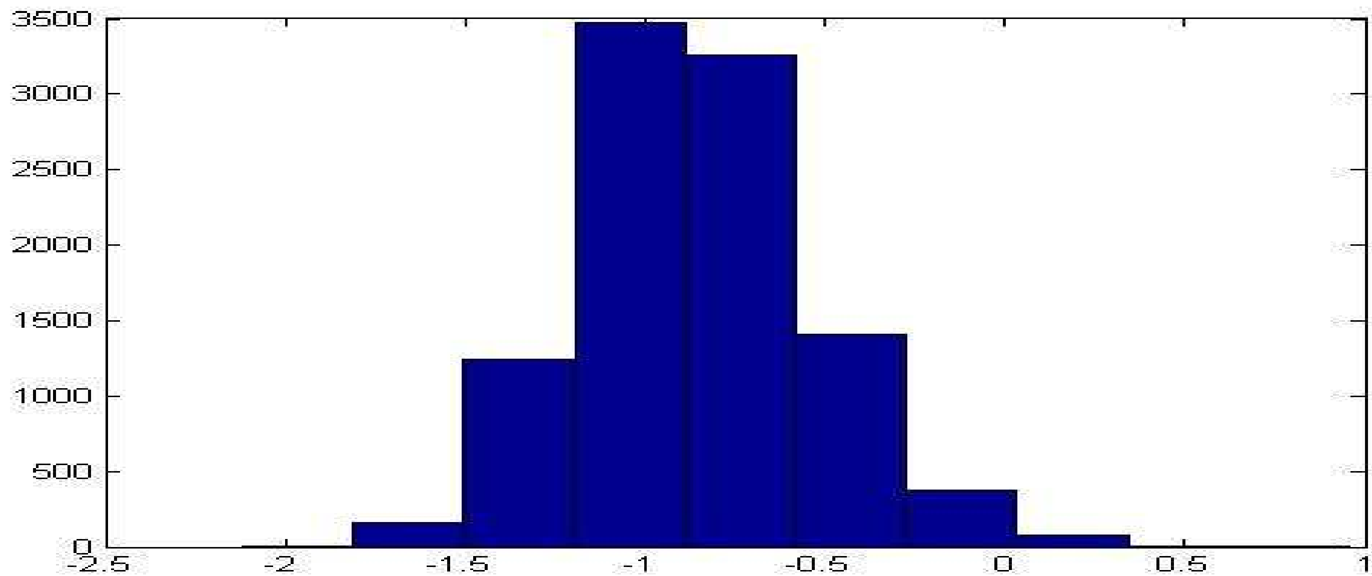
Example: EONIA swaps

Optimal terminal wealth distribution with $\gamma = 1$



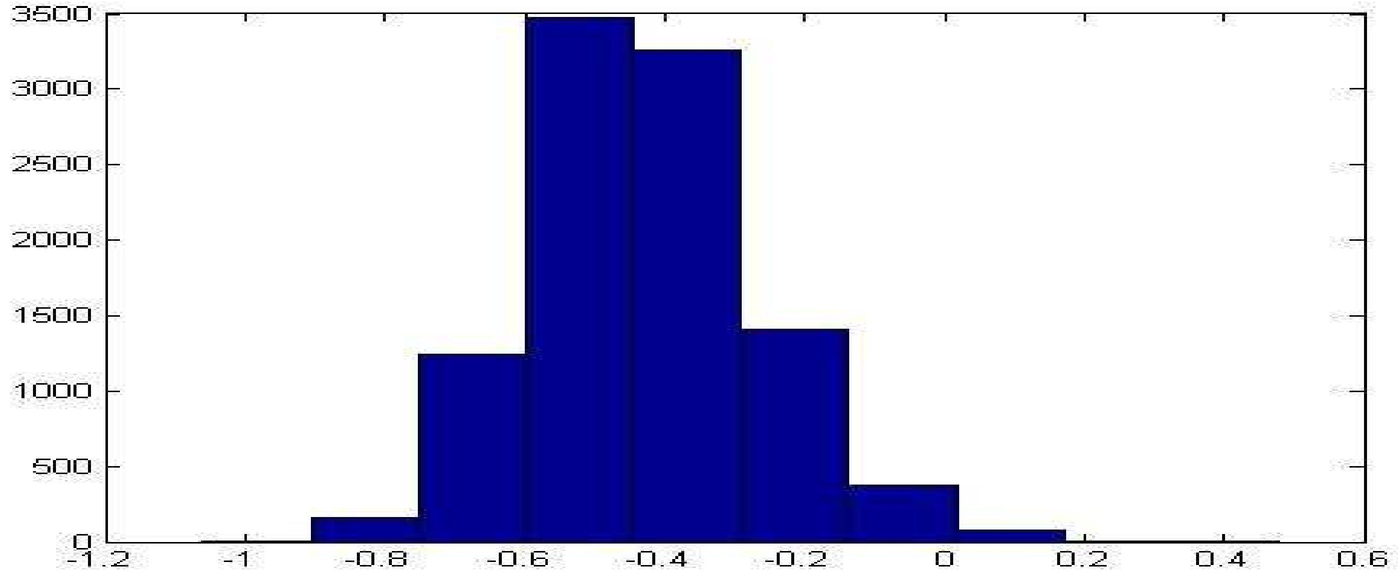
Example: EONIA swaps

Optimal terminal wealth distribution with $\gamma = 5$



Example: EONIA swaps

Optimal terminal wealth distribution with $\gamma = 10$



Example: S&P500 options

- In the second example, we study indifference prices of S&P500 index options.
- The hedging instruments are cash, S&P500 index and puts and calls all with the same maturity.
- We only consider static hedging but do account for illiquidity by trading at observed bid/ask prices.

Example: S&P500 options

- Static trading corresponds to the one period model

$$\mathcal{C} = \{(c_0, c_1) \,|\, \exists x \in \mathbb{R}^J : \sum_{j \in J} S_0^j(x^j) + c_0 \leq 0, \sum_{j \in J} S_1^j(-x^j) + c_1 \leq 0\},$$

  where $S_t^j(x^j)$ denotes the cost of buying x^j units of asset j at time $t = 0, 1$.
- We have $S_0^j(x^j) = \max\{\underline{s}_0^j x, \bar{s}_0^j x\}$, where $\underline{s}_0^j$ and $\bar{s}_0^j$ are the observed bid and ask prices, respectively.
- We assume perfect liquidity at $t = 1$ so $S_1^j(x^j) = s_1^j x^j$, where

$$s_1^j = \begin{cases} 1 & \text{if } j \text{ is cash,} \\ P_1 & \text{if } j \text{ is the index,} \\ [P_1 - K^j]^+ & \text{if } j \text{ is a call with strike } K^j, \\ [K^j - P_1]^+ & \text{if } j \text{ is a put with strike } K^j. \end{cases}$$

Example: S&P500 options

- We describe risk preferences by

$$\mathcal{V}(c) = \begin{cases} E\exp[\gamma c_1] & \text{if } c_0 \le 0, \\ +\infty & \text{otherwise.} \end{cases}$$

  where $\gamma > 0$ describes the risk aversion of the agent.

