Generalized differentiation of probability functions

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Outline

Probability constraints

- A probabilistic constraint is a constraint of the type

$$\varphi(x) := \mathbb{P}[g(x,\xi) \leq 0] \geq p, \tag{1}$$

where $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a map, $\xi \in \mathbb{R}^m$ a (multi-variate) random variable. They arise in many applications. For instance cascaded Reservoir management.

- We will however be interested in the situation:

$$\varphi(x) := \mathbb{P}[g_t(x,\xi) \leq 0,\ \forall t \in T] \geq p, , \tag{2}$$

where $g_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a map and T an “arbitrary index set”.

Unit commitment - probust

- In unit commitment problems under uncertainty, one may have to find appropriate generation levels while accounting for uncertainty on load and / or wind. This may lead to a classic probability constraint of the form

$$\varphi(x) := \mathbb{P}[Ax \geq \xi] \geq p. \tag{3}$$

- However defaults on generation may occur, leading to uncertainty on A. It may be so that such uncertainty is less well understood and it is more meaningful to consider “robust” ideas:
- we know of perturbations $A(u)$, for all $u \in \mathcal{U}$, with $\mathcal{U}$ the uncertainty set.
- Then one faces the “probust" constraint:

$$\varphi(x) := \mathbb{P}[A(u)x \geq \xi, \forall u \in \mathcal{U}] \geq p. \tag{4}$$

- See, e.g., [van Ackooij et al.(2016)].

Unit commitment - robility

- It is clear that any feasible point to:

$$\varphi(x) := \mathbb{P}[A(u)x \geq \xi, \forall u \in \mathcal{U}] \geq p. \tag{5}$$

- satisfies the “robility" constraint:

$$\varphi(x) := \mathbb{P}[A(u)x \geq \xi] \geq p, \forall u \in \mathcal{U} \tag{6}$$

- but the inverse need not hold. The latter may be seen to have a link with distributionally robust optimization.

Networks - Induced uncertainty

- In several management problems, an underlying network structure is present and ought to be accounted for.
- However the potentially arbitrary complex structure of the network “acts” on uncertainty (much like recourse).
- Uncertainty is actually a phenomenon occurring in nodes.
- Then uncertainty related to the network means, for instance, existence of a “feasible flow”.
- the probability constraint then reads: for sufficient random realizations, there exists a feasible flow.

Networks - Induced uncertainty II

- An interesting application is gas-networks, where under some structural properties on the network (tree structure or a few fundamental cycles): the implicit conditions can be recast as regular inequality systems (this is non-trivial, e.g., references in [González Gradón et al.(2017)]).
- The existence of uncertainty on friction coefficients leads again to probust constraints, since assuming knowledge of distributions of friction coefficients is not reasonable.

PDE constrained

- In certain optimization problems from engineering, e.g., optimal design of off-shore wind turbines, one deals with computing some optimal shape or structure while having to account for uncertainty.
- the given uncertainty could for instance represent stochastic loadings or environmental stress conditions
- by considering the Karhunen-Loève expansion of this uncertainty (e.g., stochastic field), one can argue that uncertainty is caused by a “finite dimensional random vector" (the uncertain coefficients in this expansion).
- However, the dynamics of the system are best described by a PDE.
- We refer to [Farshbaf-Shaker et al.(2017)] for details

PDE constrained II

- This gives for instance problems of the form :

$$\min_{x,u} \mathbb{E}\left(L(y(x,\omega), u(x))\right)$$

$$s.t. y(x,\omega) \text{ is solution to:}$$

$$-\nabla_x \cdot (\kappa(x)\nabla_x y(x,\omega)) = r(x,\omega), (x,\omega) \in D \times \Omega$$

$$n \cdot (\kappa(x)\nabla_x y(x,\omega)) + \alpha y(x,\omega) = u(x), (x,\omega) \in \partial D \times \Omega,$$

$$p \leq \mathbb{P}[\omega \in \Omega \ : y(x,\omega) \leq \bar{y}(x) \ \forall x \in C],$$

where $C \subseteq D \subseteq \mathbb{R}^3$,

- x, u belong to appropriate Sobolev spaces (u being some Boundary control)
- $\bar{y}$ is some reference state behaviour.

