Advanced decomposition methods for discrete-time stochastic optimal control problems

$\lozenge$

Dual Approximate Dynamic Programming

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March 2017

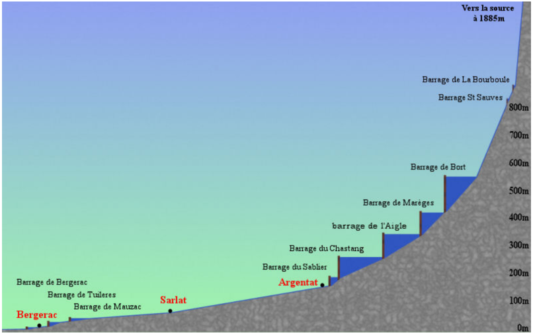
Ultimate goal of the lecture

How to to obtain "good" strategies (or cost-to-go functions) for a large scale stochastic optimal control problem in discrete time, for example a problem corresponding to the optimal management over a given time horizon of a system involving a large amount of dynamical production units.

- In order to obtain decision strategies (closed-loop controls), we have to use dynamic programming or related methods.
  - **Assumption**: Markovian case,
  - **Difficulty**: curse of dimensionality.
- To overcome the barrier of the dimension, we want to use decomposition/coordination techniques, so that we have to take into account the information pattern induced by the stochastic optimization problem.

Practical applications under consideration

Electricity production management for large hydro valleys



- *1 year time horizon*: compute each month the "values of water" (cost-to-go functions)
- *stochastic framework*: rain, market prices
- *large-scale valley*: 4 dams and much more

Lecture outline

1. Examples and mathematical background
   - Interconnected systems
   - Optimization background
   - Standard decomposition methods
2. About decomposition in stochastic optimization
   - Couplings in stochastic optimization
   - Dynamic programming and decomposition
3. Dual approximate dynamic programming (DADP)
   - Problem formulation and price decomposition
   - Subproblems resolution and coordination
   - What has been really done?
4. Hydro valleys management problem
   - DADP implementation for hydro valleys
   - Numerical results for different valleys

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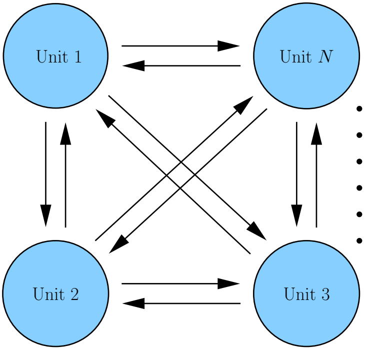
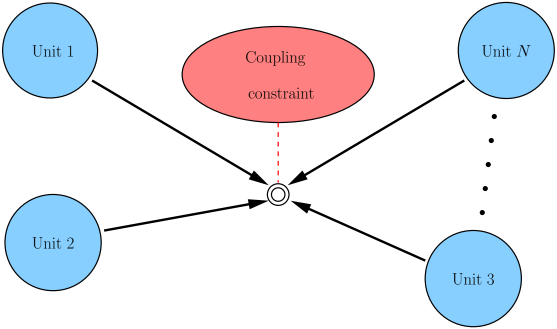
Decomposition and coordination



Interconnected units

- The "large system" to be optimized consists of interconnected subsystems: we want to use this structure in order to formulate optimization subproblems of reasonable complexity.
- But the presence of interactions requires a level of coordination.
- Coordination must provide a local model of the interactions to each subproblem: it is an iterative process.
- The ultimate goal is to obtain the solution of the overall problem by concatenation of the solutions of the subproblems.

Example: the “flower model”

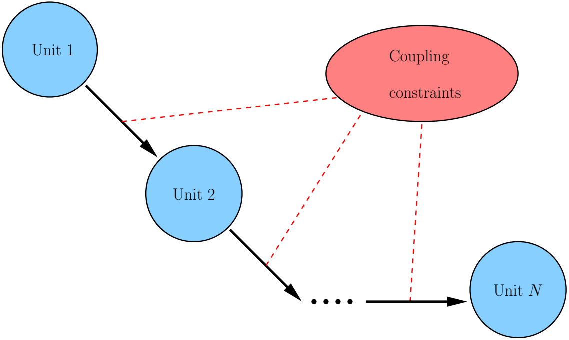


$$\min_u \sum_{i=1}^{N} J_i(u_i) \ ,$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \Theta_i(u_i) = \theta \ .$$

Unit Commitment Problem

Example: the "cascade model"

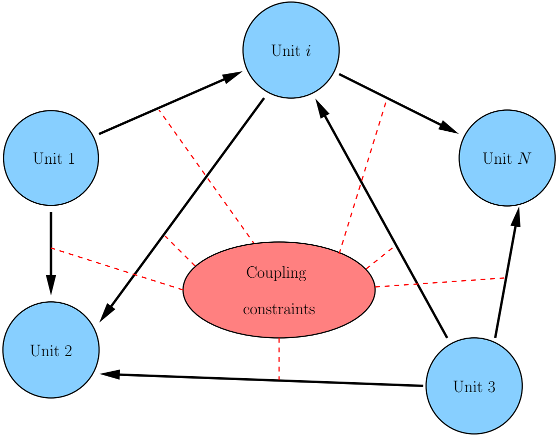


$$\min_{u,v} \sum_{i=1}^{N} J_i(u_i, v_i) ,$$
$$\text{s.t.} \quad H_i(u_i, v_i) = v_{i+1} \quad \forall i .$$

Dams Management Problem

Link with the flower model: $\Theta_i(u_i, v_i) = \big(0, \ldots, -v_i, H_i(u_i, v_i), \ldots, 0\big)^\top$.

A general model



$$\min_{u,v} \sum_{i=1}^{N} J_i\Big(u_i, \sum_{j \neq i} v_{j,i}\Big) \;,$$

$$\text{s.t.} \quad H_i\Big(u_i, \sum_{j \neq i} v_{j,i}\Big) = v_i \;.$$

Smart Grid

Motivation for theoretical developments

Mathematical ingredients needed to tackle such problems.

- Optimization mathematical framework.
- Duality theory (handling of constraints).
  - $\rightsquigarrow$ Lagrangian relaxation.
- Decomposition/coordination methods
  - $\rightsquigarrow$ Price decomposition (Walras groping).

Points already covered since the beginning of the week.

- Stochastic optimization.
- Dynamic programming.

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Optimization without explicit constraint

$$\min_{u \in \mathcal{U}^{\mathrm{ad}}} J(u) \ .$$

- $\mathcal{U}$: Hilbert space with scalar product $\langle \cdot \, , \cdot \rangle$.
  Examples: $\mathcal{U} = \mathbb{R}^n$ (vectors) or $\mathcal{U} = \mathrm{L}^2\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n\big)$ (random variables).
- $\mathcal{U}^{\mathrm{ad}}$: closed convex subset of $\mathcal{U}$.
- $J : \mathcal{U} \to \mathbb{R}$: function satisfying some properties
  $\big($convexity, continuity, differentiability, coercivity$\big)$.