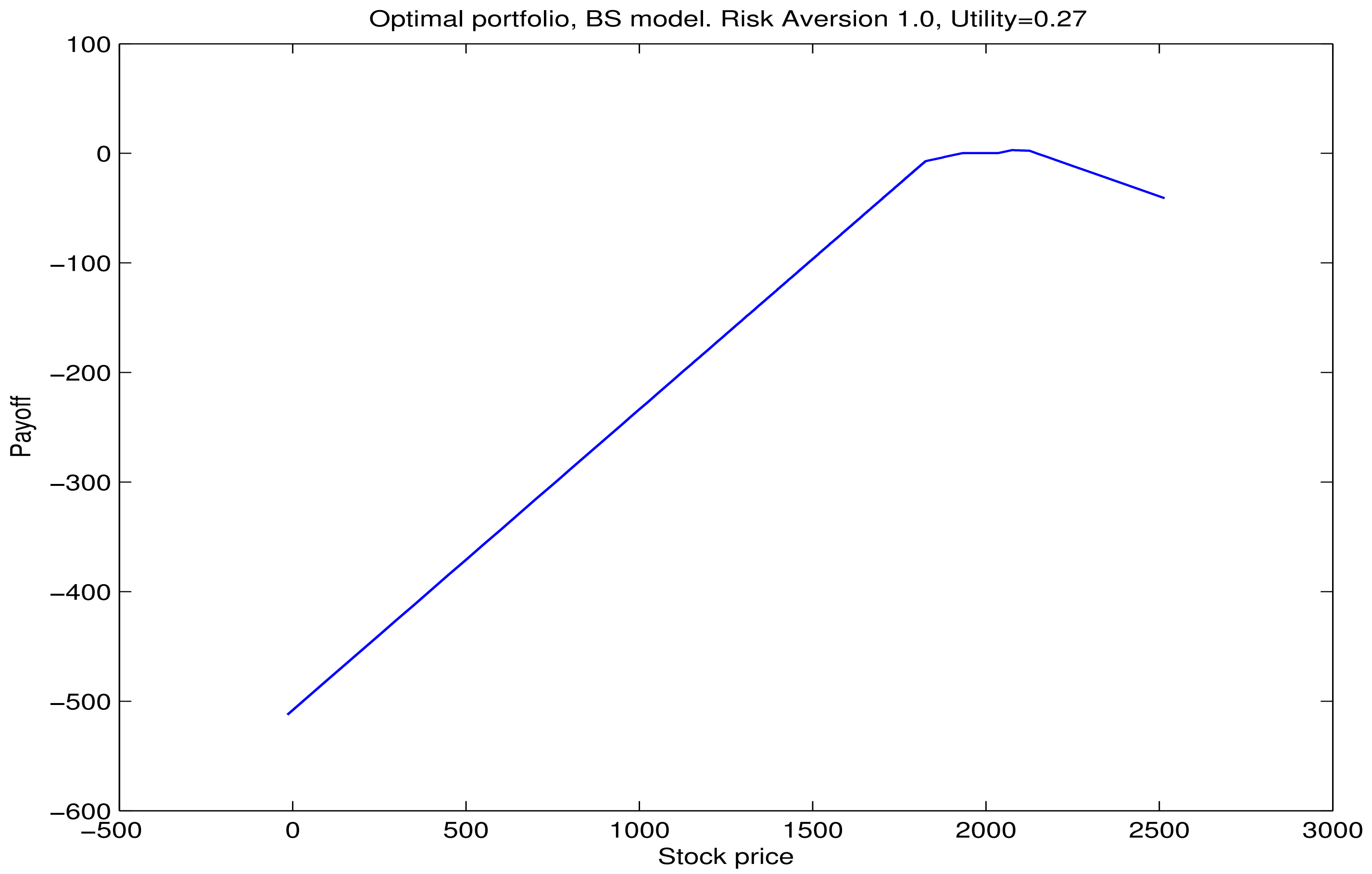
- The ALM-problem can then be written as

$$\begin{array}{lll} \text{minimize} & E\exp\left[\gamma(c_1 - s_1 \cdot x)\right] \quad \text{over} \quad x \in [-q_b, q_a] \\ \text{subject to} & \displaystyle\sum_{j\in J} S_0^j(x^j) + c_0 \le 0, \end{array}$$

  where $q_b, q_a \in \mathbb{R}^J$ are the quantities available at the best bid and best ask prices.

