A two step approach for the bidding process in electricity markets: theorerical and numerical analysis

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Workshop SESO17 - May/June 2017

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Outline of the talk:

I- *On the modelisation of the bidding process in electricity markets*

II- *Non-self quasivariational inequalities: what? and why?*

III- *Existence of projected solutions*

IV- *Application to Nash games (electicity markets)*

V- *Quasi-optimization problems*

VI- *Some ongoing results on computational aspects*

Perpignan, France

I- On the modelisation of the bidding process in electricity markets

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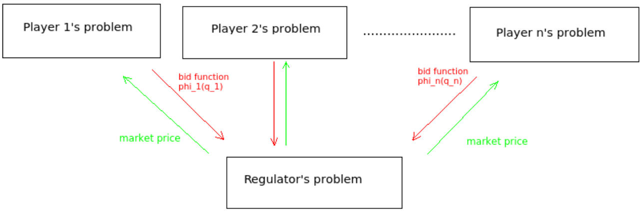
What is the current difficulty?

Models for bidding process...

A model classically used in the literature is a multi-leader-single-follower game



where the bid function is given by

$$\varphi_i(q_i) := \int_0^{q_i} \psi_i(q)dq + k_i$$

with

- $k_i \in \mathbb{R}$ is the initial payment
- ψ_i is the unit price bid function

Models with linear unit bid functions

- **Electricity markets without transmission losses:**
  X. Hu & D. Ralph, Using EPECs to Model Bilevel Games in Restructured Electricity Markets with Locational Prices, *Operations Research (2007).*
  bid-on-a-only

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- **Electricity markets with transmission losses:**
  - *Henrion, R., Outrata, J. & Surowiec, T.*, Analysis of M-stationary points to an EPEC modeling oligopolistic competition in an electricity spot market, *ESAIM: COCV (2012). M-stationary points*
  - *D. A., R. Correa & M. Marechal* Spot electricity market with transmission losses, *J. Industrial Manag. Optim (2013). existence of Nash equil., case of a two island model*
  - *D.A., M. Cervinka & M. Marechal*, Deregulated electricity markets with thermal losses and production bounds: models and optimality conditions, *RAIRO (2016) production bounds, well-posedness of model*

Some references on the topic (cont.)

- **Best response in electricity markets:**
  - *E. Anderson and A. Philpott*, Optimal Offer Construction in Electricity Markets, *Mathematics of Operations Research (2002). Linear bid function - necessary optimality cond. for local best response in time dependent case*
  - *D. Aussel, P. Bendotti and M. Pištěk, Nash Equilibrium in Pay-as-bid Electricity Market : Part 2 - Best Response of Producer, Optimization (2017) linear unit bid function, explicit formula for best response*

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- **Explicit formula for equilibria**
  D. Aussel, P. Bendotti and M. Pištěk, Nash Equilibrium in Pay-as-bid Electricity Market : Part 1 - Existence and Characterisation, Optimization (2017) explicit formula for equilibria

But also...

- **Non a priori structured bid functions**
  - *Escobar, J.F. and Jofré, A., Monopolistic competition in electricity networks with resistance losses, Econom. Theory 44 (2010).*
  - *Escobar, J.F. and Jofré, A., Equilibrium analysis of electricity auctions, preprint (2014).*
  - *E. Anderson, P. Holmberg and A. Philpott, Mixed strategies in discriminatory divisible-good auctions, The RAND Journal of Economics (2013). necessary optimality cond. for local best response*

Classical model

The multi-leader-common-follower game can be formulated as the following general equilibrium problem composed of N producer's optimization problems denoted as (P_i), $i = 1, \ldots, N$, solved simultaneously

$$(P_i) \qquad \max_{\varphi_i, q_i} \varphi_i(q_i) - Cost_i(q_i) \quad s.t. \begin{cases} q \text{ solves } ISO(\varphi) \\ \varphi_i \text{ admissible bid function,} \end{cases}$$

where the ISO problem is considered in the form

$$ISO(\varphi) \qquad \min_q \ \sum_i \varphi_i(q_i) \quad s.t. \begin{cases} \text{demand } D \text{ is satisfied: } \sum_i q_i \geq D \\ 0 \leq q_i \leq \bar{Q}_i, \quad \forall i, \end{cases}$$

where $\bar{Q}_i$ stands for the production capacity of producer i and the vector of bid functions $\varphi = (\varphi_1, \ldots, \varphi_N)$ is composed of the bid functions of all the producers.

