STOCHASTIC VARIATIONAL INEQUALITIES AND EQUILIBRIUM

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Modeling with Variational Inequalities

Targeted applications: here with stochastic ingredients

expression of conditions for optimality or equilibrium

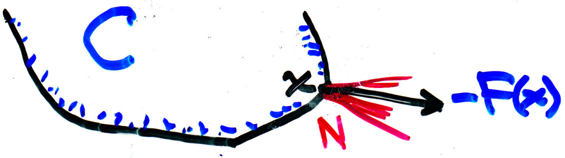
posed in a Hilbert space, finite- or infinite-dimensional

Variational inequality problem in a space $\mathcal{H}$

For $C \subset \mathcal{H}$ nonempty closed convex, $F : \mathcal{H} \to \mathcal{H}$ continuous,

$$\text{determine } x \in C \text{ such that } -F(x) \in N_C(x)$$

$$\text{i.e., } \langle F(x), x' - x \rangle \geq 0 \;\; \forall x' \in C$$



Monotone case: F monotone, hence actually maximal monotone

$$\langle F(x') - F(x), x' - x \rangle \geq 0 \text{ for all } x, x'$$

Motivations in Optimization — Without Stochastics

general V.I.: $-F(x) \in N_C(x)$

Elementary optimization: minimizing $g(x)$ over $x \in C$

$-\nabla g(x) \in N_C(x)$ $\longrightarrow$ first-order optimality

To formulate this as a V.I., take $F = \nabla g$

Lagrangian V.I.: for $l(y,z)$ on $Y \times Z$ closed convex

$$-\nabla_y l(y,z) \in N_Y(y), \quad \nabla_z l(y,z) \in N_Z(z),$$

To formulate this as a V.I., take

$$x = (x,y), \quad C = Y \times Z, \quad F(x) = (\nabla_y l(y,z), -\nabla_z l(y,z))$$

$\longrightarrow$ this encompasses KKT conditions in NLP and much more!

(also, it can model a saddle-point in a two-person game)

Motivations in Equilibrium — Without Stochastics

Game-type equilibrium: for agents $i = 1, \dots, m$

- agent i chooses $x_i \in X_i$ closed convex $\subset \boldsymbol{R}^{n_i}$
- minimization of $f_i(x_i, x_{-i})$ over $x_i \in X_i$ is desired

  where x_{-i} stands for the choices of all the other agents
- equilibrium represented by $-\nabla_{x_i} f_i(x_i, x_{-i}) \in N_{X_i}(x_i)$ for all i

  agent global optimality is replaced here by "stationarity"!

To formulate this as a V.I., take

$$x = (x_1, \dots, x_m), \qquad C = X_1 \times \cdots \times X_m$$

$$F(x) = (\nabla_{x_1} f_1(x_1, x_{-1}), \dots, \nabla_{x_m} f_m(x_m, x_{-m}))$$

Stochastic Structure with Emerging Information

capturing dynamics in stochastic optimization and equilibrium

Pattern of "decisions" and "observations" in N stages:

$$x_1,\ \xi_1,\ x_2,\ \xi_2,\dots,x_N,\ \xi_N \quad \text{with} \quad x_k \in \boldsymbol{R}^{n_k},\ \xi_k \in \Xi_k$$

$$x = (x_1,\dots,x_N) \in \boldsymbol{R}^n = \boldsymbol{R}^{n_1} \times \cdots \times \boldsymbol{R}^{n_N}$$

$$\xi = (\xi_1,\dots,\xi_N) \in \Xi \ \subset \Xi_1 \times \cdots \Xi_N$$

Interpretation: each $\xi \in \Xi$ is an information **scenario**

Nonanticipativity of decisions

x_k can respond to $\xi_1,\dots,\xi_{k-1}$ but not to $\xi_k,\dots,\xi_N$:

$$x(\xi) = (x_1, x_2(\xi_1), x_3(\xi_1,\xi_2),\dots,x_N(\xi_1,\xi_2,\dots,\xi_{N-1}))$$

Simplifying assumptions here:

- the scenario space Ξ has only finitely many elements ξ
- each scenario $\xi \in \Xi$ has known probability $p(\xi) > 0$