Generalized sub-differentials - Motivation

- Even with smooth data g and a finite index set T, φ need not be smooth.
- φ itself is never concave (on the whole space). However some transform of φ might be concave (e.g., taking the log), or φ might be concave on some portion of the space.
- That φ might fail to be smooth is not a problem. However it requires the use of “sub-differentials".
- As a potentially non-smooth, non-convex object, the sub-differential needs to differ from the usual sub-differential of convex analysis.

Generalized sub-differentials - Definitions

- Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a map and consider $\bar{x}$ such that $f(\bar{x}) < \infty$, then

$$\partial^{\mathrm{F}} f(\bar{x}) = \left\{ x^* \in X^* \ : \liminf_{u \to \bar{x}} \frac{f(u) - f(\bar{x}) - \langle x^*, u - \bar{x} \rangle}{\| u - \bar{x} \|} \geq 0 \right\}. \quad (7)$$

  is the Fréchet subdifferential of f at $\bar{x}$,

- We also introduce

$$\partial^{\mathrm{M}} f(\bar{x}) := \{ w^*\text{-}\lim x_n^* : x_n^* \in \partial^{\mathrm{F}} f(x_n), \text{ and } x_n \xrightarrow{f} \bar{x} \},$$

$$\partial^{\infty} f(\bar{x}) := \{ w^*\text{-}\lim \lambda_n x_n^* : x_n^* \in \partial^{\mathrm{F}} f(x_n), \ x_n \xrightarrow{f} \bar{x} \text{ and } \lambda_n \to 0^+ \},$$

  are the Mordukhovich and singular Mordukhovich subdifferential respectively.

Elliptical distributions

Definition

We say that the random vector $\xi \in \mathbb{R}^m$ is elliptically symmetrically distributed with mean μ, covariance matrix Σ and generator $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, notation $\xi \sim \mathcal{E}(\mu, \Sigma, \theta)$ if and only if its density $f : \mathbb{R}^n \to \mathbb{R}_+$ is given by:

$$f(x) = (\det \Sigma)^{-\frac{1}{2}} \theta((x - \mu)^\mathsf{T} \Sigma^{-1} (x - \mu)). \tag{8}$$

Variance Reducing representation of φ

- When $\xi \sim \mathcal{E}(\mu, \Sigma, \theta)$ and $\Sigma = LL^{\intercal}$ is the Cholesky decomposition of Σ, ξ admits a spherical radial decomposition
- $\xi = \mu + \mathcal{R}L\zeta$, where ζ is uniformly distributed on $\mathbb{S}^{m-1} = \{z \in \mathbb{R}^m : \|z\| = 1\}$, $\mathcal{R}$ a radial distribution independent of ζ.
- $\mathcal{R}$ possesses a density given by:

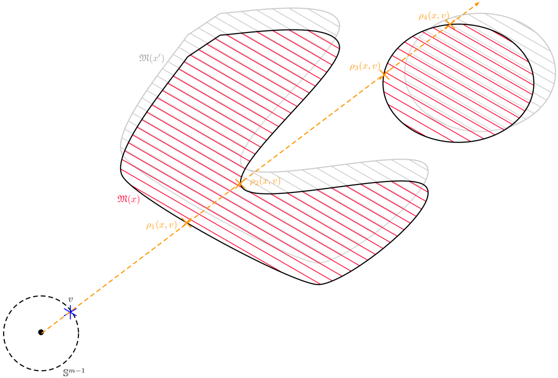
$$f_{\mathcal{R}}(r) = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} r^{m-1}\theta(r^2). \tag{9}$$

- For any Lebesgue measurable set $M \subseteq \mathbb{R}^m$ its probability may be represented as

$$\mathbb{P}(\xi \in M) = \int_{v \in \mathbb{S}^{m-1}} \mu_{\mathcal{R}}\left(\{r \geq 0 : \mu + rLv \cap M \neq \emptyset\}\right) d\mu_{\zeta}(v), \tag{10}$$

where $\mu_{\mathcal{R}}$ and μ_{ζ} are the laws of $\mathcal{R}$ and ζ, respectively.