What kind of *admissible bids?*

i) a *cumulative (unit price) bid function* $\psi_i(q_i)$ is generated by a finite set $(k = 1, \dots, N_k)$ of *block offers* with each block being characterized by a couple (*quantity,unit price*)$= (q_i^k, p_i^k)$. This cumulative bid function is an increasing step function given by

$$k_i := \psi_i(0) = p_i^1 \quad \textit{and} \quad \psi_i(q_i) := p_i^k \quad \textit{if } q_i \in]q_i^k, q_i^{k+1}]. \tag{1}$$

$\Rightarrow$ the revenue bid function φ_i is thus a piecewise linear function.

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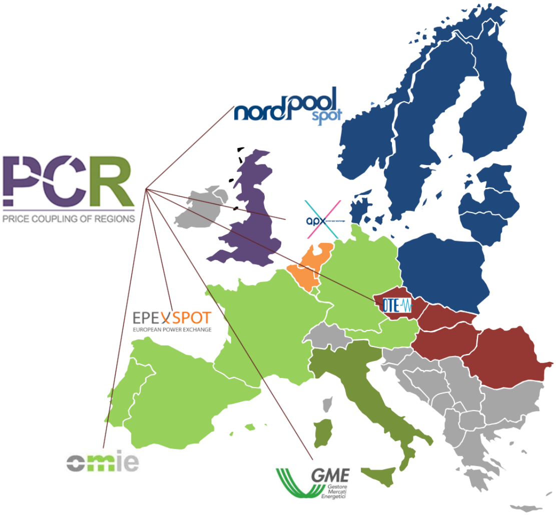
ii) a piecewise linear (unit price) bid function $\psi_i(q_i)$ is defined on $[0, \bar{Q}_i]$ by

$$k_i := \psi_i(0) = p_i^1 \quad \textit{and} \quad \psi_i(q_i) := \alpha_i^k q_i + \beta_i^k \quad \textit{if } q_i \in]q_i^k, q_i^{k+1}], \tag{2}$$

where $\mathcal{Q}_i = \{(q_i^k, p_i^k) : k = 1, \ldots, N_k\}$ is a family of couples *(quantity,unit price)* and the coefficients $\alpha_i^k = [p_i^{k+1} - p_i^k]/[q_i^{k+1} - q_i^k]$ and $\beta_i^k = p_i^k q_i^{k+1} - p_i^{k+1} q_i^k$.
$\Rightarrow$ the revenue bid function φ_i is thus a piecewise quadratic function.


nordpool spot
PCR
PRICE COUPLING OF REGIONS
apx
OTE
EPEXSPOT
EUROPEAN POWER EXCHANGE
omie
GME
Gestore Mercati Energetici

In the case of cumulative box bid

Thus the electricity market model consists in:

Finding a piecewise linear $\varphi = (\varphi_1, \dots, \varphi_n)$ solution of

$$(P_i) \qquad \max_{\varphi_i, q_i} \varphi_i(q_i) - Cost_i(q_i)$$
$$s.t. \begin{cases} q \text{ solves } ISO(\varphi) \\ \varphi_i \text{ is admissible piecewise linear,} \end{cases}$$

More precisely...

Thus the producer's optimization problems becomes (P_i), $i = 1, \ldots, N$,

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$$s.t. \begin{cases} q \text{ solves } ISO(\varphi) \\ \varphi_i \in C_i, \end{cases}$$

where the set of admissible bids C_i is given by

$$C_i = \left\{ u_i : \mathbb{R} \to \mathbb{R} \text{ such that } u_i(q_i) = \int_0^{q_i} \psi_i(q)dq + p_i^1 \text{ with } \psi_i \text{ such that } \left\{ \begin{array}{c} \psi_i \text{ cumulative box unit bid} \\ \text{function and (H) is satisfied} \end{array} \right. \right\}$$

where $\{(q_i^k, p_i^k) : k = 1, \ldots, N_k\}$ is a given family of of couples *(quantity,unit price)* satisfying

$$(H) \qquad \begin{cases} q_i^1 = 0 \quad \text{and } q_i^{N_k} = \bar{Q}_i \\ \forall\, k = 1, \ldots, N_k - 1, \quad q_i^k < q_i^{k+1} \quad \text{and } p_i^k < p_i^{k+1}. \end{cases}$$

