**Weierstrass Institute for**
**Applied Analysis and Stochastics**

# A probabilistic approach to optimization problems in gas transport networks

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## Can a certain amount of gas be transported by a given network?

Clearly, for a single pipeline with one entry and one exit on the ends, everything might be easy. What about complex networks?

Topics of this talk

- Nomination validation in stationary gas networks
- Maximization of booking capacities under probabilistic set up
- Optimization problems with nonlinear probabilistic constraints

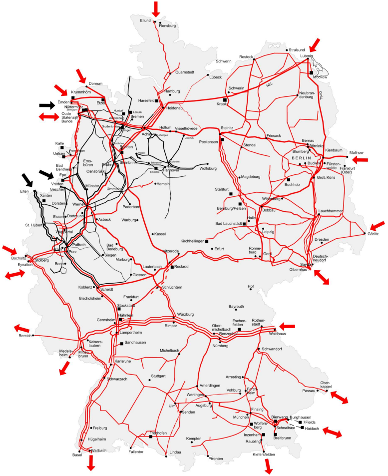


Figure: German H-gas and L-gas network system

General optimization problem

$$\boxed{\min\left\{f(x)\,\middle|\,g(x,\xi)\geq 0,\ x\in X\right\}}$$

Parameter $\xi$ fixed $\Longrightarrow$ LP, NLP, MIP depending on data

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- **Robust optimization** $\min\left\{f(x) \,\middle|\, g(x,\xi) \geq 0,\ x \in X,\ \forall \xi \in \Xi\right\}$

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- **Stochastic optimization**

1. Recourse model $\min\left\{f(x)+\mathbb{E}_{\mathbb{P}}\Phi(x,\xi)\,\middle|\,x\in X\right\}$ with

$$\Phi(x,\xi):=\inf\left\{\langle q,y\rangle\,\middle|\,y\in\mathbb{R}^m,\ Wy+g(x,\xi)\geq 0\right\}$$

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2. Chance constraints $\min\left\{f(x)\,\middle|\,\mathbb{P}\big(g(x,\xi)\geq 0\big)\geq p,\ x\in X\right\}$

Stochastic optimization problem with probabilistic constraints:

$$\min \{ f(x) \mid \varphi(x) \geq p, \; x \in X \}$$

$\varphi(x) := \mathrm{P}\big(g(x,\xi) \leq 0\big)$ probability function

$\xi$ multivariate continuously distributed random vector

$p \in (0,1]$ probability level

$\Longrightarrow$ *Robust solutions* with respect to *uncertain constraints*

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Challenges:

- Probabilistic constraints often nonsmooth and even nonconvex
- No analytical representation of the probability function $\varphi(\cdot)$
- An efficient dissolving requires (sub-)gradients of $\varphi(\cdot)$

- Analytical representation of the probability function

$$\varphi(x) = \mathbb{P}\big(g(x,\xi) \le 0\big)$$

using the spheric-radial decomposition of Gaussian distributions:

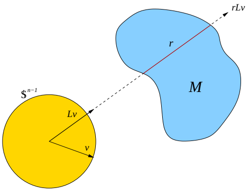



Figure: Spheric-radial decomp. for $M := \{\xi \,|\, g(x,\xi) \le 0\}$

**Spheric-Radial Decomposition**

Let $\xi \sim \mathcal{N}(0,\Sigma)$ be $n$-dimensional Gaussian distributed with zero mean and positive definite covariance matrix $\Sigma = LL^\top$. Then we have:

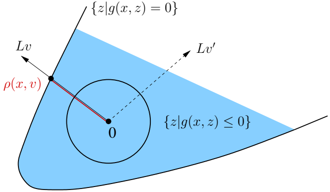
$$\varphi(x) = \int_{\mathbb{S}^{n-1}} \mu_\chi\{r \ge 0 \,|\, g(x, rLv) \le 0\} d\mu_\eta(v),$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $\mu_\eta$ denotes the law of uniform distribution on it, $\mu_\chi$ is the law of $\chi$-distribution with $n$ degree of freedom.