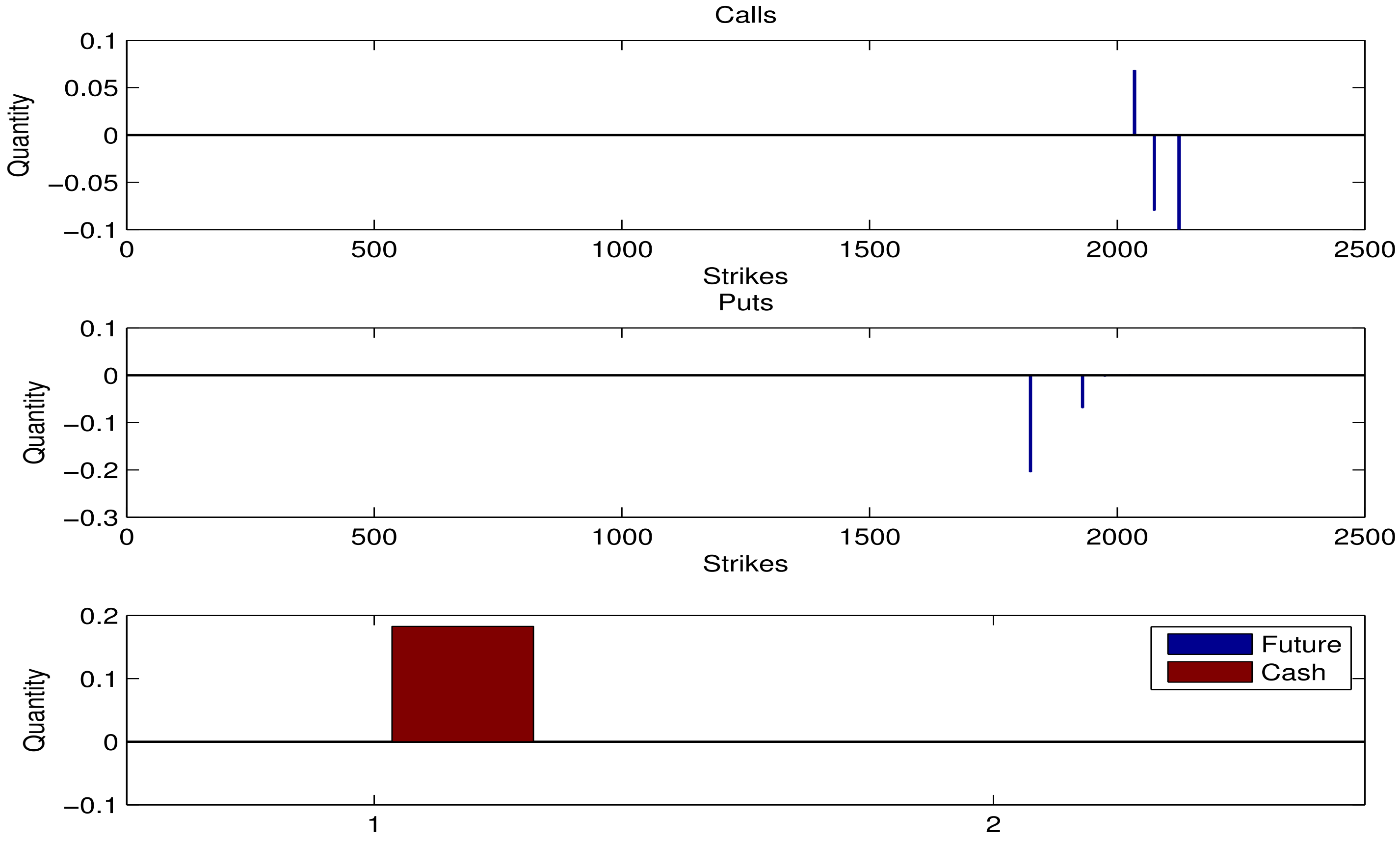
Example: S&P500 options

Optimal payout profile



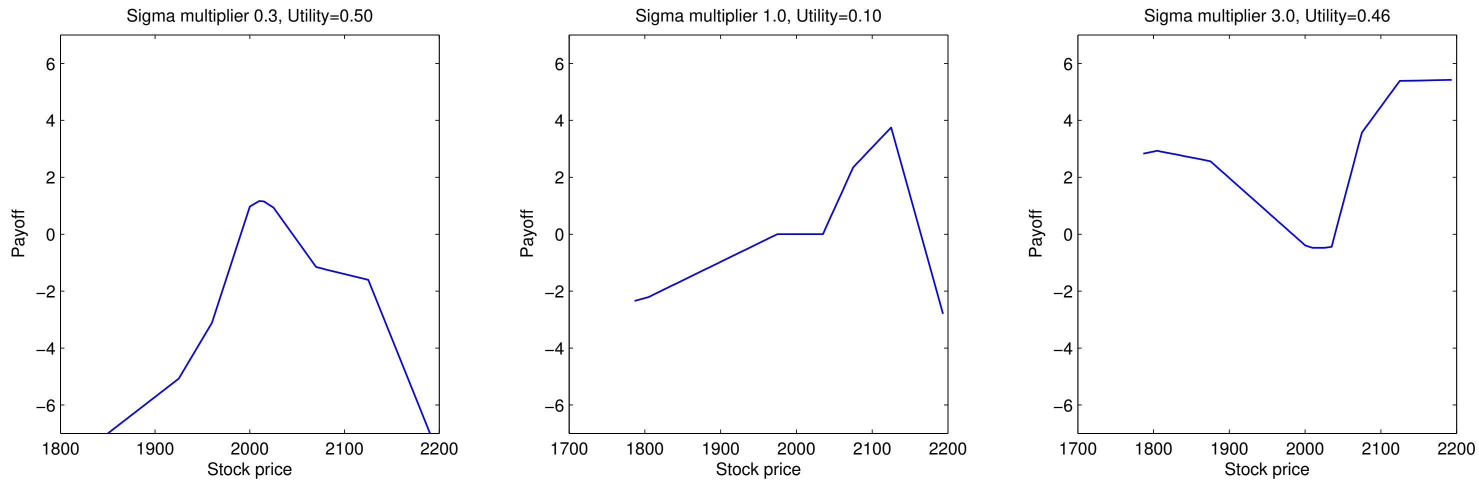
Example: S&P500 options

Optimal portfolio



Example: S&P500 options

Optimal payout profiles with increasing beliefs of volatility



Example: Defined benefit pension liabilities

- We study the ALM-problem of the Finnish private sector occupational pension system.
- The yearly claims c_t consist of aggregate old age, disability and unemployment pension benefits earned by the end of 2008 and become payable during year t.
- The claims depend on mortality and the price- and wage-inflation, etc.

Example: Defined benefit pension liabilities

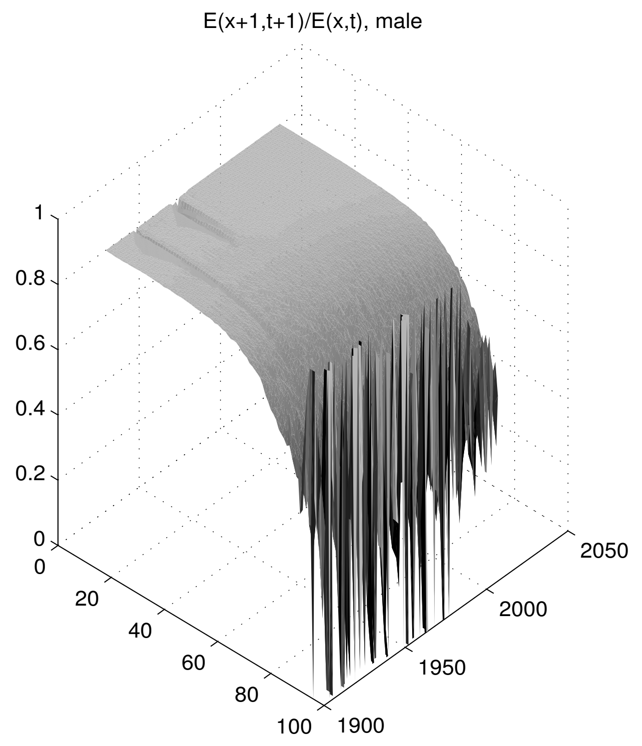
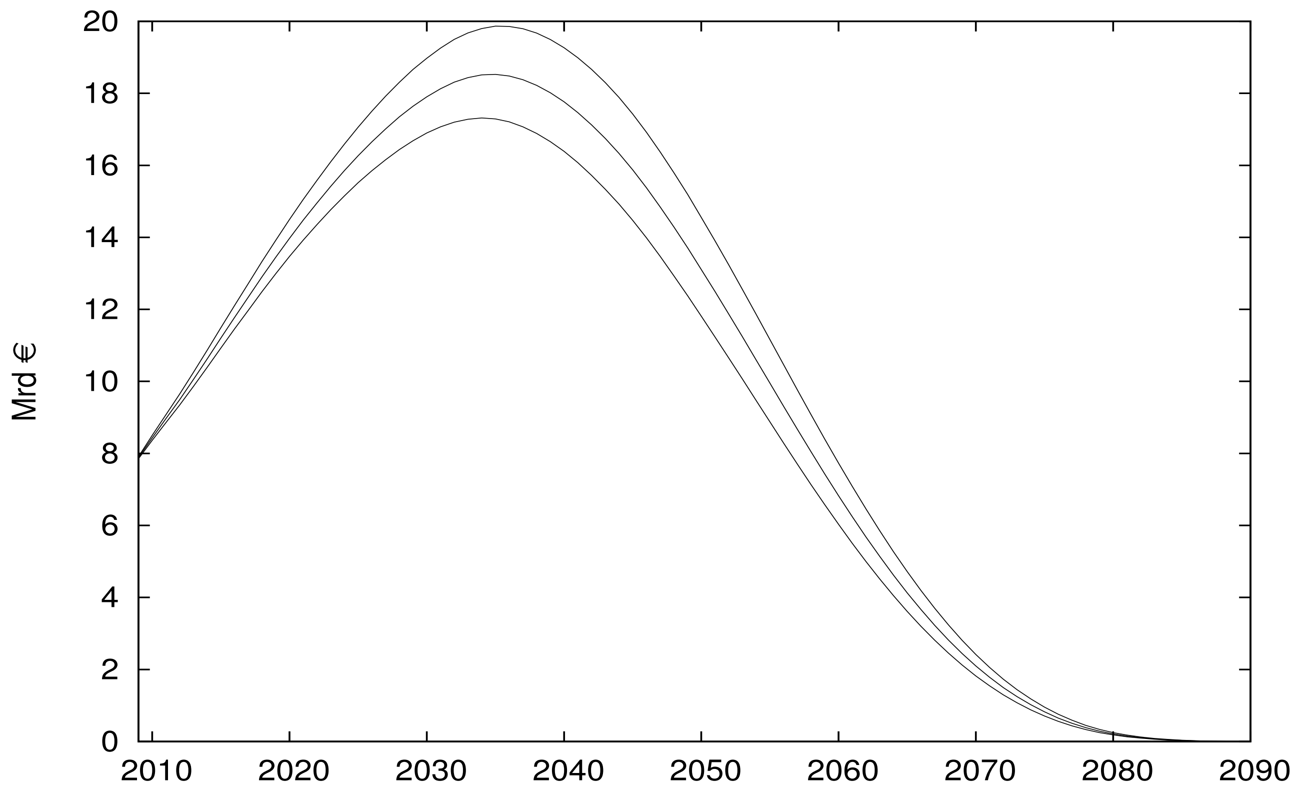


Figure 2: Survival rates of Finnish males

Example: Defined benefit pension liabilities

Figure 3: Yearly claims



Example: Defined benefit pension liabilities

- The traded assets consist of five equity indices and two bond indices.
- Yearly bond returns are modeled by

$$R_t = \exp(Y_t \Delta t - D \Delta Y_t),$$

where Y is the yield to maturity and D the duration.
- Market risk factors are modeled together with the liability risk factors (mortality, price- and wage-inflation) by a stochastic difference equation of the form

$$\Delta x_t = A x_{t-1} + b + \varepsilon_t,$$

where x is the vector of (transformed) risk factors.