Characterization of a solution $u^\sharp$ (**optimality conditions**):

$$\big\langle \nabla J(u^\sharp) \, , u - u^\sharp \big\rangle \geq 0 \quad \forall u \in \mathcal{U}^{\mathrm{ad}} \ .$$

Computation of the solution $u^\sharp$ (**projected gradient algorithm**):

$$u^{(k+1)} = \mathrm{proj}_{\mathcal{U}^{\mathrm{ad}}}\Big(u^{(k)} - \rho \nabla J(u^{(k)})\Big) \ .$$

Optimization with explicit constraints I

$$\min_{u \in \mathcal{U}^{\mathrm{ad}}} J(u) \quad \text{subject to} \quad \Theta(u) \in -C \; . \qquad (\mathcal{P})$$

- $\mathcal{U}$: Hilbert space.
- $\mathcal{U}^{\mathrm{ad}}$: closed convex subset of $\mathcal{U}$.
- $J : \mathcal{U} \to \mathbb{R}$: cost function.
- $\mathcal{V}$: another Hilbert space.
- $\Theta : \mathcal{U} \to \mathcal{V}$: constraint function satisfying some properties (convexity w.r.t. C, continuity, differentiability).
- C: cone of $\mathcal{V}$ (examples: $C = \{0\}$, $C = \{v \geq 0\}$).

An additional condition on the constraint function is needed!

Constraint Qualification Condition, e.g. $0 \in \mathrm{int}\big(\Theta(\mathcal{U}^{\mathrm{ad}}) + C\big)$.

Optimization with explicit constraints II

Karush-Kuhn-Tucker Conditions

In addition to standard conditions on J and Θ, we assume that the constraints are qualified.

Then a necessary and sufficient condition for $u^\sharp \in \mathcal{U}^{\mathrm{ad}}$ to be a solution of Problem $(\mathcal{P})$ is that there exists $\lambda^\sharp \in \mathcal{V}$ such that:

1. $\left\langle \nabla J(u^\sharp) + [\Theta'(u^\sharp)]^\top \lambda^\sharp \, , u - u^\sharp \right\rangle \geq 0 \;\; \forall u \in \mathcal{U}^{\mathrm{ad}}$,
2. $\Theta(u^\sharp) \in -C$,
3. $\lambda^\sharp \in C^\star$,
4. $\left\langle \lambda^\sharp \, , \Theta(u^\sharp) \right\rangle = 0$ (**Complementary Slackness**).

The dual cone of C is defined by: $C^\star = \left\{ \lambda \in \mathcal{V}, \; \langle \lambda \, , v \rangle \geq 0 \;\; \forall v \in C \right\}$.

Optimization with explicit constraints III

Let $L : \mathcal{U}^{\mathrm{ad}} \times C^\star \to \mathbb{R}$ be the Lagrangian associated to $(\mathcal{P})$:

$$L(u, \lambda) = J(u) + \left\langle \lambda \, , \Theta(u) \right\rangle .$$

A point $(u^\sharp, \lambda^\sharp) \in \mathcal{U}^{\mathrm{ad}} \times C^\star$ is a saddle point of L if

$$L(u^\sharp, \lambda) \leq L(u^\sharp, \lambda^\sharp) \leq L(u, \lambda^\sharp) \quad \forall (u, \lambda) \in \mathcal{U}^{\mathrm{ad}} \times C^\star .$$

- If $(u^\sharp, \lambda^\sharp)$ is a saddle point of L, then $u^\sharp$ is a solution of $(\mathcal{P})$.
- If $u^\sharp$ is a solution of $(\mathcal{P})$ and if the **KKT** conditions are met for some $\lambda^\sharp$, then $(u^\sharp, \lambda^\sharp)$ is a saddle point of L.

Moreover we have that

$$L(u^\sharp, \lambda^\sharp) = J(u^\sharp) = \min_{u \in \mathcal{U}^{\mathrm{ad}}} \max_{\lambda \in C^\star} L(u, \lambda) = \max_{\lambda \in C^\star} \min_{u \in \mathcal{U}^{\mathrm{ad}}} L(u, \lambda) .$$

Optimization with explicit constraints IV

Dealing with the dual problem

$$\max_{\lambda \in C^\star} \min_{u \in \mathcal{U}^{\mathrm{ad}}} L(u, \lambda) ,$$

paves the way for algorithmic methods. Define the dual function Φ associated to the Lagrangian L as

$$\Phi(\lambda) = \min_{u \in \mathcal{U}^{\mathrm{ad}}} L(u, \lambda) .$$

The problem of maximizing the dual function Φ is equivalent to the one of solving the dual problem:

$$\max_{\lambda \in C^\star} \Phi(\lambda) \quad \Longleftrightarrow \quad \max_{\lambda \in C^\star} \min_{u \in \mathcal{U}^{\mathrm{ad}}} L(u, \lambda) .$$

The gradient of Φ is obtained from the minimization step in u:

$$\nabla\Phi(\lambda) = \Theta(\widehat{u}_\lambda), \text{ with } \widehat{u}_\lambda \text{ unique solution of } \min_{u \in \mathcal{U}^{\mathrm{ad}}} L(u, \lambda) .$$

Optimization with explicit constraints V

In order to obtain a solution of the original constrained problem, we use a gradient algorithm for maximizing the dual function:

$$\max_{\lambda \in C^\star} \Phi(\lambda) \ .$$

The gradient of Φ at the current point $\lambda^{(k)}$ of the algorithm is obtained by minimizing $L(u, \lambda^{(k)})$ w.r.t. u.

Uzawa's Algorithm

Choose $\lambda^{(0)} \in C^\star$. At each iteration k,

1. obtain the solution $u^{(k+1)} = \arg\min_{u \in \mathcal{U}^{\mathrm{ad}}} J(u) + \langle \lambda^{(k)} , \Theta(u) \rangle$,
2. update the multiplier $\lambda^{(k+1)} = \mathrm{proj}_{C^\star} \big(\lambda^{(k)} + \rho \Theta(u^{(k+1)}) \big)$.

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Specific problem structure: additive model

Consider the optimization problem with explicit constraints:

$$\min_{u \in \mathcal{U}^{\mathrm{ad}} \subset \mathcal{U}} J(u) \quad \text{subject to} \quad \Theta(u) = \theta \in \mathcal{V} \; .$$

We assume that the space $\mathcal{U}$ writes as a Cartesian product:

$$\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_N \; , \; \text{so that } u = \big(u_1, \ldots, u_N\big) \text{ with } u_i \in \mathcal{U}_i \; .$$

We moreover assume that this space decomposition is such that

- the admissible set $\mathcal{U}^{\mathrm{ad}}$ writes as a Cartesian product:

$$\mathcal{U}^{\mathrm{ad}} = \mathcal{U}_1^{\mathrm{ad}} \times \cdots \times \mathcal{U}_N^{\mathrm{ad}} \quad \text{with} \quad \mathcal{U}_i^{\mathrm{ad}} \subset \mathcal{U}_i \; ,$$

- the functions J and Θ write additively:

$$J(u) = J_1(u_1) + \cdots + J_N(u_N) \; ,$$
$$\Theta(u) = \Theta_1(u_1) + \cdots + \Theta_N(u_N) \; .$$

Specific problem structure: additive model II

Then the original problem displays the so-called additive structure:

$$\min_{\substack{u_1 \in \mathcal{U}_1^{\mathrm{ad}} \\ \vdots \\ u_N \in \mathcal{U}_N^{\mathrm{ad}}}} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \Theta_i(u_i) - \theta = 0 \ .$$