### One-dimensional and convex case

Let be $g(x,\cdot)$ continuous and convex and $x$ chosen such that $g(x,0) < 0$. Then we have

$$\varphi(x) \;=\; \int_{v\in\mathbb{S}^{n-1}} \chi_{\mathrm{cdf}}(\rho(x,v))d\mu_\eta(v)\,,$$




where $\rho(x,v) := \sup\{r \geq 0 \,|\, g(x, rLv) \leq 0\}$.

(Notice: If $\rho(x,v) < \infty$ we obtain that $g(x,\rho(x,v)Lv) = 0$)

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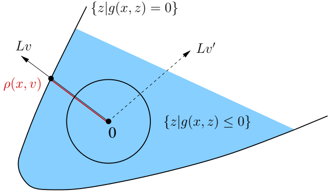


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**Theorem (Henrion, van Ackooij 2015)**

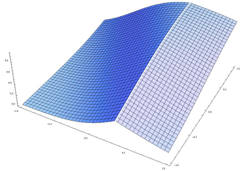
*Let be $g:\mathbb{R}^s\times\mathbb{R}^n\to\mathbb{R}$ continuous differentiable in both and convex in the second argument, $x$ chosen such that $g(x,0)<0$. If function $g(x,\cdot)$ satisfies a certain growth condition, then $\varphi(\cdot)$ is differentiable and for the gradient of $\varphi$ we obtain:*

$$\nabla\varphi(x) = \int_{v\in F(x)} -\frac{\chi_{\mathrm{pdf}}(\rho(x,v))}{\langle\nabla_\xi g(x,\rho(x,v)Lv),Lv\rangle}\nabla_x g(x,\rho(x,v)Lv)d\mu_\eta(v)$$

$(F(x) := \{v\in\mathbb{S}^{n-1}\,|\,\rho(x,v)<\infty\})$

## Nonsmoothness of the probability function

- Nice input data (e.g., smooth $g$, smooth distribution of $\xi$) do not imply nice properties (e.g., smoothness) of probability functions:



Example

$$\varphi(x) := \mathrm{P}(Mx + L\xi \geq b), \quad \xi \sim \mathcal{N}(0,1),$$

$$(M|L|b) = \left( \begin{array}{cc|c|c} 2 & 1 & -1 & 0 \\ -1 & 1 & 0 & -0.5 \end{array} \right)$$

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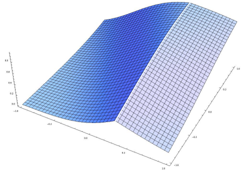

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- General case: $g : I\!R^s \times I\!R^n \to I\!R^m$ continuously differentiable in both arguments

$\Longrightarrow$ Derivatives in terms of Clarke subdifferential:

$$\partial^c \varphi(x) \subseteq \int_{v \in F(x)} \mathrm{Co} \left\{ -\frac{\chi_{\mathrm{pdf}}(\rho(x,v))}{\langle \nabla_\xi g_i(x, \rho(x,v) Lv), Lv \rangle} \nabla_x g_i(x, \rho(x,v) Lv) : i \in I(v) \right\} d\mu_\eta(v)$$

**SFB Transregio 154**

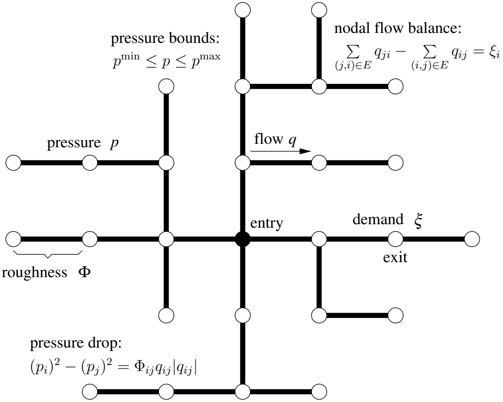
*Mathematical Modelling, Simulation and Optimization in Gas Networks*

Relevant problems for gas transport system operator

- Reliable satisfaction of random demands at exit points of the gas network
- Maximization and verification of booking capacities
- Optimal network design and optimal operation cost