Illustration of the decomposition



Hypothesis in this work

- We assume that $z \mapsto g_t(x, z)$ is convex for each t,
- We also assume that each g_t is continuously differentiable in both arguments.

Hypothesis: Consequences

- Let $D_t := \{x \in X \ : g_t(x, 0) < 0\}$
- We can entail from $g_t(x, 0) < 0$ the existence of a map $\rho_t : D_t \times \mathbb{R}^m \to \mathbb{R}_+ \cup \{\infty\}$, continuously differentiable on its domain such that

$$g_t(x, rLv) = 0 \text{ if and only if } r = \rho_t(x, v) \tag{11}$$

Variance Reducing representation of φ

Proposition (vA, Perez-Aros (2018))

Let D be defined as $D := \bigcap_{t\in T} D_t$ *. Then we define the map* $\rho : D\times\mathbb{S}^{m-1} \to \mathbb{R}_+$ *as*

$$\rho(x, v) := \inf_{t\in T} \rho_t(x, v), \tag{12}$$

where $\rho_t : D_t \times \mathbb{S}^{m-1} \to \mathbb{R}_+$ *and* D_t *is as before. Then for any* $x \in D$, $v \in \mathbb{S}^{m-1}$*, it holds that*

$$\{r \geq 0 \ : g(x, rLv) \leq 0\} = [0, \rho(x, v)], \tag{13}$$

where $[0, \infty] = [0, \infty)$ *is intended. Hence, for* $x \in D$*,*

$$\varphi(x) = \int_{v\in\mathbb{S}^{m-1}} F_{\mathcal{R}}(\rho(x, v))d\mu_\zeta(v) \tag{14}$$

Some difficulties

- The map ρ need not be solution to $g(x, rLv) = 0$ with g the supremum function!

Example

Consider the functions $g_n : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$g_n(x, z) = \begin{array}{cc} x^2 - 1 & \text{if } z_1^2 + z_2^2 \leq 1 \\ x^2 + n(z_1^2 + z_2^2 - 1)^2 - 1 & \text{if } z_1^2 + z_2^2 > 1, \end{array}$$

the supremum of this family is

$$g(x, z) = \begin{cases} x^2 - 1 & \text{if } z_1^2 + z_2^2 \leq 1, \\ +\infty & \text{if } z_1^2 + z_2^2 > 1. \end{cases}$$

Consequently, for any $v \in \mathbb{S}^{m-1}$, $\{r : g(0, rv) \leq 0\} = [0, 1]$ and there is no $r > 0$ such that $g(0, rv) = 0$. Moreover for any $x \in [0, 1)$ and $v \in \mathbb{S}^{m-1}$, we can compute $\rho_n(x, v) = \sqrt{1 + \sqrt{\frac{1-x^2}{n}}}$ and establish $\rho(x, v) = 1$.

Some difficulties II

- Although ρ is automatically u.s.c. (as the inf over a family of C1 maps), it may fail to be l.s.c. - additional assumptions will be needed.

Corollary

Moreover, for $x \in D^{\circ} := \{x \in X \ : g(x,0) < 0\}$*, one has that, if there exists* $r > 0$ *such that* $g(x, rLv) = 0$*, then* $r = \rho(x, v)$*. In particular, if* $g|_{D^{\circ} \times \mathbb{R}^m}$ *is finite valued the function* ρ *has the following alternative representation*

$$\rho(x,v) = \begin{cases} r & \text{such that } g(x, rLv) = 0 \\ +\infty & \text{otherwise} \end{cases} \qquad (15)$$

Continuity of the resolvant map

Proposition

Let x_0 be a point in X such that there exists a neighbourhood U of x_0 such that:

- *The function ρ is solution to $g(., rL.) = 0$.*
- *$g(x, 0) < 0$ for all $x \in U$.*
- *The set $\mathfrak{K} := \{(x, z) \in U \times \mathbb{R}^m : g(x, z) = 0\}$ is closed.*

Then $\rho(x_n, v_n) \to \rho(x, v)$ for every sequence $U \times \mathbb{S}^{m-1} \ni (x_n, v_n) \to (x, v) \in U \times \mathbb{S}^{m-1}$.

