Financial valuation of storages and delivery contracts

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- Many financial contracts involve optionalities that give the counterparties some control over the cash-flows: American options, convertible/callable bonds, mortgages, delivery contracts in electricity etc.
- Most of financial mathematics, however, addresses contingent claims without optionalities.
- The best known exception is the pricing theory for American options but most of that is concerned with superhedging where the counterparties accept no risk.
- In practice, however, most trades expose both counterparties to risk (in addition to possibility of returns).
- Our aim is to study
  - optimal investment with options,
  - indifference pricing of options.

- Optimal investment and asset pricing are often treated as separate problems (Markovitz vs. Black–Scholes).
- In practice, valuations have been largely disconnected from investment and risk management. This lead to large losses during 2008 e.g. with credit derivatives.
- Building on convex stochastic optimization, we describe a unified approach to optimal investment, valuation and risk management.
- The resulting valuations
  - are based on hedging costs,
  - extend and unify financial and actuarial valuations,
  - reduce to "risk neutral valuations" for perfectly liquid securities.

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Let $\mathcal{M}$ be the linear space of adapted sequences of cash-flows on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$.

- The financial market is described by a convex set $\mathcal{C} \subset \mathcal{M}$ of claims that can be superhedged without cost (i.e. each $c \in \mathcal{C}$ is freely available in the financial market).
- In models with a perfectly liquid cash-account,

$$\mathcal{C} = \{c \in \mathcal{M} \mid \sum_{t=0}^{T} c_t \in C\}$$

  where $C \subset L^0(\Omega, \mathcal{F}_T, P)$ are the claims at T that can be hedged without cost [Delbaen and Schachermayer, 2006].
- Conical $\mathcal{C}$: [Dermody and Rockafellar, 1991], [Jaschke and Küchler, 2001], [Jouini and Napp, 2001], [Madan, 2014].

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Financial markets

Example 1 (The classical model) *In the classical perfectly liquid market model with a cash-account*

$$\mathcal{C} = \{c \in \mathcal{M} \mid \exists x \in \mathcal{N} : \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\}$$

which is a convex cone. This set has been extensively studied in the literature; see e.g. [Föllmer and Schied, 2004] or [Delbaen and Schachermayer, 2006] and their references.

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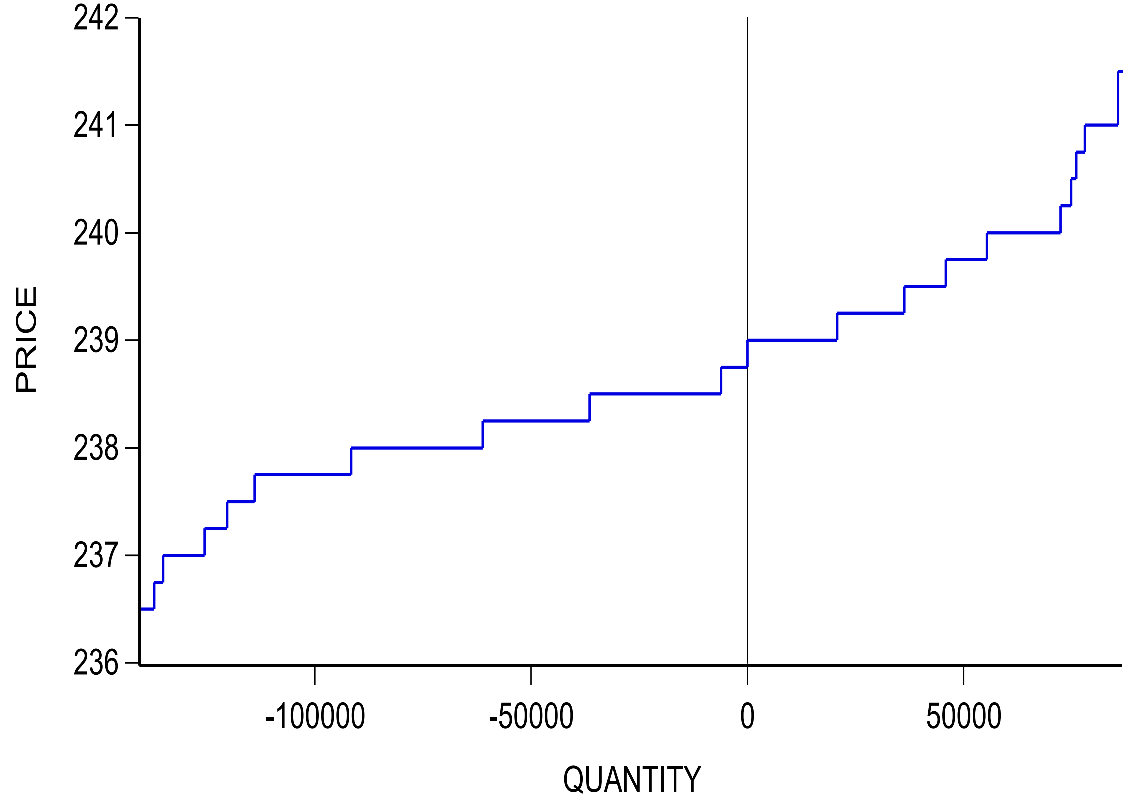
The limit order book of TDC A/S in Copenhagen Stock Exchange on January 12, 2005 at 13:58:19.43.