Note that the coupling between the i's only arises from the constraint Θ. As a matter of fact,

$$\min_{\substack{u_1 \in \mathcal{U}_1^{\mathrm{ad}} \\ \vdots \\ u_N \in \mathcal{U}_N^{\mathrm{ad}}}} \sum_{i=1}^{N} J_i(u_i) \iff \min_{u_i \in \mathcal{U}_i^{\mathrm{ad}}} J_i(u_i) \quad \forall i = 1, \ldots, N \ .$$

Additive model — Price decomposition I

$$\min_{u \in \mathcal{U}^{\mathrm{ad}}} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \Theta_i(u_i) - \theta = 0 \ .$$

1. Form the Lagrangian of the problem. The dual problem writes:

$$\max_{\lambda \in \mathcal{V}} \min_{u \in \mathcal{U}^{\mathrm{ad}}} \sum_{i=1}^{N} \Big(J_i(u_i) + \big\langle \lambda \,, \Theta_i(u_i) \big\rangle \Big) - \big\langle \lambda \,, \theta \big\rangle \ .$$

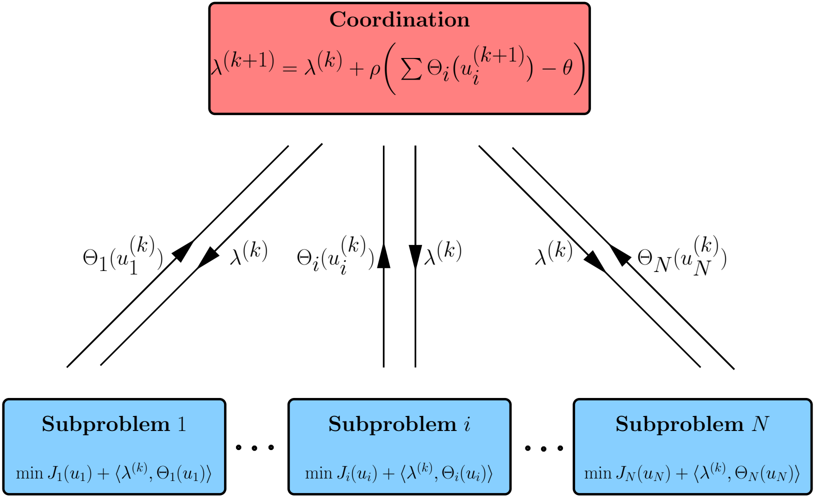
2. Solve this problem by the Uzawa algorithm:

$$u_i^{(k+1)} \in \mathop{\arg\min}_{u_i \in \mathcal{U}_i^{\mathrm{ad}}} J_i(u_i) + \big\langle \lambda^{(k)} \,, \Theta_i(u_i) \big\rangle \ , \quad i = 1 \ldots, N \ ,$$

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho \left(\sum_{i=1}^{N} \Theta_i\Big(u_i^{(k+1)}\Big) - \theta \right) .$$

$\leadsto$ *Walras groping*

Additive model — Price decomposition II



Additive model — Resource allocation I

$$\min_{u \in \mathcal{U}^{\mathrm{ad}}} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \Theta_i(u_i) - \theta = 0 \; .$$

1. Write the constraint in a equivalent manner by introducing new variables $(v_1, \ldots, v_N)$ (the so-called "allocation"):

$$\sum_{i=1}^{N} \Theta_i(u_i) - \theta = 0 \quad \Leftrightarrow \quad \Theta_i(u_i) - v_i = 0 \;\text{ and }\; \sum_{i=1}^{N} v_i - \theta = 0 \; .$$

Minimize the criterion w.r.t. u and v:

$$\min_{v \in \mathcal{V}^N} \sum_{i=1}^{N} \Big(\min_{u_i \in \mathcal{U}_i^{\mathrm{ad}}} J_i(u_i) \;\text{ s.t. }\; \Theta_i(u_i) - v_i = 0 \Big) \;\text{ s.t. }\; \sum_{i=1}^{N} v_i - \theta = 0 \; .$$

Additive model — Resource allocation II

$$\min_{v \in \mathcal{V}^N} \sum_{i=1}^{N} \underbrace{\Big(\min_{u_i \in \mathcal{U}_i^{\mathrm{ad}}} J_i(u_i) \ \text{ s.t. } \ \Theta_i(u_i) - v_i = 0 \Big)}_{G_i(v_i)} \ \text{ s.t. } \ \sum_{i=1}^{N} v_i - \theta = 0 \ ,$$

$$\Updownarrow$$

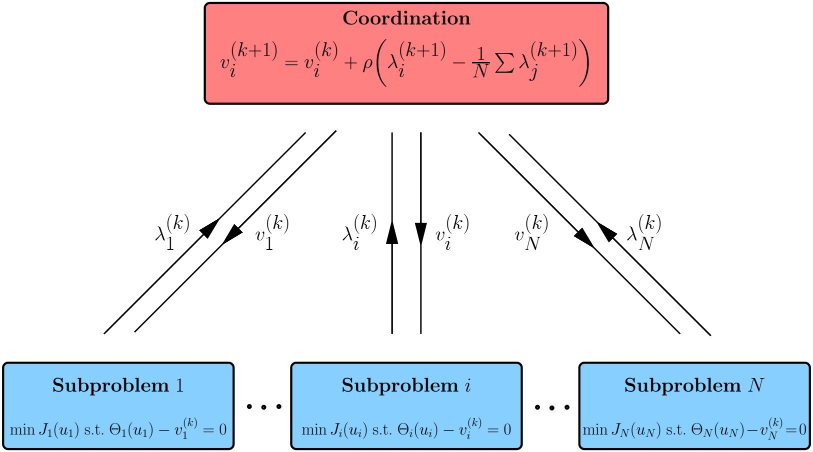
$$\min_{v \in \mathcal{V}^N} \sum_{i=1}^{N} G_i(v_i) \quad \text{s.t.} \quad \sum_{i=1}^{N} v_i - \theta = 0 \ .$$

2. Solve the last problem using a projected gradient method:

$$G_i(v_i^{(k)}) = \min_{u_i \in \mathcal{U}_i^{\mathrm{ad}}} J_i(u_i) \ \text{ s.t. } \ \Theta_i(u_i) - v_i^{(k)} = 0 \ \rightsquigarrow \ \lambda_i^{(k+1)} \ ,$$

$$v_i^{(k+1)} = v_i^{(k)} + \rho \Big(\lambda_i^{(k+1)} - \frac{1}{N} \sum_{j=1}^{N} \lambda_j^{(k+1)} \Big) \ .$$

Additive model — Resource allocation III



Additive model: conclusions

1. **Price decomposition**
   - Pros: subproblems are always feasible.
   - Cons: admissible solution once convergence achieved.

2. **Resource allocation**
   - Pros: admissible solution at each iteration.
   - Cons: potential existence of unfeasible subproblems.

Straightforward extension to inequality constraints...

Other methods are available, even for non-additive structures.

References on decomposition/coordination methods

Further readings on decomposition/coordination:

G. Cohen, "Optimisation des grands systèmes". *Cours du DEA Modélisation et Méthodes Mathématiques en Économie*, 2004.

G. Cohen, "Auxiliary Problem Principle and Decomposition of Optimization Problems". *Journal of Optimization Theory and Applications*, **32**, 1980.