$\longrightarrow$ Ξ is a probability space

Function Space Framework for Responses

$\mathcal{L}$ = **all** functions from **scenario** space Ξ to **decision** space R^n

$$\Xi \subset \Xi_1 \times \cdots \Xi_N, \qquad R^n = R^{n_1} \times \cdots \times R^{n_N}$$

Response functions as elements of this space:

$$x(\cdot) : \xi = (\xi_1, \ldots, \xi_N) \mapsto x(\xi) = (x_1(\xi), \ldots, x_N(\xi))$$

Expectation inner product giving Hilbert structure:

$$\langle x(\cdot), w(\cdot) \rangle = E_\xi[x(\xi), w(\xi)] = \sum_{\xi \in \Xi} p(\xi) \sum_{k=1}^N x_k(\xi) \cdot w_k(\xi)$$

Nonanticipativity subspace:

in terms of scenarios $\xi = (\xi_1, \ldots, \xi_{k-1}, \xi_k, \ldots, \xi_N)$,

$$\mathcal{N} = \{ x(\cdot) \in \mathcal{L} \mid x_k(\xi) \text{ depends only on } \xi_1, \ldots, \xi_{k-1} \}$$

$\longrightarrow$ $x(\cdot)$ is nonanticipative $\iff$ $x(\cdot) \in \mathcal{N}$

Complementary subspace: $\mathcal{M} = \mathcal{N}^\perp$ (source of "multipliers")

$$\mathcal{M} = \{ w(\cdot) \in \mathcal{L} \mid E_{\xi_k, \ldots, \xi_N}[w_k(\xi_1, \ldots, \xi_{k-1}, \xi_k \ldots, \xi_N)] = 0 \}$$

Multistage Stochastic Optimization in this Setting

Basic constraints beyond nonanticipativity: $x(\cdot) \in \mathcal{C}$

$$\mathcal{C} = \{x(\cdot) \in \mathcal{L} \mid x(\xi) \in C(\xi) \subset \boldsymbol{R}^n \text{ for all } \xi \in \Xi\}$$

Convexity assumption: making $\mathcal{C}$ be closed convex $\neq \emptyset$ in $\mathcal{L}$

$C(\xi)$ closed convex $\neq \emptyset$ in $\boldsymbol{R}^n$ for every scenario $\xi \in \Xi$

Stochastic programming problem in a classical format

minimize $\mathcal{G}(x(\cdot)) = E_\xi[g(x(\xi), \xi)]$ over all functions $x(\cdot) \in \mathcal{C} \cap \mathcal{N}$

Smoothness assumption: making $\mathcal{G}$ be a $\mathcal{C}^1$ function on $\mathcal{L}$

$g(\cdot, \xi)$ is a $\mathcal{C}^1$ function on $\boldsymbol{R}^n$ for every scenario $\xi \in \Xi$

Directional derivatives: $d\mathcal{G}(x(\cdot))(u(\cdot)) = \langle \nabla_x \mathcal{G}(x(\cdot)), u(\cdot) \rangle$

the gradient $\nabla \mathcal{G}(x(\cdot)) \in \mathcal{L}$ takes ξ to $\nabla g(x(\xi), \xi)$

Convex minimization case: $\mathcal{G}$ is convex if each $g(\cdot, \xi)$ is convex

Variational Inequalities in the Response Function Space $\mathcal{L}$

First-order optimality condition in stochastic programming

$$-\nabla\mathcal{G}(x(\cdot)) \in N_{\mathcal{C}\cap\mathcal{N}}(x(\cdot))$$

the V. I. for the gradient mapping $\nabla\mathcal{G}$ and the convex set $\mathcal{C}\cap\mathcal{N}$

this condition is **sufficient** in the case of **convex** minimization

Generalization: replace $\nabla\mathcal{G}$ by any mapping $\mathcal{F}:\mathcal{L}\to\mathcal{L}$ where

$\mathcal{F}(x(\cdot))$ takes ξ to $F(x(\xi),\xi)$ for functions $F(\cdot,\xi):\boldsymbol{R}^n\to\boldsymbol{R}^n$

stochastic programming had $F(\cdot,\xi)=\nabla_x g(\cdot,\xi)$

Continuity assumption: $F(\cdot,\xi)$ continuous making $\mathcal{F}$ continuous

Definition of a stochastic variational inequality, basic form

$$-\mathcal{F}(x(\cdot)) \in N_{\mathcal{C}\cap\mathcal{N}}(x(\cdot))$$

Monotone case: $\mathcal{F}$ is monotone if $F(\cdot,\xi)$ is monotone $\forall\,\xi\in\Xi$