Subdifferential estimates

Proposition (vA, Perez-Aros (2018))

Under the previous assumptions for every $x \in U$ the (regular) partial Mordukhovich sub-differential of ρ satisfies:

$$\partial^{\mathrm{M}}_x \rho(\bar{x}, v) \subseteq \left\{ x^* \in X^* \ : \ \begin{array}{c} \exists \varepsilon_n \to 0^+, \ x_n \to \bar{x}, \ \ \exists t_n \in T_{\varepsilon_n}(x_n, v), \\ \text{s.t. } \rho_{t_n}(x_n, v) \to \rho(\bar{x}, v), \\ x^* = w^*\text{-}\lim_{n\to\infty} -\frac{\nabla_x g_{t_n}(x_n, \rho_{t_n}(x_n, v) L v)}{\langle \nabla_z g_{t_n}(x_n, \rho_{t_n}(x_n, v) L v), L v\rangle} \end{array} \right\} \tag{16a}$$

and the (singular) (partial) Mordukhovich sub-differential satisfies:

$$\partial^{\infty}_x \rho(\bar{x}, v) \subseteq \left\{ x^* \in X^* \ : \ \begin{array}{c} \exists \varepsilon_n, \lambda_n \to 0^+, \ x_n \to \bar{x}, \ \ \exists t_n \in T_{\varepsilon_n}(x_n, v), \\ \text{s.t. } \rho_{t_n}(x_n, v) \to \rho(\bar{x}, v), \\ x^* = w^*\text{-}\lim_{n\to\infty} -\lambda_n \frac{\nabla_x g_{t_n}(x_n, \rho_{t_n}(x_n, v) L v)}{\langle \nabla_z g_{t_n}(x_n, \rho_{t_n}(x_n, v) L v), L v\rangle} \end{array} \right\} \tag{16b}$$

An example

- Already when $T = \{\bar{t}\}$ is a singleton, g can not be entirely arbitrary.

Example

Consider $g(x, z_1, z_2) = \alpha(x)e^{h(z_1)} + z_2 - 1$ as a map $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$. with $\alpha(x) = x^2, x \geq 0$ and 0 otherwise. Moreover $h(t) = -1 - 4\log(1 - \Phi(t))$, with Φ the c.d.f of a standard Gaussian r.v. Now with $\xi \sim \mathcal{N}(0, I)$, it follows that

- g is continuously differentiable, convex in (z_1, z_2)
- $g(0, 0, 0) < 0$
- $\varphi(x) := \mathbb{P}[g(x, \xi_1, \xi_2) \leq 0]$ is not locally Lipschitzian at $x = 0$.

Restricted growth

- This example makes it clear that some care should be taken with "unbounded directions". Hence we introduce:

Definition

For any $x \in X$ and $l > 0$, we define

$$C_l(x) := \left\{ h \in X : \langle \nabla_x g_t(x', z), h \rangle \leq l \left\| L^{-1} z \right\|^{-m} \theta^{-1}\left(\left\| L^{-1} z \right\|^2 \right) \|h\| \quad \begin{matrix} \forall x' \in \mathbb{B}_{1/l}(x) \\ \left\| L^{-1} z \right\| \geq l \end{matrix} \quad , \forall t \in T \right\} \tag{17}$$

as the uniform *l-cone of nice directions at x. Here θ^{-1} is defined as*

$$\theta^{-1}(t) = \begin{cases} \frac{1}{\theta(t)} & \text{if } \theta(t) \neq 0, \\ +\infty & \text{if } \theta(t) = 0. \end{cases} \tag{18}$$

Moreover, we recall that its polar cone is denoted as $C_l^*(x)$.

Main result - I

Theorem (vA, Perez-Aros (2018))

Let $\xi \in \mathbb{R}^m$ be an elliptically symmetrically distributed random vector with mean 0, correlation matrix $R = LL^\mathsf{T}$ and continuous generator. Consider the probability function $\varphi : X \to [0,1]$, where X is a (separable) reflexive Banach space defined as

$$\varphi(x) = \mathbb{P}[g_t(x,\xi) \leq 0, \ \forall t \in T], \tag{19}$$

where $g_t : X \times \mathbb{R}^m \to \mathbb{R}$ are continuously differentiable maps convex in the second argument and T is an arbitrary index set.