Example: Defined benefit pension liabilities

The market models is

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists h \in \mathcal{N}_D : \sum_{j \in J} h_t^j + c_t \leq \sum_{j \in J} R_t^j h_{t-1}^j\}.$$

When

$$\mathcal{V}(c) := \begin{cases} V_T(c) & \text{if } c_t \leq 0 \text{ for } t < T, \\ +\infty & \text{otherwise} \end{cases}$$

the problem can be written as

$$\begin{aligned} &\text{minimize} && \mathcal{V}_T\left(-\sum_{j \in J} h_{T,j}\right) \quad \text{over} \quad h \in \mathcal{N}_D \\ &\text{subject to} && \sum_{j \in J} h_{t,j} + c_t \leq \sum_{j \in J} r_{t,j} h_{t-1,j}. \end{aligned}$$

Example: Defined benefit pension liabilities

The Galerkin method optimizes over convex combinations of feasible trading strategies $(h^i)_{i\in I}$:

$$\begin{aligned} &\text{minimize} \quad \mathcal{V}_T\left(-\sum_{i\in I}\alpha^i\sum_{j\in J}h^i_{T,j}\right) \quad \text{over} \quad \alpha\in\mathbb{R}^I_+ \\ &\text{subject to} \quad \sum_{i\in I}\alpha^i = 1. \end{aligned}$$

- When $\mathcal{V}(W) = Ev(W)$, the objective can be approximated by integration quadratures.
- The terminal wealth $\sum_{j\in J} h^i_{T,j}$ can be evaluated independently for each strategy i and each scenario.
- (Compare with the finite element method for elliptic PDEs with nonconstant coefficients.)

Example: Defined benefit pension liabilities

Results with 529 basis strategies (buy and hold, constant proportions, portfolio insurance, target date fund).

Weight	Type	$CV@R_{97.5\%}$ (billion €)
0.665	BH	1569
0.029	BH	6567
0.104	BH	5041
0.022	CP	3324
0.039	PI	1420
0.099	PI	1907
0.042	PI	2417
	Best basis	1020
	Galerkin	251

Pre-crisis valuations

- Risk neutral valuation assumes that the payout of a claim can be replicated by trading and that the negative of the trading strategy replicates the negative claim (perfect liquidity).
- It follows that
  - there is only one sensible price for buying/selling the claim.
  - the price can be expressed as the expectation of the cash-flows under a "risk neutral measure".
  - the price does not depend on our market expectations, risk preferences or financial position.
- The independence is peculiar to redundant securities whose cash-flows can be replicated by trading other assets.

Pre-crisis valuations

- Actuarial valuations come from the opposite direction where everything is invested on the "bank account" and nothing but fixed-income instruments can be replicated.
- Actuarial valuations can be divided roughly into
  - premium principles reminiscent of indifference valuations discussed below.
  - "best estimate" which is defined as the discounted expectation of future cash-flows.
- Such valuations are not market consistent: the "best estimate" of e.g. a European call tends to be too high.
- The "best estimate" is inherently procyclical: it increases when discount rates decrease during financial crises.
- A trick question: "What discount rate should be used?"

Pre-crisis valuations

- The flaws of pre-crisis valuations are well-known so it is common to adjust the incorrect valuations:
  - Credit valuation adjustment (CVA) tries to correct for credit risk that was ignored by a pricing model.
  - Funding valuation adjustment (FVA) tries to correct for incorrect lending/borrowing rates.
  - Risk margin in Solvency II tries to correct for the the risk that is filtered out by the expectation in the "best estimate".
  - ...
- Instead of adjusting incorrect valuations, we will adjust the underlying model and derive values from hedging arguments à la Black–Scholes.

Valuation of contingent claims

- In incomplete markets, the hedging argument for valuation of contingent claims has two natural generalizations:
  - accounting value: How much cash do we need to cover our liabilities at an acceptable level of risk?
  - indifference price: What is the least price we can sell a financial product for without increasing our risk?
- The former is important in accounting, financial reporting and supervision (SII, IFRS) and in the BS-model.
- The latter is more relevant in trading.
- Classical math finance makes no distinction between the two.

Valuation of contingent claims

- In incomplete markets, the hedging argument for valuation of contingent claims has two natural generalizations:
  - accounting value: How much cash do we need to cover our liabilities at an acceptable level of risk?
  - indifference price: What is the least price we can sell a financial product for without increasing our risk?
- In general, such values depend on our views, risk preferences and financial position.
- Subjectivity is the driving force behind trading.
- Trying to avoid the subjectivity leads to inconsistencies and confusion ("What discount rate should be used?")
- In complete markets, the two notions coincide and they are independent of the subjective factors

Accounting values

- We define the accounting value for a liability $c \in \mathcal{M}$ by

$$\pi_s^0(c) = \inf\{\alpha \in \mathbb{R} \,|\, \varphi(c - \alpha p^0) \leq 0\}$$

  where $p^0 = (1, 0, \dots, 0)$.
- Similarly,

$$\pi_b^0(c) = \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\alpha p^0 - c) \leq 0\}$$

  gives the accounting value of an asset $c \in \mathcal{M}$.
- Clearly, $\pi_b^0(c) = -\pi_s^0(-c)$.
- π_s^0 can be interpreted like a risk measure in [Artzner, Delbaen, Eber and Heath, 1999]. However, we have not assumed the existence of a cash-account so π_s^0 is defined on sequences of cash-flows.

Accounting values

Define the super- and subhedging costs

$$\pi^0_{\sup}(c) := \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{C}\}, \; \pi^0_{\inf}(c) := \inf\{\alpha \,|\, \alpha p^0 - c \in \mathcal{C}\}$$

Theorem 2 *The accounting value π^0_s is convex and nondecreasing with respect to $\mathcal{C}^\infty$. We have $\pi^0_s \leq \pi^0_{\sup}$ and if $\pi^0_s(0) \geq 0$, then*

$$\pi^0_{\inf}(c) \leq \pi^0_b(c) \leq \pi^0_s(c) \leq \pi^0_{\sup}(c)$$

with equalities throughout if $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$ for $\alpha \in \mathbb{R}$.

- π^0_s is "translation invariant": if $c' - \alpha p^0 \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ (i.e. $c' \in \mathcal{M}$ is replicable with initial cash α), then

$$\pi^0(c + c') = \pi^0(c) + \alpha.$$

- In complete markets, $c - \alpha p^0 \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ always for some $\alpha \in \mathbb{R}$, so $\pi^0_s(c)$ is independent of preferences and views.