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- For duality theory, we would need the additional structure

$$\mathcal{C} = \{c \in \mathcal{M} \mid \exists x \in \mathcal{N} : \; (x, c) \in S \quad P\text{-a.s.}\},$$

where S is a random set taking values in $\mathbb{R}^n \times \mathbb{R}^{1+T}$.

- Recall that a set-valued mapping $S : \Omega \rightrightarrows \mathbb{R}^n \times \mathbb{R}^{1+T}$ is measurable if the inverse images

$$S^{-1}(O) = \{\omega \in \Omega \mid S(\omega) \cap O \neq \emptyset\}$$

of open sets $O \subset \mathbb{R}^n \times \mathbb{R}^{1+T}$ are measurable.

- We ignore this structure for now but it will become important when we get to optionalities.

Asset-Liability Management

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- Financial valuations are based on hedging costs.
- Consider an agent with liabilities $c \in \mathcal{M}$, access to $\mathcal{C}$ and a loss function $\mathcal{V} : \mathcal{M} \to \overline{\mathbb{R}}$ that measures disutility/regret/risk/... of delivering $c \in \mathcal{M}$. For example,

$$\mathcal{V}(c) = E \sum_{t=0}^{T} -u_t(-c_t).$$

- The optimum value of the hedging problem is

$$\varphi(c) := \inf_{d \in \mathcal{C}} \mathcal{V}(c - d)$$

- We assume that $\mathcal{V}$ is convex, nondecreasing and $\mathcal{V}(0) = 0$.

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Indifference pricing

- In a swap contract, an agent receives a sequence $p \in \mathcal{M}$ of premiums and delivers a sequence $c \in \mathcal{M}$ of claims.
- Examples:
  - Traditionally in mathematical finance,

    $$p = (1, 0, \dots, 0) \quad \text{and} \quad c = (0, \dots, 0, c_T).$$

  - Futures: $p = (0, \dots, 0, 1)$ and $c = (0, \dots, 0, c_T)$.
  - Swaps with a "fixed leg": $p = (1, \dots, 1)$, random c.
  - In credit derivatives (CDS, CDO, . . .) and other insurance contracts, both p and c are random.
- Claims and premiums live in the same space

$$\mathcal{M} = \{(c_t)_{t=0}^T \mid c_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R})\}.$$

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- If we already have liabilities $\bar{c} \in \mathcal{M}$, then

$$\pi(\bar{c}, p; c) := \inf\{\alpha \in \mathbb{R} \mid \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

  gives the least swap rate that would allow us to enter a swap contract without worsening our financial position.
- Similarly,

$$\pi^b(\bar{c}, p; c) := \sup\{\alpha \in \mathbb{R} \mid \varphi(\bar{c} - c + \alpha p) \leq \varphi(\bar{c})\} = -\pi(\bar{c}, p; -c)$$

  gives the greatest swap rate we would need on the opposite side of the trade.
- When $p = (1, 0, \ldots, 0)$ and $c = (0, \ldots, 0, c_T)$, we get an extension of the indifference price of [Hodges and Neuberger, 1989] to nonconical models.

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Define the super- and subhedging swap rates,

$$\pi_{\sup}(p;c) = \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}^\infty\}, \; \pi_{\inf}(p;c) = \sup\{\alpha \,|\, \alpha p - c \in \mathcal{C}^\infty\},$$

where $\mathcal{C}^\infty$ is the recession cone of $\mathcal{C}$. If $\mathcal{C}$ is conical, (like it usually is in math finance), $\mathcal{C}^\infty = \mathcal{C}$.

Theorem 2 *If $\pi(\bar{c}, p; 0) \geq 0$, then*

$$\pi_{\inf}(p;c) \leq \pi_b(\bar{c}, p; c) \leq \pi(\bar{c}, p; c) \leq \pi_{\sup}(p;c)$$

with equalities if $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$.

- Agents with identical views, preferences and financial position have no reason to trade with each other.
- Prices are independent of such subjective factors when $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$.