G. Cohen & D.L. Zhu, "Decomposition coordination methods in large scale optimization problems. The nondifferentiable case and the use of augmented Lagrangians". In J.B. Cruz (Ed.): "*Advances in Large Scale Systems*", **1**, 203-266, JAI Press, Greenwich, Connecticut, 1984.

Final remarks on decomposition methods I

The theory is available for general (infinite dimensional) Hilbert spaces, and thus applies in the stochastic framework, that is, the case where $\mathcal{U}$ is a space of random variables.

The minimization algorithm used for solving the subproblems is not specified in the decomposition process and is left to the user! It is however assumed that the user is able to solve the subproblem, for example in the price decomposition case:

$$\min_{u_i \in \mathcal{U}_i^{\mathrm{ad}}} J_i(u_i) + \left\langle \lambda^{(k)} \, , \Theta_i(u_i) \right\rangle ,$$

and to send the requested information, namely $\Theta_i(u_i^{(k+1)})$, to the coordination level.

Question: *what methods are suitable for a stochastic problem?*

Final remarks on decomposition methods II

Whatever the decomposition/coordination scheme used (price, allocation...), new variables (depending on $u^{(k)}$ and/or $\lambda^{(k)}$) are present in the subproblems arising at iteration k of the associated algorithm.

Example: subproblem i in price decomposition:

$$\min_{u_i \in \mathcal{U}_i^{\mathrm{ad}}} J_i(u_i) + \left\langle \lambda^{(k)} , \Theta_i(u_i) \right\rangle .$$

All these new variables are considered as fixed when solving the subproblems (they only depend on the iteration index k). They are nothing but constant elements in the space $\mathcal{V}$.

Question: *which consequences in the stochastic case?*

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Reminder of our ultimate goal

How to to obtain "good" strategies (or cost-to-go functions) for a large scale stochastic optimal control problem in discrete time, for example a problem corresponding to the optimal management over a given time horizon of a system involving a large amount of dynamical production units.

- In order to obtain decision strategies (closed-loop controls), we have to use dynamic programming or related methods.
  - **Assumption**: Markovian case,
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- To overcome the barrier of the dimension, we want to use decomposition/coordination techniques, so that we have to take into account the information pattern induced by the stochastic optimization problem.

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Stochastic optimal control problem in discrete time

We consider a stochastic optimal control (SOC) problem

$$\min_{\boldsymbol{U},\boldsymbol{X}} \mathbb{E}\left(\sum_{i=1}^{N} \left(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \right) \right) ,$$

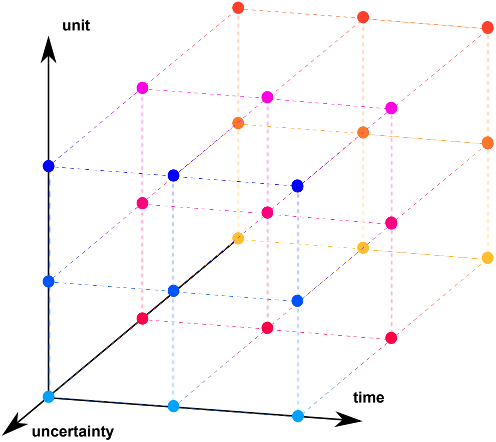
subject to the constraints:

$$\begin{aligned} \boldsymbol{X}_0^i &= f_{-1}^i(\boldsymbol{W}_0) , && i = 1 \ldots N , \\ \boldsymbol{X}_{t+1}^i &= f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) , \quad & t = 0 \ldots T-1 , \ & i = 1 \ldots N , \end{aligned}$$

$$\boldsymbol{U}_t^i = \mathbb{E}\left(\boldsymbol{U}_t^i \mid \mathcal{F}_t \right) , \qquad t = 0 \ldots T-1 , \ i = 1 \ldots N ,$$

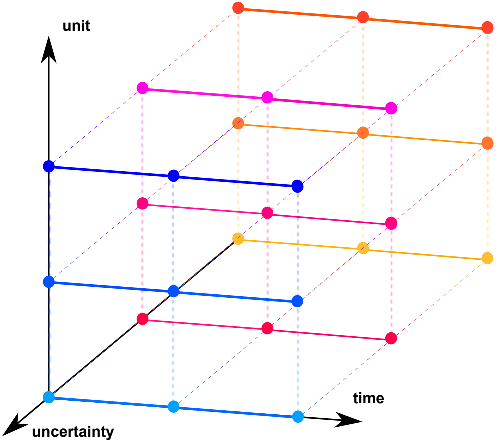
$$\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0 , \qquad t = 0 \ldots T-1 .$$

Couplings and decompositions for SOC problems I



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

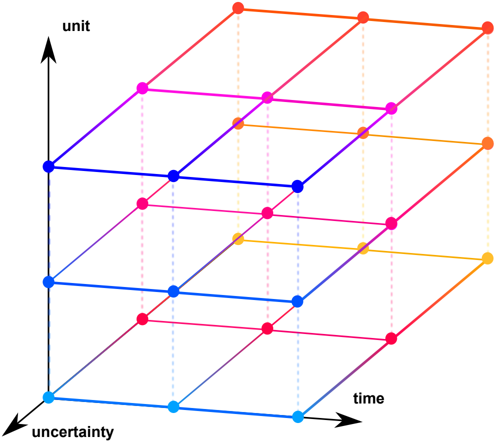
Couplings and decompositions for SOC problems II



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i - f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) = 0$$

Couplings and decompositions for SOC problems III

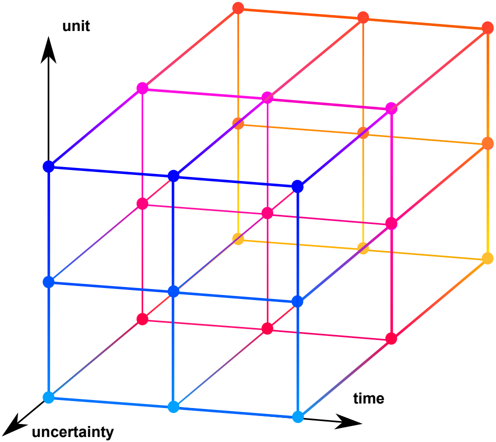


$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i - f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) = 0$$

$$\boldsymbol{U}_t^i - \mathbb{E}\big(\boldsymbol{U}_t^i \mid \mathcal{F}_t\big) = 0$$

Couplings and decompositions for SOC problems IV



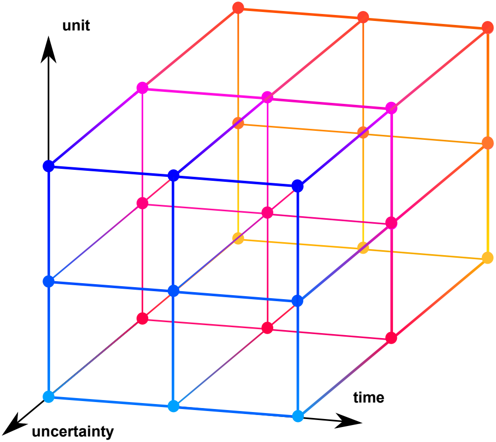
$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i - f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) = 0$$

$$\boldsymbol{U}_t^i - \mathbb{E}\big(\boldsymbol{U}_t^i \mid \mathcal{F}_t\big) = 0$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Couplings and decompositions for SOC problems V