Swap contracts

- In a swap contract, an agent receives a sequence $p \in \mathcal{M}$ of premiums and delivers a sequence $c \in \mathcal{M}$ of claims.
- Examples:
  - Swaps with a "fixed leg": $p = (1, \dots, 1)$, random c.
  - In credit derivatives (CDS, CDO, ...) and other insurance contracts, both p and c are random.
  - Traditionally in mathematical finance,

$$p = (1, 0, \dots, 0) \quad \text{and} \quad c = (0, \dots, 0, c_T).$$

- Claims and premiums live in the same space

$$\mathcal{M} = \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R})\}.$$

Swap contracts

- If we already have liabilities $\bar{c} \in \mathcal{M}$, then

$$\pi(\bar{c}, p; c) := \inf\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

  gives the least swap rate that would allow us to enter a swap contract without worsening our financial position.

- Similarly,

$$\pi^b(\bar{c}, p; c) := \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} - c + \alpha p) \leq \varphi(\bar{c})\} = -\pi(\bar{c}, p; -c)$$

  gives the greatest swap rate we would need on the opposite side of the trade.

- When $p = (1, 0, \ldots, 0)$ and $c = (0, \ldots, 0, c_T)$, we get an extension of the indifference price of [Hodges and Neuberger, 1989] to nonproportional transactions costs.

Swap contracts

Define the super- and subhedging swap rates,

$$\pi_{\sup}(p;c) = \inf\{\alpha \,|\, c-\alpha p \in \mathcal{C}^\infty\}, \; \pi_{\inf}(p;c) = \sup\{\alpha \,|\, \alpha p-c \in \mathcal{C}^\infty\}.$$

If $\mathcal{C}$ is a cone and $p = (1, 0, \dots, 0)$, we recover the super- and subhedging costs $\pi^0_{\sup}$ and $\pi^0_{\inf}$.

Theorem 3 *If $\pi(\bar{c}, p; 0) \geq 0$, then*

$$\pi_{\inf}(p;c) \leq \pi_b(\bar{c}, p; c) \leq \pi(\bar{c}, p; c) \leq \pi_{\sup}(p;c)$$

with equalities if $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$.

- Agents with identical views, preferences and financial position have no reason to trade with each other.
- Prices are independent of such subjective factors when $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$. If in addition, $p = p^0$, then swap rates coincide with accounting values.

Swap contracts

Example 4 (The classical model) *Consider the classical perfectly liquid market model where*

$$\mathcal{C} = \{c \in \mathcal{M} \mid \exists x \in \mathcal{N} : \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\}$$

and $\mathcal{C}^\infty = \mathcal{C}$*. The condition* $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ *holds if there exist* $x \in \mathcal{N}$ *such that*

$$\sum_{t=0}^{T} c_t = \alpha \sum_{t=0}^{T} p_t + \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}.$$

The converse holds under the no-arbitrage condition.

Example: EONIA swaps

The ALM-problem again:

$$\text{minimize} \quad E\exp(-\gamma x_T) \quad \text{over} \quad z \in \mathbb{R}^K,$$

where $x_t = (1 + r_{t-1})x_{t-1} + \sum_{k \in K} z_k c_{k,t} - c_t$.

- Consider a swap where the agent delivers a the floating leg c of an EONIA swap and receives a multiple $p \equiv 1$.
- The indifference swap rate

$$\pi(\bar{c}, p; c) = \inf\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

  can be found by a simple line search with respect to α by computing the optimum value $\varphi(\bar{c} + c - \alpha p)$ at each iteration.

Example: EONIA swaps

Reality check: The indifference rate of a quoted 6M swap equals the quoted rate -1.300×10^{-4}. This is independent of views and risk preferences just like the Black–Scholes formula.

Table 2: Optimal portfolios before and after the trade

OIS Maturity	before	after
1W	9.3882	9.3882
2W	-9.7979	-9.7979
3W	4.9331	4.9331
1M	-1.3731	-1.3731
2M	0.0129	0.0129
3M	0.1242	0.1242
6M	-0.0345	0.9655

Example: EONIA swaps

Indifference rate of an unquoted 100 day swap:
-1.4184×10^{-4}

Table 3: Optimal portfolios before and after the trade

OIS Maturity	before	after
1W	9.3882	9.6984
2W	-9.7979	-9.9508
3W	4.9331	4.8288
1M	-1.3731	-1.2648
2M	0.0129	-0.1825
3M	0.1242	1.0623
6M	-0.0345	0.1849

Example: EONIA swaps

Table 4: Dependence of indifference rate on the initial cash position

units of cash	ID rate
-5	4.2938×10^{-5}
0	-1.4184×10^{-4}
5	-3.1705×10^{-4}

Example: S&P500 options

The problem again

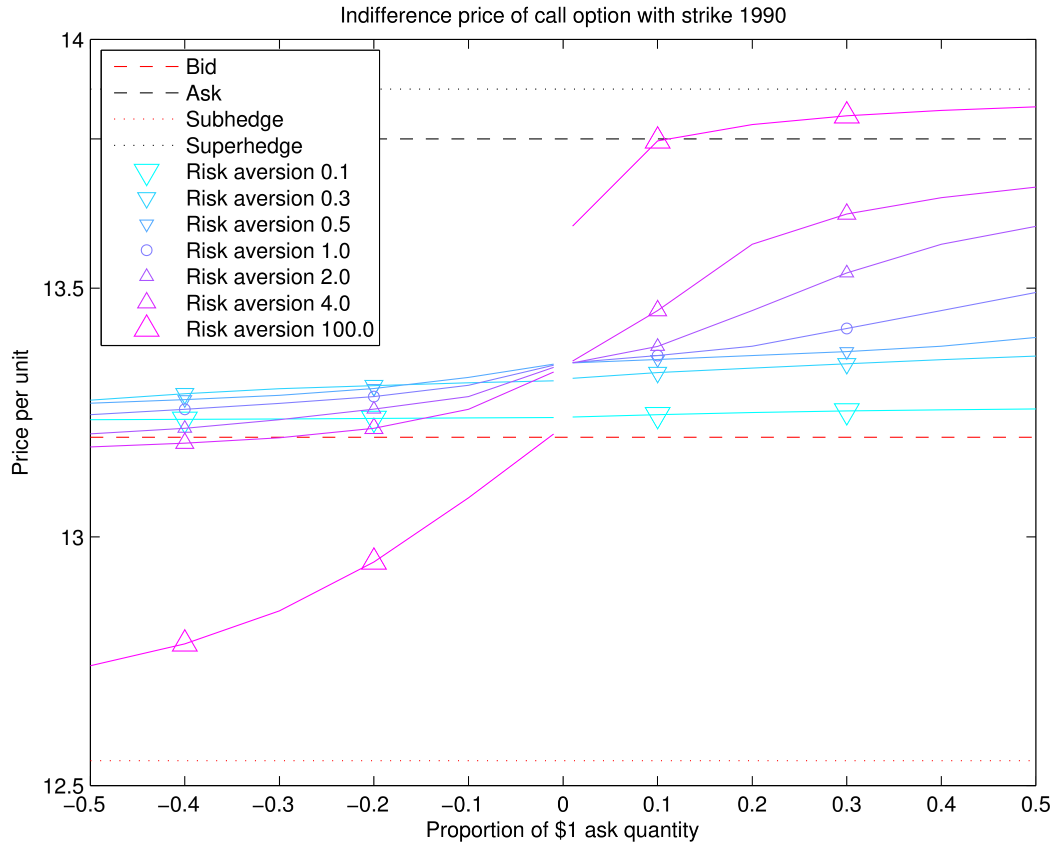
$$\begin{array}{lll} \text{minimize} & E\exp\left[\gamma(c_1 - s_1\cdot x)\right] \quad \text{over} & x\in[-q_b,q_a] \\ \text{subject to} & \sum_{j\in J} S_0^j(x^j) + c_0 \le 0, & \end{array}$$