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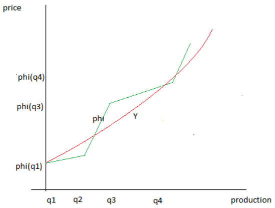
But the main problem is...non-smoothness

Solution?



approx. by quadratic bids

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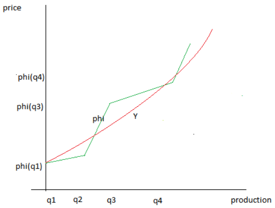

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Thus the producer's optimization problems become (P_i), $i = 1, \ldots, N$,: Find a quadratic function $y = (y_1, \ldots, y_n)$ solution of

$$(P_i) \qquad \max_{y_i, q_i} y_i(q_i) - Cost_i(q_i)$$
$$s.t. \begin{cases} q \ solves \ ISO(y) \\ y_i \in K_i \ \ (is\ a\ positive\ quadratic\ bid\ function), \end{cases}$$

$$K_i := \left\{ y_i : q_i \mapsto a_i q_i^2 + b_i q_i + c_i \text{ with } \ a_i > 0 \right\}$$

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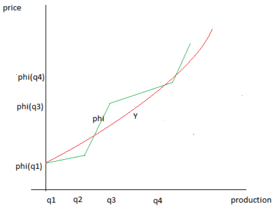

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But then there is....no longer connexion with real life bids

summarizing...

real bids	bids in model	producer's problems	
φ_i p. lin.	φ_i p. lin.	$\max_{\varphi_i, q_i} \varphi_i(q_i) - Cost_i(q_i)$ s.t. $\begin{cases} q \text{ solves } ISO(\varphi) \\ \varphi_i \text{ is admissible piecewise linear} \end{cases}$	$\Rightarrow$ nonsmoothness
φ_i p. lin.	$y_i \in K_i$ (pos. quad. bid)	$\max_{y_i, q_i} y_i(q_i) - Cost_i(q_i)$ s.t. $\begin{cases} q \text{ solves } ISO(y) \\ y_i \in K_i \end{cases}$	$\Rightarrow$ not a real life model

II- Non-self Quasivariational Inequalities

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What it is?

Why to consider that?

Let C be a non-empty subset of $\mathbb{R}^n$. Given two set-valued maps $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows C$, the quasi-variational inequality problem QVI(T, K) consists in finding $x \in C$ such that

$$x \in K(x) \quad \textit{and} \quad \exists x^* \in T(x) \textit{ with } \langle x^*, y - x\rangle \geq 0, \ \forall y \in K(x).$$

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Now what happens if the constraint map K is with values possibly not included in C?

$$K : C \rightrightarrows \mathbb{R}^n$$

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$$K : C \rightrightarrows \mathbb{R}^n$$

- if $K(C) \subsetneq C$ then, asking the solution to be a fixed point of K can be too demanding
- extreme situation: no solution if $K(C) \cap C = \emptyset$

A new concept of solution

Definition

Let C be a non-empty subset of $\mathbb{R}^n$, and $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows \mathbb{R}^n$ be two set-valued maps. A point $\bar{x}$ of C is said to be a *projected solution* of the quasi-variational inequality QVI(T, K) iff there exists $\bar{y} \in \mathbb{R}^n$ such that:

a) $\bar{x}$ is a projection of $\bar{y}$ on C;

b) $\bar{y}$ is a solution of the Stampacchia variational inequality $S(T, K(\bar{x}))$, that is, $\bar{y} \in K(\bar{x})$, and

$$\text{there exists } \bar{y}^* \in T(\bar{y}) \text{ such that } \langle \bar{y}^*, z - \bar{y} \rangle \geq 0, \quad \forall z \in K(\bar{x}).$$