**Key:** Analytical characterization of feasibility of nominations in stationary gas networks

- Given directed graph $G = (V, E)$ with $|V| = n + 1$ and $|E| = m \geq n$




**Network topology** given by *node-arc incidence matrix* $A$:

| | | |
|---|---|---|
| Kirchhoff 1: | $Aq = \xi$ | (1) |
| Kirchhoff 2: | $A^\top p^2 = -\Phi \lvert q \rvert q$ | (2) |
| Limits: | $p \in [p^{\min}, p^{\max}]$ | (3) |

**Stationary gas nets:** A demand vector $\xi$ admissible $\iff$ $\exists p, q : \; p, q$ satisfy (1)-(3)

# Explicit characterization of feasibility

- Elimination of all pressure and flow variables

$$A = \left( \begin{array}{c|c} a_B^\top & a_N^\top \\ \hline A_B & A_N \end{array} \right) \in \mathbb{R}^{n+1 \times m}$$ basis/non-basis decomposition of incidence matrix

TRR 154

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**Theorem**

*A balanced nomination vector $\xi$ is feasible, iff there is a $z$ such that*

$$\begin{aligned} A_N^\top h(\xi, z) &= \Phi_N |z| z & (1) \\ \min_{i=1,\ldots,|V|} \left[ (p_i^{max})^2 + h_i(\xi, z) \right] &\geq \max_{i=1,\ldots,|V|} \left[ (p_i^{min})^2 + h_i(\xi, z) \right] \\ (p_0^{min})^2 &\leq \min_{i=1,\ldots,|V|} \left[ (p_i^{max})^2 + h_i(\xi, z) \right] \\ (p_0^{max})^2 &\geq \max_{i=1,\ldots,|V|} \left[ (p_i^{min})^2 + h_i(\xi, z) \right] \end{aligned}$$

*Definition:* $h(u, v) := \left(A_B^\top\right)^{-1} \Phi_B \left| A_B^{-1}(u - A_N v) \right| \left( A_B^{-1}(u - A_N v) \right)$

$\Longrightarrow$ The complexity rises with the number of cycles $=$ number of non-basis variables



WIAS

Theorem requires $\rightarrow \quad \exists z : \; A_N^\top h(b, z) = \Phi_N |z| z$

With definition of $h(\cdot,\cdot)$ this is equivalent to solving the algebraic equation:

$$\mathcal{F}(b, z) := A_N^\top \left(A_B^\top\right)^{-1} \Phi_B \left| A_B^{-1}(b - A_N z)\right|^* - \Phi_N |z|^* = 0$$

Notation: $|a|^* := |a|a$ Variables: $z$ Parameters: $b$

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Analytical properties of $\mathcal{F}(b,\cdot)$

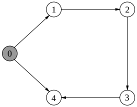
- $\mathcal{F}(b,\cdot) : \mathbb{R}^{|N|} \rightarrow \mathbb{R}^{|N|}$ is continuous and (strongly) coercive
- For every $b \in \mathbb{R}^{|V|-1}$ there exists a (unique) solution $z(b)$ with $\mathcal{F}(b, z(b)) = 0$
- System of $|N|$ multivariate polynomial equations of degree 2 with $|N|$ indeterminates
- Cycle Network: As long as cycles are disjoint, for fixed $b$, $\mathcal{F}(b, z) = 0$ separates into "highschool quadratic equations"

$\Longrightarrow$ Parametric Solution: $z(\cdot) : \mathbb{R}^{|V|-1} \rightarrow \mathbb{R}^{|N|}$



WIAS

- Computing the probability of feasible nominations under Gaussian distribution




$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

**Figure:** Network graph and incidence matrix of the network

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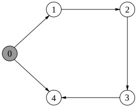


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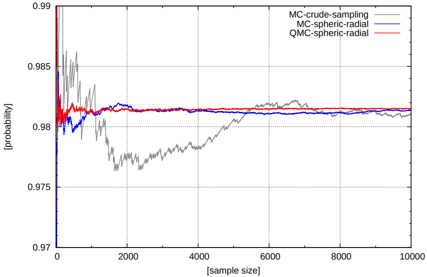


**Figure:** Average of probability vs. sample size for both crude sampling and spheric-radial decomposition. Monte Carlo (MC) and Quasi-Monte Carlo (QMC) number generators have been used.