Stochastic Decomposition

Basic S.V.I. to understand further: $-\mathcal{F}(x(\cdot)) \in N_{\mathcal{C}\cap\mathcal{N}}(x(\cdot))$

$\mathcal{C} = \{x(\cdot) \mid x(\xi) \in C(\xi),\ \forall\xi\}$, $\mathcal{N}$ = nonanticipativity subspace

Calculus of normals: if $\exists\,\tilde{x}(\cdot) \in \mathcal{N}$ with $\tilde{x}(\xi) \in \mathrm{ri}\, C(\xi)\ \forall\xi$, then

$N_{\mathcal{C}\cap\mathcal{N}}(x(\cdot)) = N_{\mathcal{C}}(x(\cdot)) + N_{\mathcal{N}}(x(\cdot))$ where $N_{\mathcal{N}}(x(\cdot)) = \mathcal{N}^{\perp} = \mathcal{M}$

and moreover $v(\cdot) \in N_{\mathcal{C}}(x(\cdot)) \iff v(\xi) \in N_{C(\xi)}(x(\xi))\ \forall\xi$

Definition of a stochastic variational inequality, extensive form

$$x(\cdot)\in\mathcal{N} \text{ and } \exists w(\cdot)\in\mathcal{M}: \quad -F(x(\xi),\xi)-w(\xi) \in N_{C(\xi)}(x(\xi)),\ \forall\xi$$

the basic and extensive forms of are "essentially equivalent"

Meaning in stochastic programming: when $F(\cdot,\xi) = \nabla_x g(\cdot,\xi)$

$x(\xi)$ minimizes $g(\cdot,\xi) + \langle\cdot, w(\xi)\rangle$ over $C(\xi)$ for each scenario ξ

the multiplier vectors $w(\xi)$ give the **price of information**

having $w(\cdot) \in \mathcal{M}$ is a "martingale-like" condition

Projection Tool for Aggregating Responses

Recalling the structure of the complementary subspaces:

$$\mathcal{N} = \{x(\cdot) \in \mathcal{L} \mid x_k(\xi) \text{ depends only on } \xi_1, \ldots, \xi_{k-1}\}$$
$$\mathcal{M} = \{w(\cdot) \in \mathcal{L} \mid E_{\xi_k,\ldots,\xi_N}[w_k(\xi_1, \ldots, \xi_{k-1}, \xi_k \ldots, \xi_N)] = 0\}$$

Aggregation: let $\mathcal{P}$ = projection onto $\mathcal{N}$

then $\mathcal{I} - \mathcal{P}$ = projection onto $\mathcal{M}$, since $\mathcal{M} = \mathcal{N}^\perp$

Execution relative to the information structure:

- Scenarios $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$ are at stage k information-equivalent if $(\xi_1, \ldots, \xi_{k-1}) = (\xi'_1, \ldots, \xi'_{k-1})$
- Let $A_k(\xi)$ = kth-stage equivalence class containing ξ
- Then $x(\cdot) = \mathcal{P}(\bar{x}(\cdot))$ has its kth-stage component given by

$$x_k(\xi) = \sum_{\xi' \in A_k(\xi)} p(\xi')\bar{x}_k(\xi') / \sum_{\xi' \in A_k(\xi)} p(\xi')$$

thus $x_k(\xi)$ is the **conditional expectation** of $\bar{x}_k(\xi)$ relative to the kth-stage information-equivalence class containing ξ

Progressive Hedging in Stochastic Programming

Rock. & Wets, 1991: stochastic decomposition realized iteratively

Algorithm statement in the convex case with parameter $r > 0$

Having $x^\nu(\cdot) \in \mathcal{N}$ and $w^\nu(\cdot) \in \mathcal{M}$, get $\bar{x}^\nu(\cdot) \in \mathcal{L}$ by

$$\bar{x}^\nu(\xi) = \operatorname{argmin}_{x \in C(\xi)} \left\{ g(x,\xi) + x \cdot w^\nu(\xi) + \frac{r}{2} ||x - x^\nu(\xi)||^2 \right\}$$