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Example 3 (The classical model) *Consider the classical perfectly liquid market model where*

$$\mathcal{C} = \{c \in \mathcal{M} \mid \exists x \in \mathcal{N} : \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\}$$

and $\mathcal{C}^\infty = \mathcal{C}$. *The condition* $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ *holds if there exist* $x \in \mathcal{N}$ *such that*

$$\sum_{t=0}^{T} c_t = \alpha \sum_{t=0}^{T} p_t + \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}.$$

The converse holds under the no-arbitrage condition. When $p = (1, 0, \ldots, 0)$ *this is the classical attainability condition.*

Financial contracts with optionalities

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- An option allows its owner to choose a sequence $c = (c_t)_{t=0}^T$ of cash-flows subject to the constraint that there is an exercise strategy $e = (e_t)_{t=0}^T$ with $(e, c) \in C$ for a given random set C.
- The values of c_t and e_t have to be chosen by time t.
- We assume e_t takes values in $\mathbb{R}^d$ so C is a set in $\mathbb{R}^{(1+T)(1+d)}$.

Example 4 *An American option on $X = (X_t)_{t=0}^T$ corresponds to*

$$C = \{(e, c) \,|\, c_t \le e_t X_t,\ \sum_{t=0}^T e_t \le 1,\ e_t \in \{0, 1\}\}.$$

The strategy e corresponds to a stopping time τ with $\tau = t$ if and only if $e_t = 1$.

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Below, we assume that the owner can buy/sell energy at market price X.

Example 5 *A delivery contract (swing option) with tariff K corresponds to*

$$C = \{(e,c) \,|\, c_t \le e_t(X_t - K),\ \sum_{t=0}^{T} e_t \le E,\ e_t \in [l_t, u_t]\}.$$

Example 6 *A storage with capacity E corresponds to*

$$C = \{(e,c) \,|\, c_t \le -\Delta e_t X_t,\ e_t \in [0,E],\ \Delta e_t \in [l_t, u_t]\}.$$

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Buyer's problem

Given access to the financial markets and to the payouts of the option, the buyer's ALM problem becomes

$$\text{minimize} \quad \mathcal{V}(c-d-d') \quad \text{over } \; d\in\mathcal{C},\; d'\in\mathcal{M}_C,$$

where

$$\mathcal{M}_C := \{c\in\mathcal{M} \,|\, \exists e\in\mathcal{N}:\; (e,c)\in C\;\; P\text{-a.s.}\}.$$

- This has the same structure as the earlier ALM problem.
- We will denote the optimum value by

$$\varphi_C(c) := \inf_{d\in\mathcal{C},\; d'\in\mathcal{M}_C} \mathcal{V}(c-d-d') = \inf_{d'\in\mathcal{M}_C} \varphi(c-d').$$

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The indifference swap rate for a long position in C is given by

$$\pi_l(\bar{c},p;C) := \sup\{\alpha\in\mathbb{R} \,|\, \inf_{c\in\mathcal{M}_C}\varphi(\bar{c}+\alpha p-c)\leq\varphi(\bar{c})\}.$$

If the infimum is attained for every $\bar{c}$ and $\alpha\in\mathbb{R}$ (we have reasonable conditions for this), this may be written as

$$\pi_l(\bar{c},p;C) = \sup_{c\in\mathcal{M}_C}\pi_l(\bar{c},p;c),$$

where

$$\pi_l(\bar{c},p;c) := \sup\{\alpha\in\mathbb{R} \,|\, \varphi(\bar{c}-c+\alpha p)\leq\varphi(\bar{c})\}$$

is the indifference rate for a long position in the swap (c,p).

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Theorem 2 thus gives

$$\sup_{c\in\mathcal{M}_C}\pi_{\inf}(c)\le\sup_{c\in\mathcal{M}_C}\pi_l(\bar c,p;c)\le\sup_{c\in\mathcal{M}_C}\pi_s(\bar c,p;c)\le\sup_{c\in\mathcal{M}_C}\pi_{\sup}(c).$$

In complete markets, the indifference rate is thus given by

$$\sup_{c\in\mathcal{M}_C}\pi_{\sup}(c),$$

which is independent of the buyer's views and risk preferences.

Seller's problem

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- The seller of the option does not know the counter party's strategy but only observes (c_t, e_t) at time $t = 0, \ldots, T$.
- Being Bayesian, the seller models the sequence (e, c) as an $\mathbb{R}^{(1+T)(1+d)}$-valued random variable on $(\Omega, \mathcal{F}, P)$.
- The seller's information at time t is thus given by the sigma-algebra $\mathcal{F}_t^{e,c} \subset \mathcal{F}$ generated by $\mathcal{F}_t$ and the random variables (c_s, e_s), $s = 0, \ldots, t$.
- This reduces the option to a nonoptional claim so we can simply apply the existing theory and techniques.