Let $\bar{x} \in X$ be such that ...

Then the following formulæ hold true: ...

Main result - I

Theorem (vA, Perez-Aros (2018))

Let $\xi \in \mathbb{R}^m$ be an elliptically ... Let $\bar{x} \in X$ be such that

1. *a neighbourhood U of $\bar{x}$ can be found such that $g_{|_{U \times \mathbb{R}^m}}$ is finite valued and $\sup_{t \in T} g_t(x', 0) < 0$ for all $x' \in U$.*
2. *the set $\{(x, z) : g(x, z) = 0\}$ is closed in $U \times \mathbb{R}^m$*
3. *the outer-estimate $\mathcal{S}$ of $\partial^{\mathrm{M}}_x \rho(x, v)$ is locally bounded at $\bar{x}$, $v \in \mathbb{S}^{m-1}$ such that $\rho(\bar{x}, v) < \infty$.*
4. *Either there exists $l > 0$ such that $C_l(\bar{x})$ has non-empty interior, or $M(\bar{x}) := \{z \in \mathbb{R}^m : g(\bar{x}, z) \leq 0\}$ is bounded.*

Then the following formulæ hold true: ...

Main result - I

Theorem (vA, Perez-Aros (2018))

Let $\xi \in \mathbb{R}^m$ be an elliptically ...

Let $\bar{x} \in X$ be such that ...
Then the following formulæ hold true:

- [(i)] $\partial^{\mathrm{M}}\varphi(\bar{x}) \subseteq \mathrm{cl}^* \left\{ \int_{v \in \mathbb{S}^{m-1}} \partial^{\mathrm{M}}_x e(\bar{x}, v) d\mu_\zeta(v) - C_l^*(\bar{x}) \right\}$
- [(ii)] Provided that X is finite-dimensional,
$$\partial^{\mathrm{M}}\varphi(\bar{x}) \subseteq \int_{v \in \mathbb{S}^{m-1}} \partial^{\mathrm{M}}_x e(\bar{x}, v) d\mu_\zeta(v) - C_l^*(\bar{x}).$$
- [(iii)] $\partial^{\infty}\varphi(\bar{x}) \subseteq -C_l^*(\bar{x})$.
- [(vi)] $\partial^{\mathrm{C}}\varphi(\bar{x}) \subseteq \overline{\mathrm{co}} \left\{ \int_{v \in \mathbb{S}^{m-1}} \partial^{\mathrm{M}}_x e(\bar{x}, v) d\mu_\zeta(v) - C_l^*(\bar{x}) \right\}$,

where ∂^{M}, ∂^{C} and ∂^{∞} refer respectively to the limiting (or Mordukhovich), the Clarke and (limiting) singular sub-differential sets of a map.

Moreover, the set $C_l^*(\bar{x})$ can be replaced by $\{0\}$ if $M(\bar{x})$ is bounded.

Main result - I

Theorem (vA, Perez-Aros (2018))

Finally, for every $v \in F(\bar{x}) = \mathcal{D}\mathrm{om}(\rho(x,.))$

$$\partial_x^{\mathrm{M}} e(\bar{x},v) \subseteq f_{\mathcal{R}}(\rho(\bar{x},v))\mathcal{S}^{\mathrm{M}}(\bar{x},v)$$

with

$$\mathcal{S}^{\mathrm{M}}(\bar{x},v) \subseteq \left\{ x^* \in X^* \ : \ \begin{array}{c} \exists \varepsilon_n \to 0^+,\ x_n \to \bar{x},\ \ \exists t_n \in T_{\varepsilon_n}(x_n,v), \\ s.t.\ \rho_{t_n}(x_n,v) \to \rho(\bar{x},v), \\ x^* = w^*\text{-}\lim_{n\to\infty} -\frac{\nabla_x g_{t_n}(x_n,\rho_{t_n}(x_n,v)Lv)}{\left\langle \nabla_z g_{t_n}(x_n,\rho_{t_n}(x_n,v)Lv),Lv\right\rangle} \end{array} \right\} \tag{19}$$