- The sales of a European option is a swap where the floating leg is $(0, c_T)$ and the premium is a multiple of $p = (1,0)$.
- The indifference price

$$\pi(\bar{c}, p; c) = \inf\{\alpha\in\mathbb{R} \,|\, \varphi(\bar{c}+c-\alpha p) \le \varphi(\bar{c})\}$$

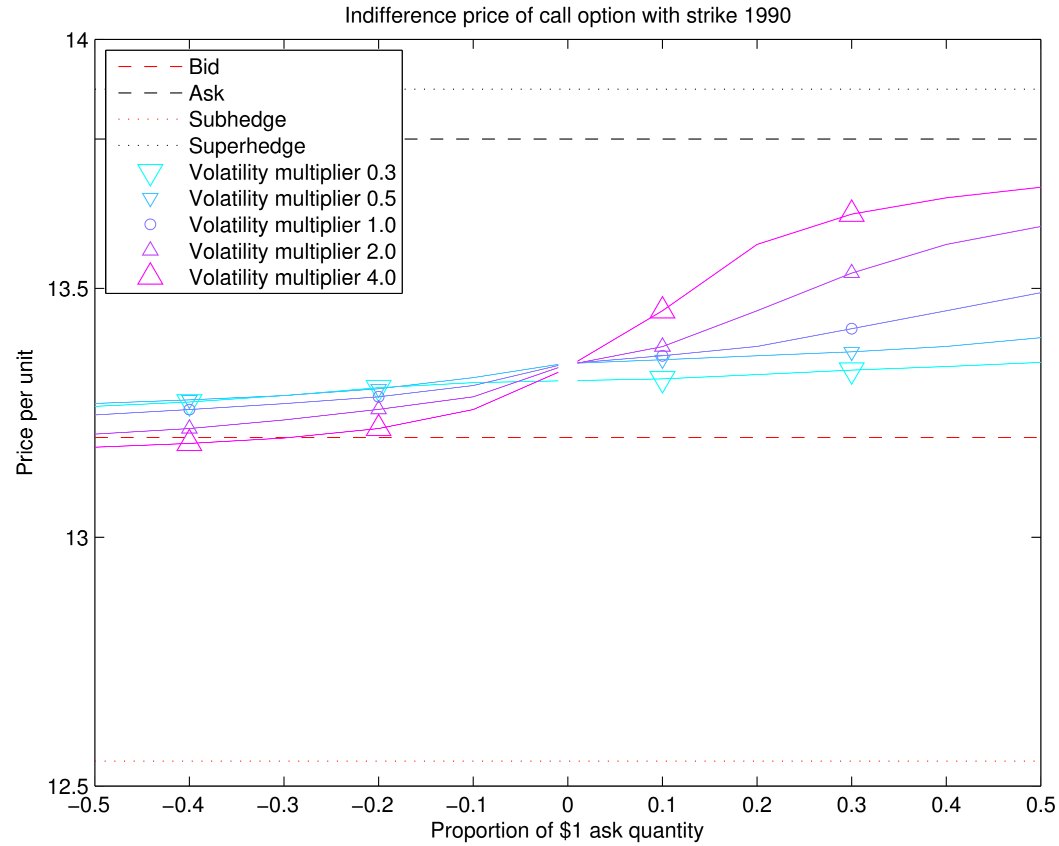
can be found by line search and numerical evaluation of $\varphi(\bar{c}+c-\alpha p)$ at each iteration.

Example: S&P500 options



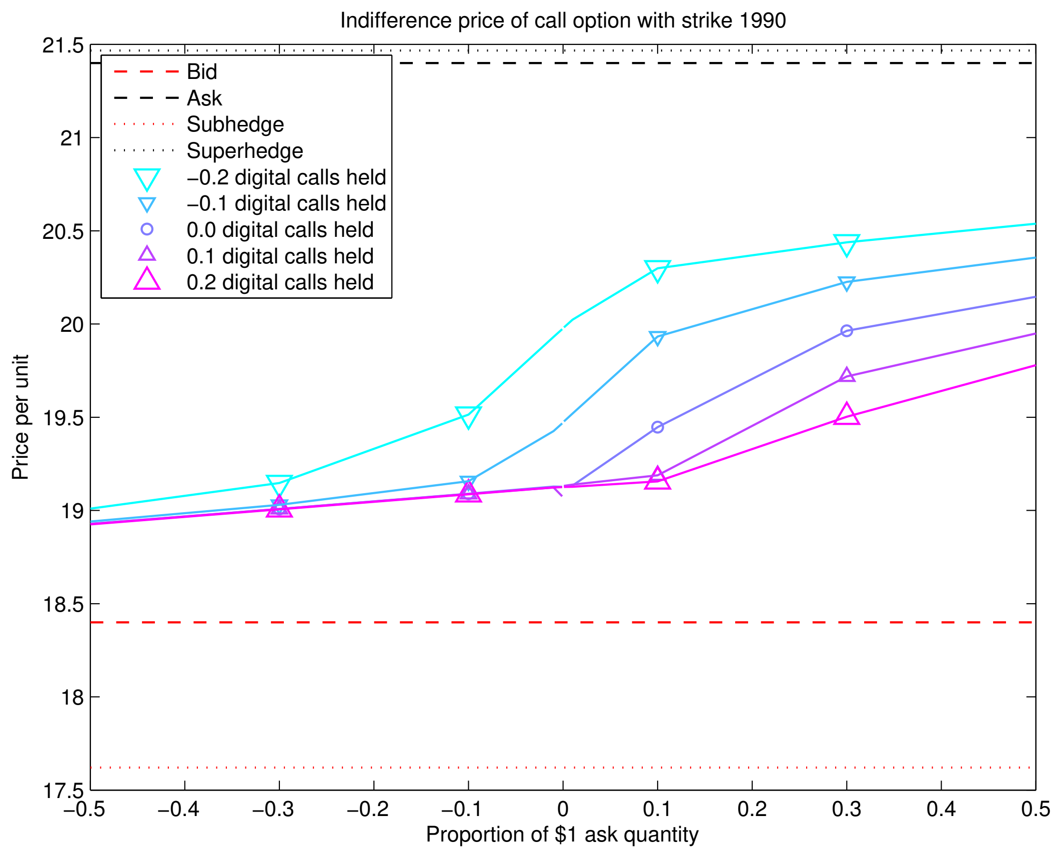
For high risk aversion, indifference prices approach super/subhedging costs.

Example: S&P500 options



As the assumed volatility increases, the indifference prices again approach super/subhedging costs.

Example: S&P500 options



- Our initial position is λ units of a digital call with strike 2000.
- Lower the λ, more we value the call as a hedge for our position

Example: Defined benefit pension liabilities

The problem again

$$\begin{aligned} &\text{minimize} \quad && \mathcal{V}_T\left(-\sum_{j\in J} h_{T,j}\right) \quad \text{over} \quad h \in \mathcal{N}_D \\ &\text{subject to} \quad && \sum_{j\in J} h_{t,j} + c_t \le \sum_{j\in J} r_{t,j} h_{t-1,j}. \end{aligned}$$

- We will compute the minimal accounting value for the Finnish private sector pension liabilities effective in 2010.
- We find the minimum reserve

$$\pi^0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \le 0\}$$

by numerical optimization and line search.