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- If the market is described by a random set S the seller's ALM problem can be written as

$$\begin{array}{ll} \text{minimize} & \mathcal{V}(c-d) \quad \text{over} \quad (x,d) \in \mathcal{N}^{e,c} \\ \text{subject to} & (x,d) \in C \;\; P\text{-a.s.} \end{array}$$

  where $\mathcal{N}^{e,c}$ denotes the feasible trading strategies adapted to the enlarged filtration $(\mathcal{F}_t^{e,c})_{t=0}^T$.
- We will denote the optimum value of the above by $\varphi^{e,c}$.
- The seller's indifference price is given by

$$\pi_s^{e,c}(\bar{c}, p; c) := \inf\{\alpha \in \mathbb{R} \,|\, \varphi^{e,c}(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\},$$

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If $\varphi^{e,c}(\bar{c}) = \varphi(\bar{c})$, Theorem 2 gives

$$\pi^{e,c}_{\inf}(p;c) \le \pi^{e,c}_{l}(\bar{c},p;c) \le \pi^{e,c}_{s}(\bar{c},p;c) \le \pi^{e,c}_{\sup}(p;c),$$

where (assuming, for simplicity, that S is conical)

$$\pi^{e,c}_{\sup}(p;c) := \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}^{e,c}\}$$

and

$$\mathcal{C}^{e,c} = \{d \in \mathcal{M} \,|\, \exists x \in \mathcal{N}^{e,c} : \ (x,d) \in S \quad P\text{-a.s.}\}.$$

Clearly,

$$\pi^{e,c}_{\sup}(p;c) \le \sup_{(e,c)\in L^0(C)} \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}^{e,c}\}.$$

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- How is the above related to the theory of American options?
- By Doob-Dynkin lemma, x_t is $\mathcal{F}_t^{c,e}$-measurable iff there is an $(\mathcal{F}_t \otimes \mathcal{B}_t)$-measurable function $\tilde{x}_t$ such that $x_t = \tilde{x}_t \circ g$. Here $g(\omega) := (\omega, c(\omega), e(\omega))$ and $\mathcal{B}_t$ is the sigma algebra on $\mathbb{R}^{(1+T)(1+d)}$ generated by the projections $(c, e) \mapsto (c^t, e^t)$.
- The representation $\tilde{x}_t$ is unique $P \circ g^{-1}$-almost surely.
- Thus, the space $\mathcal{N}^{c,e}$ is isomorphic to the space

$$\{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega \times \mathbb{R}^{(1+T)(1+d)}, \mathcal{F}_t \otimes \mathcal{B}_t, P \circ g^{-1})\},$$

which is a quotient space of the linear space $\tilde{\mathcal{N}}$ of functions

$$(\omega, c, e) \mapsto \tilde{x}(\omega, c, e)$$

such that $\tilde{x}_t$ is $\mathcal{F}_t \otimes \mathcal{B}_t$-measurable.

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We have

$$\pi^{e,c}_{\sup}(p;c) \le \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}^{e,c} \quad \forall (e,c) \in L^0(C)\}$$
$$\le \inf\{\alpha \in \mathbb{R} \,|\, \exists \tilde{x} \in \tilde{\mathcal{N}} : \ (\tilde{x}(e,c), c - \alpha p) \in S \text{ a.s. } \forall (e,c) \in L^0(C)\}.$$

Example 7 (American options) *In [Föllmer and Schied, 2004], a self-financing trading strategy $x^a \in \mathcal{N}$ whose value process (liquidation value) dominates X is called a superhedging strategy for X. Given such an x^a, the functions $\tilde{x}_t(\omega, e, c) = x^a_t(\omega)\mathbb{1}_{\{t<\tau\}}$ are $(\mathcal{F}_t \otimes \mathcal{B}_t)$-measurable and, for any $(e,c) \in L^0(C)$,*

$$x(\omega) = \tilde{x}(e(\omega), c(\omega), \omega)$$

superhedges c.

Summary

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- Optimal investment with liabilities (ALM) provides a unifying framework for economic valuations.
- Convex stochastic optimization allows for extending the classical theory to nonlinear market models with portfolio constraints, nonlinear illiquidity effects, etc.
- Convex duality (not discussed in this talk) extends the "fundamental theorem of asset pricing" to general convex market models and indifference pricing.
- Financial contracts with optionalities can be reduced to nonoptional ones.
- Our formulation extends the theory of American options to more general financial contracts, general convex market models and beyond superhedging.