The set of projected solutions will be denoted by $PQVI(T, K)$

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Any (classical) solution is a projected solution:

$$QVI(T, K) \subset PQVI(T, K).$$

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Note the variational inequality depends on the expected "projected solution"

A simple example

Let us consider the subset $C = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1 \text{ and } x + y \geq 1\}$ of $\mathbb{R}^2$ and the constraint map $K : C \rightrightarrows \mathbb{R}^2$, defined by

$$K(x, y) := \left\{ \frac{2}{\|(x, y)\|}(x, y) + (u, v) \ : \ 0 \leq u \leq 1, \ 0 \leq v \leq 1 \right\}.$$

- This set-valued map K is clearly non-self since $C \cap K(C) = \emptyset$;

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- but it has the following set of projected solutions:

$$\mathcal{P} = \{(1, 0), (1, 1), (0, 1)\};$$

A simple (modified) example

Let us consider the subset $C = \{(x,y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1 \text{ and } x + y \geq 1\}$ of $\mathbb{R}^2$ and the constraint map $K : C \rightrightarrows \mathbb{R}^2$, defined by

$$K(x,y) := \left\{ \frac{\sqrt{2}}{\|(x,y)\|}(x,y) + (u,v) \ : \ 0 \leq u \leq 1,\ 0 \leq v \leq 1 \right\}.$$

- it has the same set of projected solutions:

$$\mathcal{P} = \{(1,0),(1,1),(0,1)\};$$

- and the unique (classical) solution $(\bar{x}, \bar{y}) = (1,1)$.

III- Existence of projected solutions

A first existence result

Theorem

Let C be a non-empty, closed and convex subset of $\mathbb{R}^n$. Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows \mathbb{R}^n$ be two set-valued maps where $K(C)$ is relatively compact. Then, QVI(T, K) admits at least a projected solution if the following properties hold:

(i) *K is closed, lower semicontinuous and convex valued map with $\mathrm{int} K(x) \neq \emptyset$ for all $x \in C$;*

(ii) *T is locally upper sign-continuous or lower sign-continuous on $\mathrm{conv} K(C)$;*

(iii) *T is pseudomonotone on $\mathrm{conv} K(C)$.*

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(iii) *T is pseudomonotone on $\mathit{conv} K(C)$.*

Recall that a set-valued operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called a *lower sign-continuous* on a convex subset $K \subseteq \mathbb{R}^n$ iff for any $x, y \in K$,

$$\forall t \in]0,1[, \quad \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x \rangle \geq 0 \Rightarrow \inf_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0,$$

Proof is based on

Theorem (Lassonde (90))

Let K be a non-empty and convex subset of a locally convex topological vector space X. Suppose that $\Gamma : K \rightrightarrows K$ is a Kakutani factorizable set-valued map such that $\Gamma(K)$ is relatively compact. Then, Γ has a fixed point.

A set-valued map $\Gamma : K \rightrightarrows K$ is *Kakutani factorizable* if $\Gamma = \Gamma_N \circ \Gamma_{N-1} \circ \cdots \circ \Gamma_0$, that is, if there is a diagram $\Gamma : K = K_0 \overset{\Gamma_0}{\rightrightarrows} K_1 \overset{\Gamma_1}{\rightrightarrows} K_2 \rightrightarrows \cdots \overset{\Gamma_N}{\rightrightarrows} K_{N+1} = K$, where for $i = 0, 1, \cdots N$, each Γ_i is a non-empty, compact and convex valued upper semi-continuous set-valued map and K_i is a convex subset of X.

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Instead of using this theorem in the proof, one can apply a Kakutani fixed point theorem to the map $G : C \times K(C) \to C \times K(C)$ defined by $G(x, y) := (P_C(y), S(T, K(x)))$, where $P_C(y)$ is the projection set of y on C. However, then convexity of the set $K(C)$ would be required in addition to the assumptions of the theorem.