- Formulation of gas specific optimization problems in terms of:
  - uncertain parameters (e.g. exit demand, roughness, free booked capacities)
  - constraints representing technical feasibility (depending on net parameters)

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**Probabilistic approach**

$$\mathbb{P}_z\big(g_i(x, z) \geq 0\big) \geq p$$

**Robust approach**

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**New class of problems**

**Joint robust/probabilistic model**

$$\min \left\{ f(x) \,\middle|\, \mathbb{P}_\xi\big(g(x,y,\xi) \geq 0 \quad \forall y \in \mathcal{U}\big) \geq p \right\}$$

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Challenge: Problem involves an infinite system of random constrains!

**Problem:** Maximizing booking capacities in a robust/probabilistic setting

Demand (exit load) $\xi$ is of stochastic nature (due to historical data). A network operator aims to enlarge capacities while guaranteeing a reliable network operation with high probability level $p$:

$$\mathrm{P}\Big(g_{kl}(\xi + y) \geq 0;\ k, l = 0, \ldots, |V|\Big) \geq p$$

The additional nomination $y$ is considered to be uncertain as well. In particular, we assume

$$y \in [0, x],$$

where $x$ is the maximal available free booked capacity. This motivates to consider a joint robust/probabilistic model for maximizing available *booking capacities* in the network:

$$\max\ c^\top x \qquad \text{s.t.}$$
$$\mathrm{P}\Big(g_{kl}(\xi + y) \geq 0;\ k, l = 0, \ldots, |V|;\ \forall y \in [0, x]\Big) \geq p \quad (x \geq 0)$$

## Constraints: Single entry tree network

A demand vector $z$ admissible, if and only if it holds

$$g_{kl}(z) := (p_k^{\max})^2 + \alpha_k(z) - (p_l^{\min})^2 - \alpha_l(z) \geq 0 \qquad (k, l = 0, \ldots, |V|)\,,$$

where $\alpha_k(\cdot)$ is defined as

$$\alpha_k(z) := \sum_{e \in \Pi(k)} \Phi_e \left( \sum_{t \in V : t \geq h(e)} z_t \right)^2 \qquad (k = 0, \ldots, |V|)\,.$$

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## Theorem (passive network, tree, single entry)

*For the feasibility set* $M(x) := \{z \,|\, g_{kl}(z + x) \geq 0\}$ *we have:* $M(x) \subseteq A \cup B$, *where*

*(i)* $A$ *satisfies the Rank-2-Constraint-Qualification (R2CQ)*

*(ii)* *For* $B$ *it holds* $\text{codim}(B) = 2$

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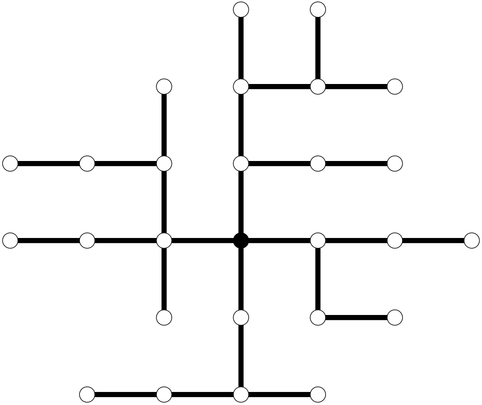
$\Longrightarrow$ **Gradients of the probability function exist**

**Computation:** Free capacities for a network with $27$ nodes and random exit demand $\xi$:



**Assumption**

$$\xi \in \mathcal{TN}(\mu, \Sigma, [0, L])$$

Probability level

$$p = 0.97$$

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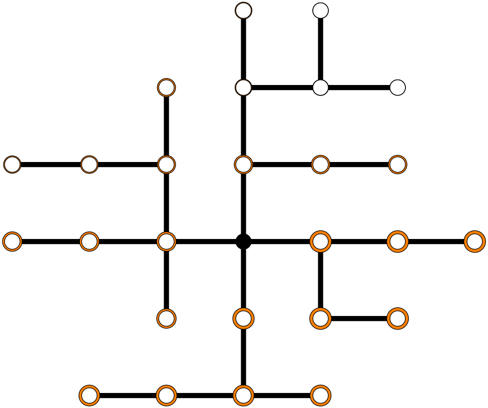