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

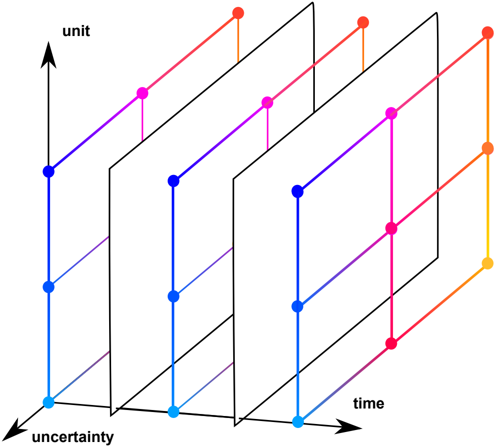
s.t. $\boldsymbol{X}_{t+1}^i - f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) = 0$

$$\boldsymbol{U}_t^i - \mathbb{E}\big(\boldsymbol{U}_t^i \mid \mathcal{F}_t\big) = 0$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

3 additive structures!

Couplings and decompositions for SOC problems VI



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

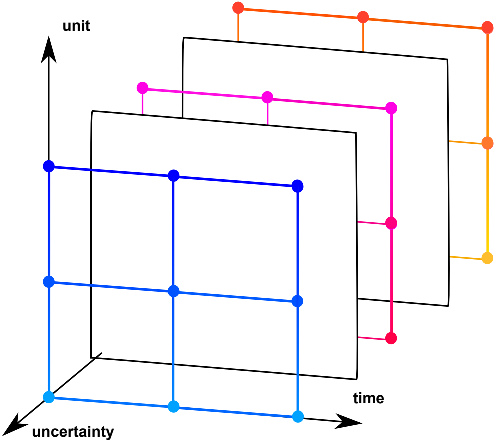
$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i - f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) = 0$$

$$\boldsymbol{U}_t^i - \mathbb{E}\left(\boldsymbol{U}_t^i \mid \mathcal{F}_t\right) = 0$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Time decomposition

Couplings and decompositions for SOC problems VII



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

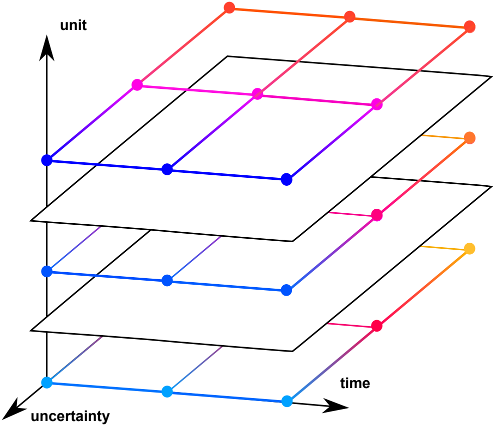
$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i - f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) = 0$$

$$\boldsymbol{U}_t^i - \mathbb{E}\left(\boldsymbol{U}_t^i \mid \mathcal{F}_t\right) = 0$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Progressive hedging

Couplings and decompositions for SOC problems VIII



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

s.t. $$\boldsymbol{X}_{t+1}^i - f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) = 0$$

$$\boldsymbol{U}_t^i - \mathbb{E}\left(\boldsymbol{U}_t^i \mid \mathcal{F}_t\right) = 0$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Purpose of DADP

1. Examples and mathematical background
   - Interconnected systems
   - Optimization background
   - Standard decomposition methods
2. About decomposition in stochastic optimization
   - Couplings in stochastic optimization
   - Dynamic programming and decomposition
3. Dual approximate dynamic programming (DADP)
   - Problem formulation and price decomposition
   - Subproblems resolution and coordination
   - What has been really done?
4. Hydro valleys management problem
   - DADP implementation for hydro valleys
   - Numerical results for different valleys

Mixing spatial decomposition and dynamic programming

Consider the SOC problem (the spatial structure is highlighted)

$$\min_{\boldsymbol{U},\boldsymbol{X}} \sum_{i=1}^{N} \left(\mathbb{E}\left(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \right) \right) ,$$

subject to the constraints:

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) , \qquad t = 0 \ldots T-1 , \quad i = 1 \ldots N ,$$

$$\boldsymbol{U}_t^i = \mathbb{E}\left(\boldsymbol{U}_t^i \mid \mathcal{F}_t\right) , \qquad t = 0 \ldots T-1 , \quad i = 1 \ldots N ,$$

$$\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0 , \qquad t = 0 \ldots T-1 \qquad \rightsquigarrow \qquad \boldsymbol{\Lambda}_t ,$$

and assume that the random variables $\boldsymbol{W}_t$ are independent (**white noise** assumption).

Dynamic programming yields centralized controls

Under the white noise assumption, it is possible to use dynamic programming (**DP**) in order to solve the SOC problem.

The true optimal control $\boldsymbol{U}_t^i$ of unit i is a feedback of the whole system state, that is, a function of all $\boldsymbol{X}_t^i$'s:

$$\boldsymbol{U}_t^i = \gamma_t^i(\boldsymbol{X}_t^1, \dots, \boldsymbol{X}_t^N) \ .$$

Of course, a straightforward use of **DP** is prohibited for N large (curse of dimensionality), and decomposition is needed!

But the feedback γ_t^i structurally induces a coupling between all the units! Moreover, a naive decomposition of the problem should lead to decentralized feedbacks:

$$\boldsymbol{U}_t^i = \widehat{\gamma}_t^i(\boldsymbol{X}_t^i) \ ,$$

which, in most cases, are far from being optimal. . .

Price decomposition in the stochastic case

Apply price decomposition to the SOC problem by dualizing the spatial coupling constraint. Then a dual multiplier $\boldsymbol{\Lambda}_t^{(k)}$ appears in each subproblem i at each iteration k:

$$\min_{\boldsymbol{U}^i, \boldsymbol{X}^i} \mathbb{E}\bigg(\sum_{t=0}^{T-1} \big(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \boldsymbol{\Lambda}_t^{(k)} \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \big) + K^i(\boldsymbol{X}_T^i) \bigg) .$$

The $\boldsymbol{\Lambda}_t^{(k)}$'s are fixed random variables at this step of the algorithm. Subproblem i encompasses two noise variables, namely $\boldsymbol{W}_{t+1}$ and $\boldsymbol{\Lambda}_t^{(k)}$, but the $\boldsymbol{\Lambda}_t^{(k)}$'s may be correlated in time, in which case the white noise assumption fails!

Otherwise stated, the original state $\boldsymbol{X}_t^i$ is not a “good” state for subproblem i: the feature which seemed to have been won by decomposition is actually lost again by dynamic programming.

Summary

- On the one hand, it seems that dynamic programming cannot be decomposed in a straightforward manner.
- On the other hand, applying a decomposition scheme to a SOC problem introduces coordination instruments in the subproblems, e.g. the multipliers $\boldsymbol{\Lambda}_t^{(k)}$ in the case of price decomposition. They correspond to additional fixed random variables whose time structure is unknown,[1] so that dynamic programming cannot be used for solving the subproblems

Question: how to handle these coordination instruments in order to be able to use dynamic programming and to obtain (at least an *approximation* of) the overall optimum of the SOC problem?