Then get $x^{\nu+1}(\cdot) \in \mathcal{N}$ and $w^{\nu+1}(\cdot) \in \mathcal{M}$ by aggregation:

$$x^{\nu+1}(\cdot) = \mathcal{P}(\bar{x}^\nu(\cdot)), \qquad w^{\nu+1}(\cdot) = w^\nu(\cdot) + r[\bar{x}^\nu(\cdot) - x^{\nu+1}(\cdot)]$$

Convergence theorem — when a solution pair $x(\cdot)$, $w(\cdot)$, exists

The sequence $\{(x^\nu(\cdot), w^\nu(\cdot))\}_{\nu=1}^{\infty}$ generated by the algorithm will always converge to a particular solution pair $(x^*(\cdot), w^*(\cdot))$, with

$$\begin{aligned} &||x^{\nu+1}(\cdot) - x^*(\cdot)||^2 + r^{-2}||w^{\nu+1}(\cdot) - w^*(\cdot)||^2 \\ &\qquad \le ||x^\nu(\cdot) - x^*(\cdot)||^2 + r^{-2}||w^\nu(\cdot) - w^*(\cdot)||^2 \end{aligned}$$

Progressive Hedging for Stochastic Variational Inequalities

Recall extensive form of S.V.I: $x(\cdot) \in \mathcal{N}$, $w(\cdot) \in \mathcal{M}$, and
$-F(x(\xi), \xi) - w(\xi) \in N_{C(\xi)}(x(\xi))$ for every scenario $\xi \in \Xi$

Algorithm statement in the monotone case with parameter $r > 0$

Having $x^\nu(\cdot) \in \mathcal{N}$ and $w^\nu(\cdot) \in \mathcal{M}$, get $\bar{x}^\nu(\cdot) \in \mathcal{L}$ by solving for **each individual scenario** $\xi \in \Xi$ the V.I.

$$-F^\nu(x, \xi) \in N_{C(\xi)}(x)$$

with respect to $x \in \boldsymbol{R}^n$ to get $x^\nu(\xi)$, where

$$F^\nu(x, \xi) = F(x, \xi) + w^\nu(\xi) + r[x - x^\nu(\xi)].$$

Then get $x^{\nu+1}(\cdot) \in \mathcal{N}$ and $w^{\nu+1}(\cdot) \in \mathcal{M}$ by aggregation:

$$x^{\nu+1}(\cdot) = \mathcal{P}(\bar{x}^\nu(\cdot)), \qquad w^{\nu+1}(\cdot) = w^\nu(\cdot) + r[\bar{x}^\nu(\cdot) - x^{\nu+1}(\cdot)]$$

Convergence theorem — when a solution pair $x(\cdot)$, $w(\cdot)$, exists

same result again!

The Role of Monotonicity and its Prospects

Technical basis for the algorithm:

proximal point algorithm applied to a certain $T : \mathcal{L} \rightrightarrows \mathcal{L}$

T is derived from a "twisting" of the ingredients of the S.V.I.

Global convergence: T must be a maximal monotone mapping

Local convergence: T maximal monotone just around solution

Prospects for extended stochastic programming:

- convexity of the cost functions can be weakened to a second-order optimality condition right at a solution
- expectation may be replaced by risk measure in objective

Prospects for game models and equilibrium

- game versions of multistage stochastic programming
- local convergence in circumstances of "moderate interaction"
- an augmented Lagrangian technique may actually elicit that!

Some References

[1] R.T. Rockafellar, R.J -B Wets (2016) "Stochastic variational inequalities: single-stage to multistage," *Mathematical Programming B,* forthcoming.

[2] R.T. Rockafellar and R.J-B Wets (1991) "Scenarios and policy aggregation in optimization under uncertainty," *Mathematics of Operations Research* 16, 119–147.

[3] R.T. Rockafellar and R.J-B Wets (1976) "Nonanticipativity and L^1-martingales in stochastic optimization problems," *Mathematical Programming Study* 6, 170-187.

website: **www.math.washington.edu/∼rtr/mypage.html**