Another existence result

Theorem

Let C be a non-empty, closed and convex subset of $\mathbb{R}^n$. Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows \mathbb{R}^n$ be two set-valued maps, where $K(C)$ is relatively compact. Then, $QVI^*(T, K)$ admits at least a projected solution if the following properties hold:

(i) *K is a closed, lower semi-continuous and convex valued map with $intK(x) \neq \emptyset$, for all $x \in C$;*

(ii) *T is quasimonotone, locally upper sign-continuous and dually lower semi-continuous on $convK(C)$.*

Recall that T is called *dually lower semi-continuous* on a set K iff, for any $x \in K$ and any sequence $(y_k)_k$ of K with $y_k \to y$, the following implication holds:

$$\liminf_k \sup_{y_k^* \in T(y_k)} \langle y_k^*, x - y_k \rangle \leq 0 \Rightarrow \sup_{y^* \in T(y)} \langle y^*, x - y \rangle \leq 0.$$

IV- Back to GNEP and electricity market

General GNEP context

Definition

For any $\nu = 1, \dots, p$, let C_ν be a non-empty subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ and $K_\nu : C \rightrightarrows \mathbb{R}^{n_\nu}$, where $C = \prod_{\nu=1}^{p} C_\nu$. A point $\bar{x} := (\bar{x}^1 \dots, \bar{x}^p)$ of $C = \prod_\nu C_\nu$ is said to be a *projected solution* of the generalized Nash equilibrium problem $GNEP(\theta_\nu, K_\nu)$ iff there exists $\bar{y} := (\bar{y}^1 \cdots, \bar{y}^p) \in \mathbb{R}^n$ such that:

a) $\bar{x}$ is a projection of $\bar{y}$ on C;

b) $\bar{y}$ is a solution of the Nash equilibrium problem defined by the functions $(\theta_\nu)_\nu$ and the constraint sets $(K_\nu(\bar{x}))_\nu$, that is, for any ν, $\bar{y}^\nu \in K_\nu(\bar{x})$ is a solution of the following optimization problem

$$P_\nu(\bar{y}^{-\nu}, \bar{x}) \qquad \min_{y^\nu} \theta_\nu(y^\nu, \bar{y}^{-\nu}), \quad \textit{subject to} \quad y^\nu \in K_\nu(\bar{x}).$$

Existence of projected Nash equilibria

Theorem

For any $\nu = 1, \ldots, p$, *let* C_ν *be a non-empty, closed and convex subset of* $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ *and* $K_\nu : C = \prod_{\nu=1}^{p} C_\nu \rightrightarrows \mathbb{R}^{n_\nu}$. *Then, the GNEP*$(\theta_\nu, K_\nu)$ *admits a projected Nash Equilibrium* $\bar{x} \in C$ *if*

a) *the functions* θ_ν *are continuously differentiable and convex with respect to the* x^ν *variable;*

b) *for each* ν, *the maps* K_ν *are closed and lower semi-continuous with* $K_\nu(C)$ *being relatively compact;*

c) *for each* ν, *the maps* K_ν *are either single-valued or convex valued map with* $\mathrm{int}K_\nu(x) \neq \emptyset$, $\forall x \in C$.

In the case of electricity market model...

real bids	bids in model	producer's problems	
φ_i p. lin.	φ_i p. lin.	$\max_{\varphi_i, q_i} \varphi_i(q_i) - Cost_i(q_i)$ s.t. $\begin{cases} q \text{ solves } ISO(\varphi) \\ \varphi_i \text{ is admissible piecewise linear} \end{cases}$	$\Rightarrow$ nonsmoothness
φ_i p. lin.	$y_i \in K_i$ (pos. quad. bid)	$\max_{y_i, q_i} y_i(q_i) - Cost_i(q_i)$ s.t. $\begin{cases} q \text{ solves } ISO(y) \\ y_i \in K_i \end{cases}$	$\Rightarrow$ not a real life model
φ_i p. lin.	$y_i \in K_i(\varphi)$	$\max_{\mathbf{y_i}, q_i} y_i(q_i) - Cost_i(q_i)$ s.t. $\begin{cases} q \text{ solves } ISO(\varphi) \\ y_i \in K_i(\varphi) \end{cases}$	$\Rightarrow$ non self constraint map

where $K_i(\varphi) := \left\{ y_i : q_i \mapsto a_i q_i^2 + b_i q_i + c_i \text{ with } a_i > 0 \text{ and } c_i \geq \mathbf{p_i^1} \right\}$.