[1] One can only say that $\boldsymbol{\Lambda}_t^{(k)}$ is measurable with respect to $(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$.

1 Examples and mathematical background
- Interconnected systems
- Optimization background
- Standard decomposition methods

2 About decomposition in stochastic optimization
- Couplings in stochastic optimization
- Dynamic programming and decomposition

3 Dual approximate dynamic programming (DADP)
- Problem formulation and price decomposition
- Subproblems resolution and coordination
- What has been really done?

4 Hydro valleys management problem
- DADP implementation for hydro valleys
- Numerical results for different valleys

1. Examples and mathematical background
   - Interconnected systems
   - Optimization background
   - Standard decomposition methods
2. About decomposition in stochastic optimization
   - Couplings in stochastic optimization
   - Dynamic programming and decomposition
3. Dual approximate dynamic programming (DADP)
   - Problem formulation and price decomposition
   - Subproblems resolution and coordination
   - What has been really done?
4. Hydro valleys management problem
   - DADP implementation for hydro valleys
   - Numerical results for different valleys

Optimization problem

We recall the SOC problem under consideration.

$$\min_{\boldsymbol{U},\boldsymbol{X}} \sum_{i=1}^{N} \left(\mathbb{E}\left(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \right) \right) , \tag{1a}$$

subject to dynamics constraints

$$\boldsymbol{X}_0^i = f_{-1}^i(\boldsymbol{W}_0) , \tag{1b}$$

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) , \tag{1c}$$

to measurability constraints

$$\boldsymbol{U}_t^i \preceq \sigma(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t) , \tag{1d}$$

and to spatial constraints

$$\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0 . \qquad \textbf{Coupling} \text{ constraints} \tag{1e}$$

Assumptions

Assumption 1 (White noise)

Noises $\boldsymbol{W}_0, \ldots, \boldsymbol{W}_T$ are independent over time.

Note that we also assume full noise observation:

$$\boldsymbol{U}_t^i \preceq \sigma(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t) \ .$$

As a consequence of these assumptions, there is no optimality loss to seek the controls $\boldsymbol{U}_t^i$ as a function of the state at time t rather than a function of the past noises:

$$\boldsymbol{U}_t^i = \gamma_i^t(\boldsymbol{X}_t^1, \ldots, \boldsymbol{X}_t^N) \ .$$

We are in the so-called Markovian case, and **DP** applies.

Lagrangian formulation

We dualize the coupling constraints and obtain the Lagrangian

$$\mathcal{L}(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{\Lambda}) = \sum_{i=1}^{N} \left(\mathbb{E}\left(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i) \right.\right.$$
$$\left.\left. + \sum_{t=0}^{T-1} \boldsymbol{\Lambda}_t \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \right) \right) ,$$

where the $\boldsymbol{\Lambda}_t$'s are $\sigma(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$ - measurable random variables.

We assume that a saddle point of $\mathcal{L}$ exists,[2] so that

$$\min_{\boldsymbol{U},\boldsymbol{X}} \max_{\boldsymbol{\Lambda}} \mathcal{L}(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{\Lambda}) = \max_{\boldsymbol{\Lambda}} \min_{\boldsymbol{U},\boldsymbol{X}} \mathcal{L}(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{\Lambda}) .$$

[2]Such an assumption requires going beyond the **Hilbert setting**...

Uzawa algorithm

At iteration k of the algorithm,[3]

1. Solve subproblem i, $i = 1, \dots, N$, with $\boldsymbol{\Lambda}^{(k)}$ fixed:
$$\min_{\boldsymbol{U}^i, \boldsymbol{X}^i} \mathbb{E}\left(\sum_{t=0}^{T-1} \Big(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \boldsymbol{\Lambda}_t^{(k)} \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \Big) + K^i(\boldsymbol{X}_T^i) \right) ,$$
subject to
$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) ,$$
$$\boldsymbol{U}_t^i \preceq \sigma(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t) .$$
The subproblem solution is denoted $\big(\boldsymbol{U}^{i,(k+1)}, \boldsymbol{X}^{i,(k+1)}\big)$.

2. Update the multipliers $\boldsymbol{\Lambda}_t$:
$$\boldsymbol{\Lambda}_t^{(k+1)} = \boldsymbol{\Lambda}_t^{(k)} + \rho_t \left(\sum_{i=1}^{N} \Theta_t^i\big(\boldsymbol{X}_t^{i,(k+1)}, \boldsymbol{U}_t^{i,(k+1)}\big) \right) .$$

[3] The convergence of this algorithm is **not guaranteed** in this context. . .

1. Examples and mathematical background
   - Interconnected systems
   - Optimization background
   - Standard decomposition methods
2. About decomposition in stochastic optimization
   - Couplings in stochastic optimization
   - Dynamic programming and decomposition
3. Dual approximate dynamic programming (DADP)
   - Problem formulation and price decomposition
   - Subproblems resolution and coordination
   - What has been really done?
4. Hydro valleys management problem
   - DADP implementation for hydro valleys
   - Numerical results for different valleys

Main idea of the approximation

As already seen, $\boldsymbol{\Lambda}_t^{(k)}$ depends on $(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$, so that solving a subproblem by **DP** is as complex as solving the initial problem.

In order to overcome the difficulty, we choose at each time t a random variable $\boldsymbol{Y}_t$ which is measurable w.r.t. the past noises $(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t)$. We call $\boldsymbol{Y} = (\boldsymbol{Y}_0, \ldots, \boldsymbol{Y}_{T-1})$ the information process associated to the constraint.

The core idea is to replace the multiplier $\boldsymbol{\Lambda}_t^{(k)}$ at iteration k by its conditional expectation w.r.t. $\boldsymbol{Y}_t$, that is, $\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t)$.

This will lead to a "good" approximation if

$\boldsymbol{Y}_t$ is "sufficiently" correlated to the random variable $\boldsymbol{\Lambda}_t$.

Note that we require that the information process is not influenced by controls.

Subproblem approximation

Following this idea, we replace subproblem i in Uzawa algorithm by:

$$\min_{\boldsymbol{U}^i,\boldsymbol{X}^i} \mathbb{E}\bigg(\sum_{t=0}^{T-1} \Big(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \Big) + K^i(\boldsymbol{X}_T^i) \bigg) ,$$

$$\text{subject to} \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) ,$$

$$\boldsymbol{U}_t^i \preceq \sigma(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_t) .$$

The conditional expectation $\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t)$ corresponds to a given function μ_t of the variable $\boldsymbol{Y}_t$, so that subproblem i now involves the white noise process $\boldsymbol{W}$ and the information process $\boldsymbol{Y}$. If the process $\boldsymbol{Y}$ follows a Markovian dynamics, e.g.

$$\boldsymbol{Y}_{t+1} = h_t\big(\boldsymbol{Y}_t, \boldsymbol{W}_{t+1}\big) ,$$

then $\big(\boldsymbol{X}_t^i, \boldsymbol{Y}_t\big)$ is a valid state for subproblem i and **DP** applies.