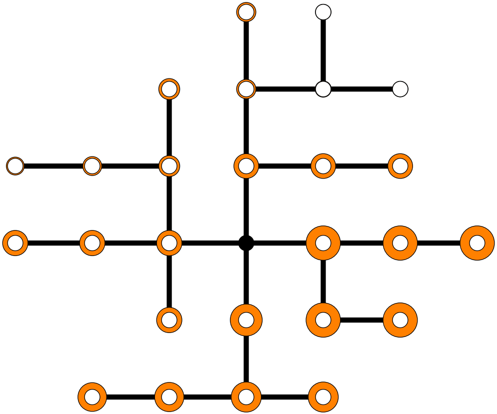
**Assumption**

$$\xi \in \mathcal{TN}(\mu, \Sigma, [0, L])$$

Probability level

$$p = 0.95$$

**Computation:** Free capacities for a network with $27$ nodes and random exit demand $\xi$:



**Assumption**

$$\xi \in \mathcal{TN}(\mu, \Sigma, [0, L])$$

Probability level

$$p = 0.90$$

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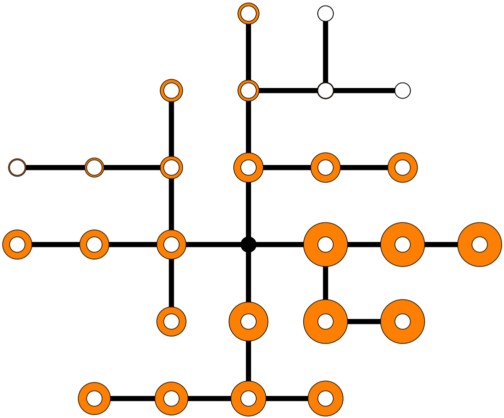


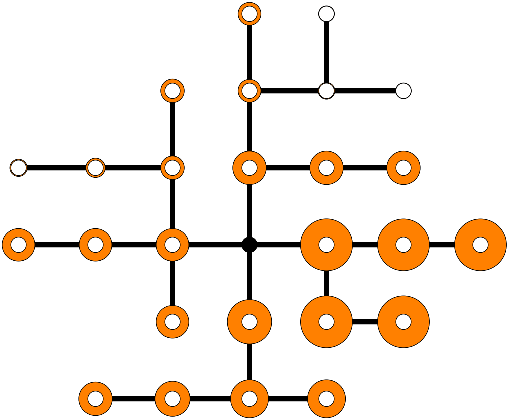
**Assumption**

$$\xi \in \mathcal{TN}(\mu, \Sigma, [0, L])$$

Probability level

$$p = 0.85$$

**Computation:** Free capacities for a network with $27$ nodes and random exit demand $\xi$:



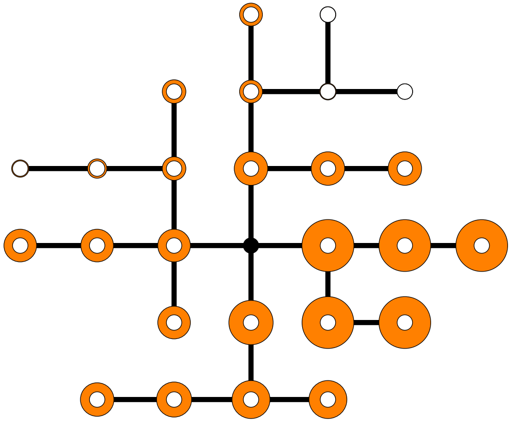
**Assumption**

$$\xi \in \mathcal{TN}(\mu, \Sigma, [0, L])$$

Probability level

$$p = 0.80$$

**Out-of-sample test:** Capacity problem with fixed probability level $p = 0.80$:



**Assumption**

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Optimal solution

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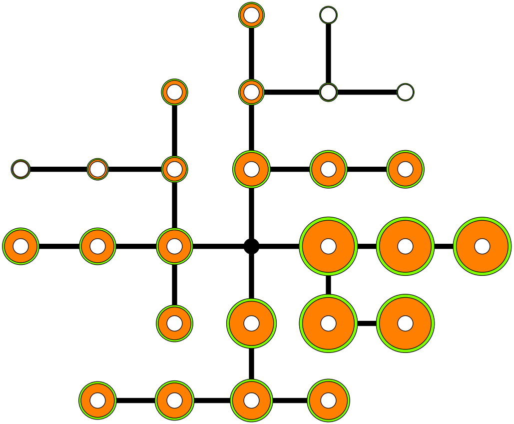