Discussion of the assumptions

When the family $\{g_t\}_{t\in T}$ is uniformly locally Lipschitzian at $\bar{x}$, i.e., iff at every $\bar{z} \in \mathbb{R}^m$:

$$\limsup_{z\to\bar{z}} \sup\{\|\nabla g_t(x,z)\| \mid x \in U,\ t \in T\} < \infty. \tag{20}$$

Then

- if $g(\bar{x}, 0)$ is finite, then on some neighbourhood U : $g_{U\times\mathbb{R}^m}$ is finite valued
- if $g(\bar{x}, 0) < 0$ then this holds on a neighbourhood
- the set $\mathcal{S}$ is locally bounded

Motivation

- The previous Theorem has given us already a first formula: an outer-estimate of the various subdifferentials.
- The outer estimate involves $\mathcal{S}$, related to special limits
- If $\mathcal{S}$, can be replaced by a smaller set, better formulæ may result.
- This will require some additional assumptions

Equicontinuous subdifferentiability

Definition

Let $f_t : X \to \mathbb{R} \cup \{\infty\}$ be a family of l.s.c. functions indexed by $t \in T$. The family is called strongly equicontinuously subdifferentiable at $\bar{x} \in X$ if for any weak-* neighbourhood V^* of the origin in X^* there is some $\varepsilon > 0$ such that

$$\partial^{\mathrm{M}} f_t(x) \subseteq \partial^{\mathrm{M}} f_t(\bar{x}) + V^*, \tag{21}$$

for all $t \in T_\varepsilon(x)$ $x \in \mathbb{B}_\varepsilon(\bar{x})$, with $|f_t(x) - f(\bar{x})| \leq \varepsilon$ where $T_\varepsilon(x)$ refers to the ε-active index set related to the supremum function of the family f_t.

Main Result - II

Theorem (vA, Perez-Aros (2018))

Let $\xi \in \mathbb{R}^m$ be an elliptically symmetrically distributed random vector with mean 0, correlation matrix $R = LL^{\mathsf{T}}$ and continuous generator. Consider the probability function $\varphi : X \to [0,1]$, where X is a reflexive Banach space defined as

$$\varphi(x) = \mathbb{P}[g_t(x,\xi) \leq 0, \ \forall t \in T], \tag{22}$$

where $g_t : X \times \mathbb{R}^m \to \mathbb{R}$ are continuously differentiable maps convex in the second argument and T is an arbitrary index set. Then let $\bar{x} \in X$ be such that the assumptions 1-4 as before hold and in addition:

1. *that at any $v \in \mathbb{S}^{m-1}$, the family of resolvant mappings $\{\rho_t(.,v)\}_{t \in T}$ is strongly equicontinuously subdifferentiable at $\bar{x}$.*

Then in the previous formulæ we may consider

$$\partial_x^{\mathrm{M}} e(\bar{x},v) \subseteq f_{\mathcal{R}}(\rho(\bar{x},v)) \bigcap_{\varepsilon>0} \mathrm{cl}^{\mathrm{w}^*} \left\{ -\frac{\nabla_x g_t(\bar{x}, \rho_t(\bar{x},v)Lv)}{\langle \nabla_z g_t(\bar{x}, \rho_t(\bar{x},v)Lv), Lv \rangle} : \begin{array}{c} x \in \mathbb{B}(\bar{x},v), \ t \in T_\varepsilon^\rho(x,v) \\ \text{with } |\rho_t(x,v) - \rho(\bar{x},v)| \leq \varepsilon \end{array} \right\}.$$

at $v \in F(\bar{x}) = \mathcal{D}\mathrm{om}(\rho(\bar{x},v))$.

Some compactness assumptions

Assumption

Let T be a metric space and there exists a neighbourhood U of $\bar{x}$ such that:

1. $g_{|_{U\times\mathbb{R}^m}}$ is finite valued.
2. $g(x,0) < 0$ for all $x \in U$.
3. The function $G : T \times U \times \mathbb{R}^m \to X \times X^* \times \mathbb{R}^m$ given by $G(t,x,z) = (g_t(x,v), \nabla_x g_t(x,z), \nabla_z g_t(x,z))$ is continuous.
4. The active index set $T^g(x,z)$ is non-empty for every $(x,z) \in \mathfrak{K} = \{(x,z) \in U \times \mathbb{R}^m : g(x,z) = 0\}$.
5. The set $\bigcup_{(x,z)\in\mathfrak{K}} T^g(x,z)$ is relatively compact.