Projected solution for the bid process

It consists in finding a vector of bid functions $\bar{\varphi} = (\bar{\varphi}_1, \dots, \bar{\varphi}_N)$, for which there exists a vector of quadratic bid functions $\bar{y} = (\bar{y}_i)_i$, characterized by the matrix $((\bar{a}_i, \bar{b}_i, \bar{c}_i))_i$, such that:

a) the vector of bid functions $\bar{\varphi}$ is, between all possible vectors of bid functions of $C = \Pi_{i=1}^N C_i$, the best approximation in the sense of L^2-norm of the vector of quadratic bid functions $\bar{y}$;

$$\inf_{\varphi \in C} \sum_{i=1}^{N} \int_0^{\bar{Q}_i} |\bar{y}_i(q_i) - \varphi(q_i)|^2 \, dq_i,$$

or in other words, $\bar{\varphi}$ is a projection of $\bar{y}$ on C.

b) for each producer i, looking for its maximum benefit, $\bar{y}_i : q_i \mapsto a_i q_i^2 + b_i q_i + c_i$ solves the following optimization problem

$$P_i(\bar{y}_{-i}, \bar{\varphi}) \qquad \begin{array}{ll} \max\limits_{y_i, q_i} & y_i(q_i) - (A_i q_i^2 + B_i q_i) \\ s.t. & y_i \in K_i(\bar{\varphi}) \text{ and } q = (q_j)_{j \in \mathcal{N}} \text{ solves } ISO(y_i, \bar{y}_{-i}). \end{array} \tag{3}$$

QVI reformulation

Actually, under some suitable additional conditions, such a vector of bid functions $\bar{\varphi}$ will also be a projected solution of the quasi-variational inequality QVI(T, K) for the maps K and T defined as follows:

$K : C \rightrightarrows L^2([0, \bar{Q}], \mathbb{R})$ *is defined by* $K(\varphi) := \Pi_{i=1}^{N} K_i(\varphi)$

(where $C = \Pi_{i=1}^{N} C_i$ and $Q = \max_i Q_i$) and the map is defined as

$T : L^2([0, \bar{Q}], \mathbb{R}) \rightrightarrows L^2([0, \bar{Q}], \mathbb{R})$ *is given by* $T(y) := \prod_{i=1}^{N} \nabla_i \theta_i(\cdot, y_{-i})(y_i)$

where $\theta_i(\cdot, y_{-i})(y_i) := (a_i q_i^2(y) + b_i q_i(y) + c_i) - (A_i q_i^2(y) + B_i q_i(y))$.

An example of electricity market model

We assume that, for any i:

1) the approximated bid function $y_i = a_i q_i^2 + b_i q_i + c_i$ of the producer i is such that
   a) $a_i = A_i$, which means that the bid curve y_i is forced to be "relatively close" to the curve of real cost of production $A_i q_i^2 + B_i q_i$;
   b) b_i is bounded, $b_i \in [\underline{b}_i, \overline{b}_i]$, where $0 \leq \underline{b}_i \leq \overline{b}_i$;
   c) $c_i = p_i^1$, that is, the minimal value of the bid curve y_i is equal to the minimal value p_i^1 at which producer i is willing to produce electricity;
2) $0 < q_i < \bar{Q}_i$, which means that each producer of the market is active (produces electricity) at equilibrium but none of them reaches his maximum capacity of production.

Then there exist a vector $\bar{\varphi}$ of revenue bid functions and a vector $\bar{y}$ of quadratic bids such that, at the same time, $\bar{y}$ is a Nash equilibrium associated to the family of problem (3) and $\bar{\varphi}$ is the "real bid" which is the closest to $\bar{y}$.

V- Quasi-optimization problems

Definition

It corresponds actually to a constraint optimization problem, in which the constraint set depends on the solution.

This concept has been introduced in [Facchinei-Kanzow (2007)].