Example: Defined benefit pension liabilities

	Confidence level				
	95%	90%	85%	80%	66%
Best basis	296	284	273	261	239
Optimized	288	271	254	236	202

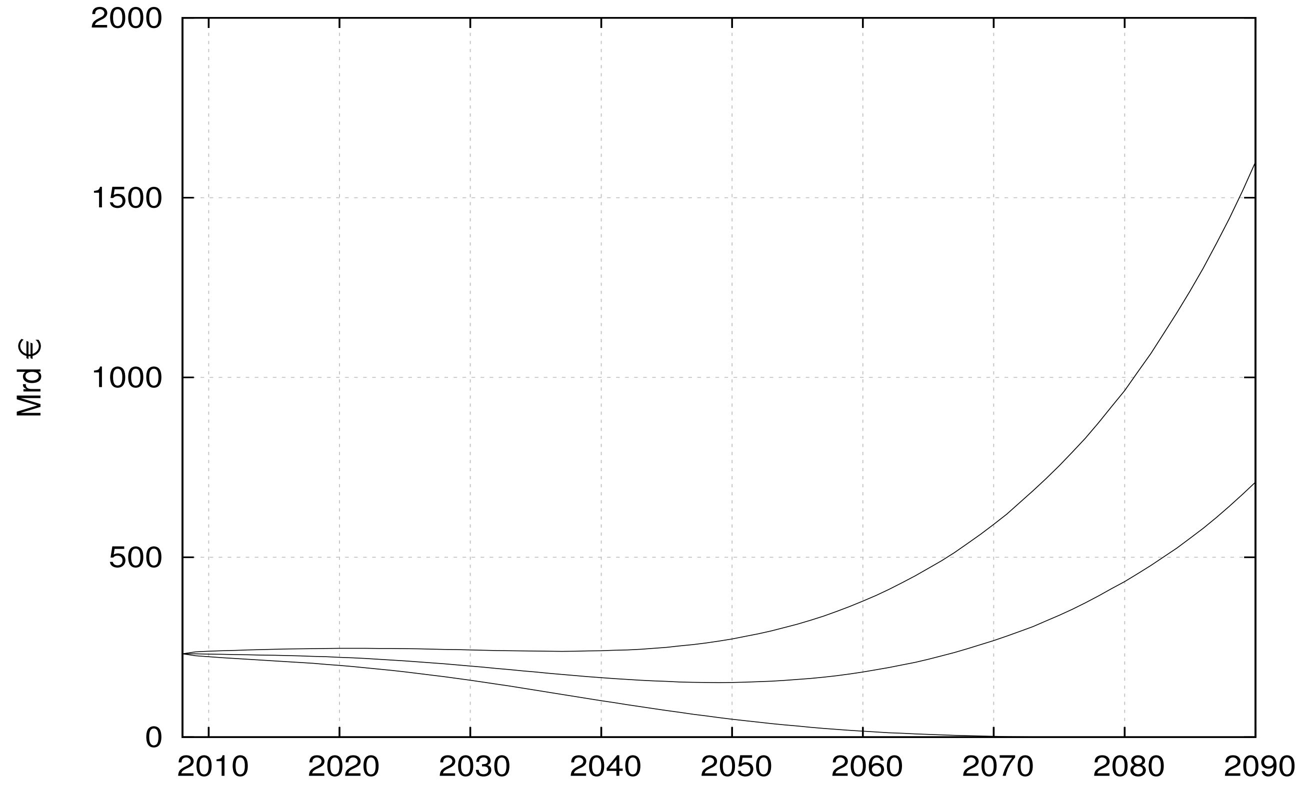
Table 5: Liability values with varying risk tolerances

	Confidence level				
	95%	90%	85%	80%	66%
Best basis	24.3	25.4	26.4	27.6	30.1
Optimized	25.0	26.6	28.3	30.5	35.6

Table 6: Corresponding funding ratios

Example: Defined benefit pension liabilities

Figure 4: The development of 34%, 50%- and 66%-quantiles of net wealth when $\pi_0(c)$ is defined with $\mathcal{V} = V@R_{66\%}$.



Existence of solutions

From now on we assume that

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \; S_t(\Delta x_t) + c_t \leq 0 \quad \forall t\},$$

where $\mathcal{N}_D = \{x \in \mathcal{N} \,|\, x_t \in D_t, \; x_T = 0\}$ and for each (t, ω)

- $S_t(x, \omega)$ is the cost (in cash) of buying a portfolio $x \in \mathbb{R}^J$,
- $D_t(\omega)$ is the portfolio constraint.

We assume that S_t and D_t are $\mathcal{F}_t$-measurable, closed and convex so, in particular, $\mathcal{C}$ is a convex set with $\mathcal{M}_- \subset \mathcal{C}$.

- If $S_t(\cdot, \omega)$ are sublinear and $D_t(\omega)$ are conical, then $\mathcal{C}$ is a cone.

Existence of solutions

Given a market model (S, D), let

$$S_t^\infty(x,\omega) = \sup_{\alpha>0} \frac{S_t(\alpha x,\omega)}{\alpha} \quad \text{and} \quad D_t^\infty(\omega) = \bigcap_{\alpha>0} \alpha D_t(\omega).$$

If S is sublinear and D is conical, then $S^\infty = S$ and $D^\infty = D$

Theorem 5 *Assume that $\mathcal{V}(c) = E\sum_{t=0}^T V_t(c_t)$, where V_t are bounded from below. If the cone*

$$\mathcal{L} := \{x \in \mathcal{N}_{D^\infty} \mid S_t^\infty(\Delta x_t) \leq 0\}$$

is a linear space, then φ is lower semicontinuous in L^0 (in particular, $\mathcal{C}$ is closed).

The lower bound can be replaced by RAE; [Perkkiö, 2014].

Existence of solutions

Example 6 *In the classical perfectly liquid market model*

$$\mathcal{L} = \{x \in \mathcal{N} \mid s_t \cdot \Delta x_t \leq 0,\ x_T = 0\},$$

so the linearity condition becomes the no-arbitrage condition and we recover the key lemma from [Schachermayer, 1992].

Example 7 *When $D \equiv \mathbb{R}^J$, the linearity condition becomes the robust no-arbitrage condition: there exists a positively homogeneous arbitrage-free cost process $\tilde{S}$ with*

$$\tilde{S}_t(x,\omega) \leq S_t^\infty(x,\omega) \quad \forall x \in \mathbb{R}^J,$$

$$\tilde{S}_t(x,\omega) < S_t^\infty(x,\omega) \quad \forall x \notin \operatorname{lin} S_t(\cdot,\omega);$$

see [Schachermayer, 2004].

Existence of solutions

The linearity condition can hold even under arbitrage.

Example 8 *If* $S_t^\infty(x,\omega) > 0$ *for* $x \notin \mathbb{R}^J_-$, *then* $\mathcal{L} = \{0\}$.

Example 9 *In [Çetin and Rogers, 2007],*

$$S_t(x,\omega) = x^0 + s_t(\omega)\psi(x^1)$$

so $S_t^\infty(x,\omega) = x^0 + s_t(\omega)\psi^\infty(x^1)$. *If* $\inf \psi' = 0$ *and* $\sup \psi' = \infty$ *we have* $\psi^\infty = \delta_{\mathbb{R}_-}$, *so the condition in Example 8 holds.*

Example 10 *If* $S_t(\cdot,\omega) = s_t(\omega) \cdot x$ *for a componentwise strictly positive price process* s *and* $D_t^\infty(\omega) \subseteq \mathbb{R}^J_+$ *(infinite short selling is prohibited), then* $\mathcal{L} = \{0\}$.