Implications of these assumptions

Lemma

Under the compactness Assumptions one has that:

1. *the set $\mathfrak{K}$ is closed.*
2. *for every $T \times U \times \mathbb{S}^{m-1} \ni (t_n, x_n, v_n) \to (t, x, v) \in T \times U \times \mathbb{S}^{m-1}$, $\rho_{t_n}(x_n, v_n) \to \rho_t(x, v)$.*
3. *the set $T_\varepsilon^\rho(x, v)$ is closed for every $(x, v) \in U \times \mathbb{S}^{m-1}$.*

Explicit growth condition

Definition

Let $\theta_{\mathcal{R}} : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing mapping such

$$\lim_{r \to +\infty} r f_{\mathcal{R}}(r) \theta_{\mathcal{R}}(r) = 0. \tag{23}$$

We say that $\{g_t : t \in T\}$ satisfies the $\theta_{\mathcal{R}}$-growth condition uniformly on T at $\bar{x}$ if for some $l > 0$

$$\|\nabla_x g_t(x,z)\| \leq l \theta_{\mathcal{R}}\left(\frac{\|z\|}{\|L\|}\right) \text{ for all } x \in \mathbb{B}_{1/l}(\bar{x}) \ \ \forall z : \|z\| \geq l; \ \ \forall t \in T. \tag{24}$$

Main Result - III

Theorem (vA, Perez-Aros (2018))

Let $\xi \in \mathbb{R}^m$ be an elliptically symmetrically distributed random vector with mean 0, correlation matrix $R = LL^\top$ and continuous generator. Consider the probability function $\varphi : X \to [0,1]$, where X is a reflexive Banach space defined as

$$\varphi(x) = \mathbb{P}[g_t(x,\xi) \leq 0, \ \forall t \in T], \tag{25}$$

where $g_t : X \times \mathbb{R}^m \to \mathbb{R}$ are continuously differentiable maps convex in the second argument and T is a metric space. Then let $\bar{x} \in X$ be such that

1. *the compactness assumptions hold*
2. *Either $\{g_t : t \in T\}$ satisfies the $\theta_{\mathcal{R}}$-growth condition uniformly on T at $\bar{x}$, or $M(\bar{x}) := \{z \in \mathbb{R}^m : g(\bar{x}, z) \leq 0\}$ is bounded.*

Then φ is locally Lipschitz at $\bar{x}$ and the following formulæ hold true:

$$\partial^{\mathrm{M}}\varphi(\bar{x}) \subseteq \mathrm{cl}^{\mathrm{w}^*} \int_{v \in F(\bar{x})} \left\{ -f_{\mathcal{R}}(\rho(x,v)) \frac{\nabla_x g_t(x, \rho(x,v)Lv)}{\langle \nabla_z g_t(x, \rho(x,v)Lv), Lv \rangle} : t \in T^\rho(x,v) \right\} d\mu_\zeta(v) \tag{26a}$$

$$\partial^{\mathrm{C}}\varphi(\bar{x}) \subseteq \int_{v \in F(\bar{x})} \mathrm{Co} \left\{ -f_{\mathcal{R}}(\rho(x,v)) \frac{\nabla_x g_t(x, \rho(x,v)Lv)}{\langle \nabla_z g_t(x, \rho(x,v)Lv), Lv \rangle} : t \in T^\rho(x,v) \right\} d\mu_\zeta(v), \tag{26b}$$

where ∂^{M} refers to the limiting (or Mordukhovich) sub-differential and ∂^{C} to the Clarke-subdifferential.

Summary

In this talk we have discussed recent results on differentiation for probability functions acting on infinite systems. The results have been taken from:

- W. van Ackooij and P. Pérez-Aros. Generalized differentiation of probability functions acting on an infinite system of constraints. *Submitted draft - available soon on arxiv*, pages 1–24, 2018

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