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Let C be a non-empty subset of $\mathbb{R}^n$. Now, for a given real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ and a set-valued operator $K : C \rightrightarrows C$, the quasi-optimization problem QOpt(f, K) consists in finding $x_0 \in C$ such that

$$x_0 \in K(x_0) \quad \text{and} \quad f(x_0) = \min_{z \in K(x_0)} f(z).$$

Existence of projected solutions

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ *be a continuous quasiconvex function such that* $intS_a \neq \emptyset$ *for all* $a > \inf f$. *Suppose that :*

- C *is a non-empty, closed and convex subset of* $\mathbb{R}^n$
- $K : C \rightrightarrows \mathbb{R}^n$ *is a set-valued map such that* $K(C)$ *is relatively compact and* $convK(C) \subseteq \mathbb{R}^n \setminus \arg\min_{\mathbb{R}^n} f$.

Then, there exists at least a projected solution to $QOpt(f, K)$ *if the following conditions hold:*

(a) K *is closed, lower semi-continuous and convex valued map with* $intK(x) \neq \emptyset$ *for all* $x \in C$;

(b) *The normal operator* N_f^a *is dually lower semi-continuous on* $convK(C)$.

VI- Ongoing results on computational aspects

Aim...

To compute some projected solutions of set-valued quasi-variational inequalities.

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To compute some projected solutions of set-valued quasi-variational inequalities.
Formally, the naive algorithm for finding a Projected solution is the following:

Algorithm for Projected solution

(Initialization) Choose $x_0 \in \bar{C}$, *set* $k := 1$ *and choose* ε ;

Find $y_1 \in S(T, K(x_0))$ *(using PATH solver);*
Compute $x_1 = P_{\bar{C}}(y_1)$;
while $\|x_{k-1} - x_k\| \leq \varepsilon$ **do**
  Find $y_k \in S(T, K(x_{k-1}))$ *(using PATH solver);*
  Compute $x_k = P_{\bar{C}}(y_k)$;
  $k \to k+1$
end

Assumption (H) on the constraint map

$C \subset \mathbb{R}^n$ is a nonempty subset and $K : C \rightrightarrows \mathbb{R}^n$ is a set-valued map with nonempty closed convex values with a special structure as

$$K(\lambda) = P \cap \{x : \langle a, x \rangle \leq h(\lambda)\}$$

where $P \subset \mathbb{R}^n$ is a polyhedral set given as

$$P = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq b_i\}$$

where $a \in \mathbb{R}^n$ and h is a function from C to $\mathbb{R}$.

A convergence result

Theorem

Let C be a nonempty compact convex subset of $\mathbb{R}^n$ and $x_0 \in C$. Assume that

(i) *The map $K : C \rightrightarrows \mathbb{R}^n$ is nonempty closed and compact valued with structure (H). Consider $I = \{i : \widehat{(a_i, a)} \in [-\pi/2, \pi/2]\}$, where $\widehat{(a_i, a)}$ is angle between a_i and a. Assume that for all $i \in \{1, 2, .., p\}/I$, $\widehat{(a_i, a)} \geq \pi/6$. Let h to be k_{λ_0}-locally lipschitz at $\lambda_0 \in C$ with $k_{\lambda_0} \in]0, 1/2[$.*

(ii) *The map $T : \mathbb{R}^n \to \mathbb{R}^n$ be a α-strongly monotone on $\mathbb{R}^n$ and L-lipschitz function on X with $\alpha = 1$ and $L = 1$, where X is closed convex neighborhood of $x_0 \in K(x_0)$. Fixing $\gamma \in]0, \alpha/L^2]$, then*
(a) there exist a neighborhood U of λ_0 and $\bar{k} \in]0, 1[$ such that,

$$\|S(T, K(x)) - S(T, K(x^{'}))\| \leq \bar{k}\|x - x^{'}\|, \quad \forall \quad x, x^{'} \in U \cap C. \tag{4}$$

(b) Consider a closed set $\bar{C} \subset C \cap U$ and $x_{n+1} = G(x_n)$, where $G = G_1 \circ G_0$ with $G_0(x) = S(T, K(x))$ and $G_1(x) = P_{\bar{C}}(x)$. Then the sequence $\{x_n\}$ converges to a point in $PQVI(T, K)$.

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