Existence of solutions

Proposition 11 *Under the linearity condition, the conditions*

- $p^0 \notin \mathcal{C}^\infty$,
- $\pi^0(0) > -\infty$,
- $\pi^0(c) > -\infty$ *for all* $c \in \mathcal{M}$,

are equivalent and imply that π^0 *is proper and lower semicontinuous on* $\mathcal{M}$ *and that the infimum*

$$\pi^0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \leq 0\}$$

is attained for every $c \in \mathcal{M}$.

Existence of solutions

Proposition 12 *Assume the linearity condition. Then, for every $\bar{c} \in \operatorname{dom}\varphi$ and $p \in \mathcal{M}$, the conditions*

- $p \notin \mathcal{C}^\infty$,
- $\pi(\bar{c}, p; 0) > -\infty$,
- $\pi(\bar{c}, p; c) > -\infty$ *for all* $c \in \mathcal{M}$,

are equivalent and imply that $\pi(\bar{c}, p; \cdot)$ is proper and lower semicontinuous on $\mathcal{M}$ and that the infimum

$$\pi(\bar{c}, p; c) = \inf\{\alpha \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

is attained for every $c \in \mathcal{M}$.

Duality

- Let $\mathcal{M}^p = \{c \in \mathcal{M} \,|\, c_t \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R})\}$.
- The bilinear form

$$\langle c, y \rangle := E \sum_{t=0}^{T} c_t y_t$$

puts $\mathcal{M}^1$ and $\mathcal{M}^\infty$ in separating duality.

- The conjugate of a function f on $\mathcal{M}^1$ is defined by

$$f^*(y) = \sup_{c \in \mathcal{M}^1} \{\langle c, y \rangle - f(c)\}.$$

- If f is proper, convex and lower semicontinuous, then

$$f(y) = \sup_{y \in \mathcal{M}^\infty} \{\langle c, y \rangle - f^*(y)\}.$$

Duality

We assume from now on that

$$\mathcal{V}(c) = E\sum_{t=0}^{T} V_t(c_t)$$

for convex random functions $V_t : \mathbb{R} \times \Omega \to \overline{\mathbb{R}}$ with $V_t(0) = 0$.

Theorem 13 *If $S_t(x, \cdot) \in L^1$ for all $x \in \mathbb{R}^J$, then*

$$\varphi^*(y) = \mathcal{V}^*(y) + \sigma_{\mathcal{C}}(y)$$

where $\mathcal{V}^*(y) = E\sum_{t=0}^{T} V_t^*(y_t)$ and $\sigma_{\mathcal{C}}(y) = \sup_{c \in \mathcal{C}} \langle c, y \rangle$. Moreover,

$$\sigma_{\mathcal{C}}(y) = \inf_{v \in \mathcal{N}^1} E\sum_{t=0}^{T} [(y_t S_t)^*(v_t) + \sigma_{D_t}(E[\Delta v_{t+1} | \mathcal{F}_t])]$$

where the infimum is attained for all $y \in \mathcal{M}^\infty$.

Duality

Example 14 *If* $S_t(\omega, x) = s_t(\omega) \cdot x$ *and* $D_t(\omega)$ *is a cone,*

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty \,|\, E[\Delta(y_{t+1}s_{t+1})\,|\mathcal{F}_t] \in D_t^*\}.$$

Example 15 *If* $S_t(\omega, x) = \sup\{s \cdot x \,|\, s \in [s_t^b(\omega), s_t^a(\omega)]\}$ *and* $D_t(\omega) = \mathbb{R}^J$*, then*

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty \,|\, ys \text{ is a martingale for some } s \in [s^b, s^a]\}.$$

Example 16 *In the classical model,* $\mathcal{C}^*$ *consists of positive multiples of martingale densities.*

Duality

Theorem 17 *Assume the linearity condition, the Inada condition* $V_t^\infty = \delta_{\mathbb{R}_-}$ *and that* $p^0 \notin \mathcal{C}^\infty$ *and* $\inf \varphi < 0$. *Then*

$$\pi^0(c) = \sup_{y\in\mathcal{M}^\infty} \{ \langle c, y\rangle - \sigma_{\mathcal{C}}(y) - \sigma_{\mathcal{B}}(y) \,|\, y_0 = 1 \} ,$$

where $\mathcal{B} = \{c \in \mathcal{M}^1 \,|\, \mathcal{V}(c) \le 0\}$. *In particular, when* $\mathcal{C}$ *is conical and* $\mathcal{V}$ *is positively homogeneous,*

$$\pi^0(c) = \sup_{y\in\mathcal{M}^\infty} \{ \langle c, y\rangle \,|\, y \in \mathcal{C}^* \cap \mathcal{B}^*,\ y_0 = 1 \} .$$

- Extends good deal bounds to sequences of cash-flows.

Duality

Theorem 18 *Assume the linearity condition, the Inada condition and that* $p \notin \mathcal{C}^\infty$ *and* $\inf \varphi < \varphi(\bar{c})$*. Then*

$$\pi(\bar{c}, p; c) = \sup_{y \in \mathcal{M}^\infty} \left\{ \langle c, y \rangle - \sigma_{\mathcal{C}}(y) - \sigma_{\mathcal{B}(\bar{c})}(y) \,\middle|\, \langle p, y \rangle = 1 \right\},$$

where $\mathcal{B}(\bar{c}) = \{c \in \mathcal{M}^1 \,|\, \mathcal{V}(\bar{c} + c) \leq \varphi(\bar{c})\}$*. In particular, if* $\mathcal{C}$ *is conical,*

$$\pi(\bar{c}, p; c) = \sup_{y \in \mathcal{M}^\infty} \left\{ \langle c, y \rangle - \sigma_{\mathcal{B}(\bar{c})}(y) \,\middle|\, u \in \mathcal{C}^*, \; \langle p, y \rangle = 1 \right\}.$$

Duality

Example 19 *In the classical model, with* $p = (1, 0, \dots, 0)$ *and* $V_t = \delta_{\mathbb{R}_-}$ *for* $t < T$*, we get*

$$\pi(\bar{c}, p; c) = \sup_{Q \in \mathcal{Q}} \sup_{\alpha > 0} E^Q \left\{ \sum_{t=0}^{T} (\bar{c}_t + c_t) - \alpha \left[V_T^*(\frac{dQ}{dP}/\alpha) - \varphi(\bar{c}) \right] \right\}$$

where $\mathcal{Q}$ is the set of absolutely continuous martingale measures; see [Biagini, Frittelli, Grasselli, 2011] for a continuous-time version.

Duality

Theorem 20 (FTAP) *Assume that S^∞ is finite-valued and that $D \equiv \mathbb{R}^J$. Then the following are equivalent*

1. *S satisfies the robust no-arbitrage condition.*
2. *There is a strictly consistent price system: adapted processes y and s such that $y > 0$, $s_t \in \operatorname{ri} \operatorname{dom} S_t^*$ and ys is a martingale.*

- In the classical linear market model, $\operatorname{ri} \operatorname{dom} S_t^* = \{1, \tilde{s}_t\}$ so we recover the Dalang–Morton–Willinger theorem.
- The robust no-arbitrage condition means that there exists a sublinear arbitrage-free cost process $\tilde{S}$ with $\operatorname{dom} \tilde{S}_t^* \subseteq \operatorname{ri} \operatorname{dom} S_t^*$.

Summary

- Post-crisis FM is subjective: optimal investment and valuations depend on views, risk preferences, financial position and trading expertise.
- ALM brings pricing, accounting and risk management under a single consistent framework.
- Not a quick solution but a coherent and universal approach based on risk management.
- Requires techniques from statistics, optimization, and computer science.
- With some convex analysis, classical "fundamental theorems" can be extended to illiquid market models.