Possible choices for the information process

1. **Perfect memory**: $\boldsymbol{Y}_t = \big(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t\big)$.
   - $\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) = \boldsymbol{\Lambda}_t^{(k)}$: no approximation!
   - A valid state for each subproblem is $\big(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t\big)$.
2. **Minimal information**: $\boldsymbol{Y}_t \equiv \mathrm{cste}$.
   - $\boldsymbol{\Lambda}_t^{(k)}$ is approximated by its expectation $\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)})$.
   - The information variable does not deliver online information.
   - A valid state for subproblem i is $\boldsymbol{X}_t^i$.
3. **Static information**: $\boldsymbol{Y}_t = h_t\big(\boldsymbol{W}_t\big)$.
   - A part of $\boldsymbol{W}_t$ mostly "explains" the optimal multiplier.
   - A valid state for subproblem i is $\boldsymbol{X}_t^i$.
4. **Dynamic information**: $\boldsymbol{Y}_{t+1} = h_t\big(\boldsymbol{Y}_t, \boldsymbol{W}_{t+1}\big)$.
   - A number of possibilities: mimicking the state of another unit, adding a hidden dynamics...
   - A valid state for subproblem i is $\big(\boldsymbol{X}_t^i, \boldsymbol{Y}_t\big)$.

Dynamic programming equation

In the case of dynamic information, the **DP** equation associated to subproblem i writes:

$$
\begin{aligned}
V_T^i(x,y) &= K^i(x) \; , \\
V_t^i(x,y) &= \min_u \mathbb{E}\bigg(\Big(L_t^i(x,u,\boldsymbol{W}_{t+1}) \\
&\qquad\qquad + \mathbb{E}\big(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t = y\big) \cdot \Theta_t^i(x,u) \\
&\qquad\qquad\qquad + V_{t+1}^i\big(\boldsymbol{X}_{t+1}^i, \boldsymbol{Y}_{t+1}\big)\Big)\bigg) \; ,
\end{aligned}
$$

subject to the dynamics:

$$
\begin{aligned}
\boldsymbol{X}_{t+1}^i &= f_t^i(x,u,\boldsymbol{W}_{t+1}) \; , \\
\boldsymbol{Y}_{t+1} &= h_t(y,\boldsymbol{W}_{t+1}) \; .
\end{aligned}
$$

About the coordination: standard way

The task of coordination is performed thanks to scenarios.

- A set of noise scenarios is drawn once for all. Trajectories of the information process $\boldsymbol{Y}$ are simulated along the scenarios.
- At iteration k, the optimal trajectories of the state process $\boldsymbol{X}^{i,(k+1)}$ and of the control process $\boldsymbol{U}^{i,(k+1)}$ are simulated along the noise scenarios, for all subsystems.
- The dual multipliers are updated along the noise scenarios according to the formula:
$$\boldsymbol{\Lambda}_t^{(k+1)} = \boldsymbol{\Lambda}_t^{(k)} + \rho_t \left(\sum_{i=1}^{N} \Theta_t^i \big(\boldsymbol{X}_t^{i,(k+1)}, \boldsymbol{U}_t^{i,(k+1)} \big) \right) .$$
- The conditional expectations $\mathbb{E}(\boldsymbol{\Lambda}_t^{(k+1)} \mid \boldsymbol{Y}_t)$ are obtained by regression of the trajectories of $\boldsymbol{\Lambda}_t^{(k+1)}$ on those of $\boldsymbol{Y}_t$.

About the coordination: information based way II

One may perform the coordination by dealing with functions of $\boldsymbol{Y}_t$.

- Compute the optimal trajectories of the state process $\boldsymbol{X}^{i,(k+1)}$ and of the control process $\boldsymbol{U}^{i,(k+1)}$ along the noise scenarios.
- Compute the conditional expectation of the gradient:

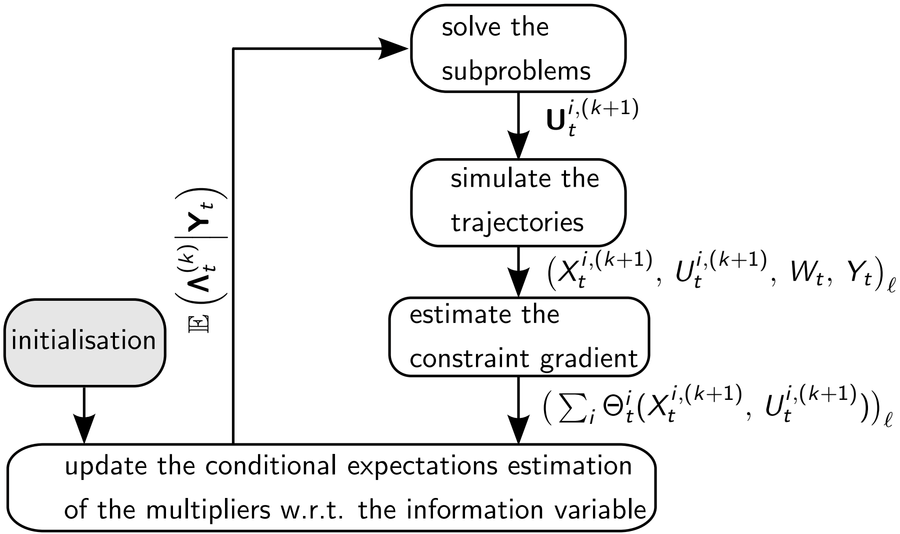
$$\mathbb{E}\left(\sum_{i=1}^{N} \Theta_t^i\big(\boldsymbol{X}_t^{i,(k+1)}, \boldsymbol{U}_t^{i,(k+1)}\big) \;\middle|\; \boldsymbol{Y}_t \right) .$$

- Update the conditional expectation of the multipliers:

$$\mathbb{E}(\boldsymbol{\Lambda}_t^{(k+1)} \mid \boldsymbol{Y}_t) = \mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) + \rho_t \, \mathbb{E}\left(\sum_{i=1}^{N} \Theta_t^i\big(\boldsymbol{X}_t^{i,(k+1)}, \boldsymbol{U}_t^{i,(k+1)}\big) \;\middle|\; \boldsymbol{Y}_t \right) .$$

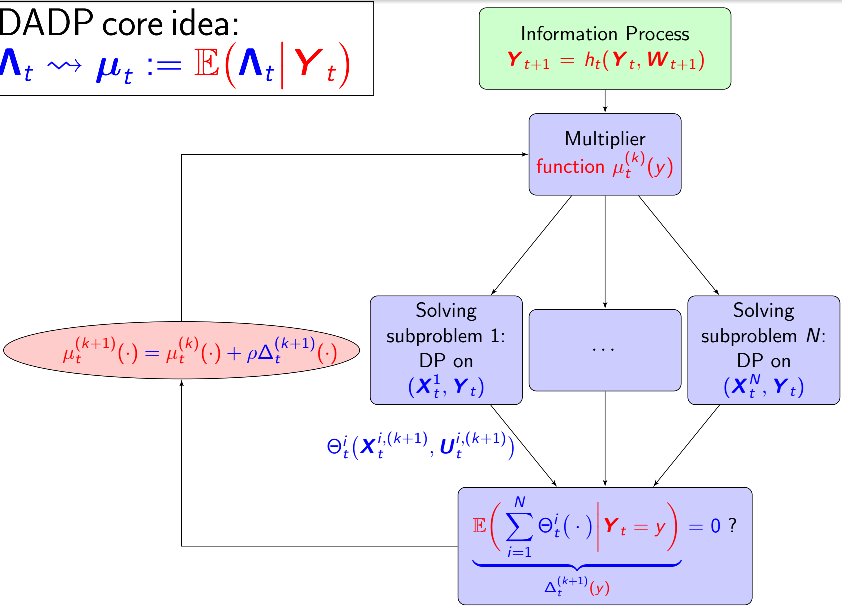
Many numerical advantages if the support of $\boldsymbol{Y}_t$ is finite.