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*Sample[1] – feasible*

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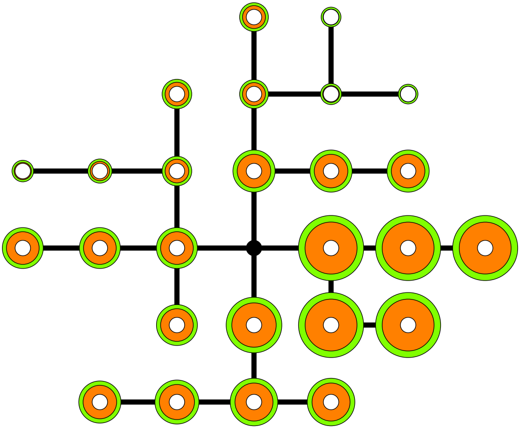

**Assumption**

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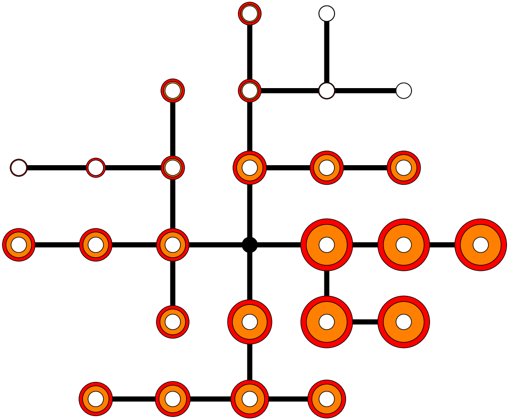
*Sample[2] – feasible*

**Out-of-sample test:** Capacity problem with fixed probability level $p = 0.80$:

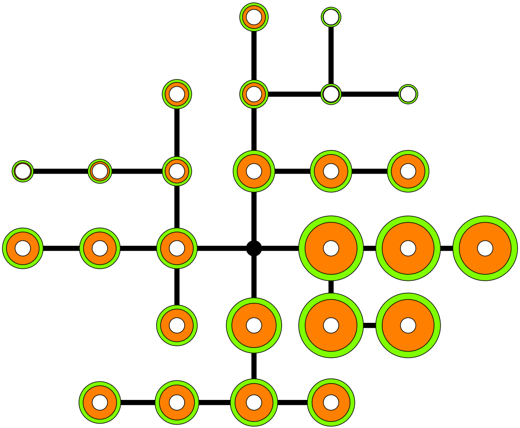


**Assumption**

$$\xi \in \mathcal{TN}(\mu, \Sigma, [0, L])$$

*Sample[3] – infeasible*

**Out-of-sample test:** Capacity problem with fixed probability level $p = 0.80$:

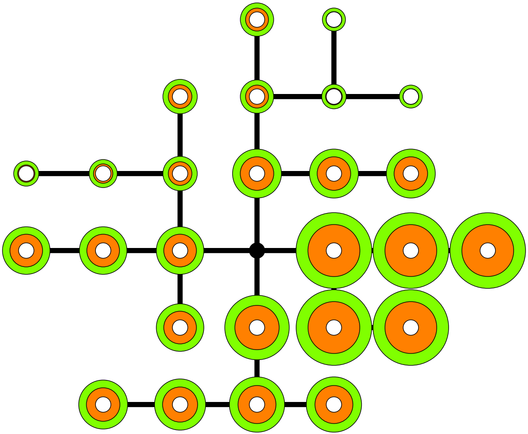


**Assumption**

$$\xi \in \mathcal{TN}(\mu, \Sigma, [0, L])$$

*Sample[4] – feasible*

**Out-of-sample test:** Capacity problem with fixed probability level $p = 0.80$:



**Assumption**

$\xi \in \mathcal{TN}(\mu, \Sigma, [0, L])$

*Sample[5] – feasible*

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