DADP flowchart



DADP core idea:

$$\boldsymbol{\Lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\big(\boldsymbol{\Lambda}_t \,\big|\, \boldsymbol{Y}_t\big)$$



1. Examples and mathematical background
   - Interconnected systems
   - Optimization background
   - Standard decomposition methods
2. About decomposition in stochastic optimization
   - Couplings in stochastic optimization
   - Dynamic programming and decomposition
3. Dual approximate dynamic programming (DADP)
   - Problem formulation and price decomposition
   - Subproblems resolution and coordination
   - What has been really done?
4. Hydro valleys management problem
   - DADP implementation for hydro valleys
   - Numerical results for different valleys

Interpretation of DADP

The approximation made on the dual process gives us a tractable way of computing strategies for the subsystems. Let us examine precisely the consequences in terms of constraints.

Consider a relaxed problem derived from (1):

$$\min_{\boldsymbol{U},\boldsymbol{X}} \mathbb{E}\bigg(\sum_{i=1}^{N}\Big(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i)\Big)\bigg) , \tag{2a}$$

subject to the modified coupling constraints:

$$\mathbb{E}\Big(\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \;\Big|\; \boldsymbol{Y}_t\Big) = 0 . \tag{2b}$$

Interpretations of DADP II

Proposition 1

Suppose the Lagrangian associated with Problem (2) has a saddle point. Then the DADP algorithm can be interpreted as the Uzawa algorithm applied to Problem (2).

Proof. Since the duality term $\mathbb{E}\big(\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i)\big)$ which appears in the cost function of subproblem i in DADP can be written:

$$\mathbb{E}\big(\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \boldsymbol{Y}_t) \cdot \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i)\big) = \mathbb{E}\big(\boldsymbol{\Lambda}_t^{(k)} \cdot \mathbb{E}(\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \mid \boldsymbol{Y}_t)\big) ,$$

the global constraint really handled by DADP is:

$$\mathbb{E}\Big(\sum_{i=1}^{N} \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \;\Big|\; \boldsymbol{Y}_t\Big) = 0 . \qquad \square$$

DADP thus consists in replacing an almost-sure constraint by its conditional expectation w.r.t. the information variable $\boldsymbol{Y}_t$.

Interpretations of DADP III

- DADP as an approximation of the optimal multiplier

$$\boldsymbol{\Lambda}_t \quad \rightsquigarrow \quad \mathbb{E}\big(\boldsymbol{\Lambda}_t \bigm| \boldsymbol{Y}_t\big) \ .$$

- DADP as a decision-rule approach for the dual problem

$$\max_{\boldsymbol{\Lambda}} \ \min_{\boldsymbol{U},\boldsymbol{X}} \ \mathcal{L}\big(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\Lambda}\big) \quad \rightsquigarrow \quad \max_{\boldsymbol{\Lambda}_t \preceq \boldsymbol{Y}_t} \ \min_{\boldsymbol{U},\boldsymbol{X}} \ \mathcal{L}\big(\boldsymbol{X},\boldsymbol{U},\boldsymbol{\lambda}\big) \ .$$

- DADP as a constraint relaxation for the primal problem

$$\sum_{i=1}^{N} \Theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i\big) = 0 \quad \rightsquigarrow \quad \mathbb{E}\Big(\sum_{i=1}^{N} \Theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i\big) \Bigm| \boldsymbol{Y}_t\Big) = 0 \ .$$

Thanks to the last interpretation, the optimal value given by DADP is a guaranteed lower bound for the original problem value.

Summary

To summarize, **DADP** leads to solve the approximated problem

$$\min_{\boldsymbol{U},\boldsymbol{X}} \mathbb{E}\bigg(\sum_{i=1}^{N}\sum_{t=0}^{T-1}\Big(L_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t)+K^i(\boldsymbol{X}_T^i)\Big)\bigg) \text{ s.t. } \mathbb{E}\bigg(\sum_{i=1}^{N}\Theta_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i)\ \bigg|\ \boldsymbol{Y}_t\bigg)=0\ ,$$

whereas the true problem is

$$\min_{\boldsymbol{U},\boldsymbol{X}} \mathbb{E}\bigg(\sum_{i=1}^{N}\sum_{t=0}^{T-1}\Big(L_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t)+K^i(\boldsymbol{X}_T^i)\Big)\bigg) \text{ s.t. } \sum_{i=1}^{N}\Theta_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i)=0\ .$$

Questions:

⋆ What is the suitable theoretical framework of the algorithm?
 ◇ Existence of a multiplier ?
 ◇ Convergence of the algorithm ?

⋆ Does the approximate solution converge to the true solution?

⋆ How to obtain a feasible solution from the approximate solution?

Some questions

⋆ What is the suitable theoretical framework of the algorithm?

The convergence of Uzawa's algorithm is granted provided that:

- the problem is posed in Hilbert spaces,
- and a saddle point exists.

It thus seems natural to place ourselves in a Hilbert space. But it is known (works by Rockafellar and Wets) that a saddle point doesn't exist in Hilbert spaces for such problems... *(See V. Leclère thesis.)*

⋆ Does the approximate solution converge to the true solution?

Epiconvergence results are available w.r.t. the information given by $\boldsymbol{Y}_t$. But epiconvergence raises technical problems when addressed to stochastic optimization problems. *(See V. Leclère thesis.)*

⋆ How to obtain a feasible solution from the approximate solution?

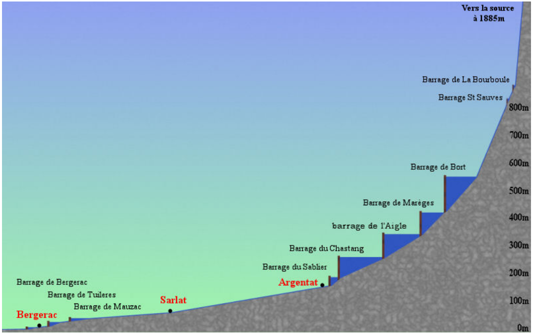
Use an appropriate heuristic! *(See J.-C. Alais thesis.)*

1. Examples and mathematical background
   - Interconnected systems
   - Optimization background
   - Standard decomposition methods
2. About decomposition in stochastic optimization
   - Couplings in stochastic optimization
   - Dynamic programming and decomposition
3. Dual approximate dynamic programming (DADP)
   - Problem formulation and price decomposition
   - Subproblems resolution and coordination
   - What has been really done?
4. Hydro valleys management problem
   - DADP implementation for hydro valleys
   - Numerical results for different valleys

Some French hydro valleys

Motivation

Electricity production management for large hydro valleys



- *1 year time horizon*: compute each month the "values of water" (cost-to-go functions)
- *stochastic framework*: rain, market prices
- *large-scale valley*: 4 dams and more

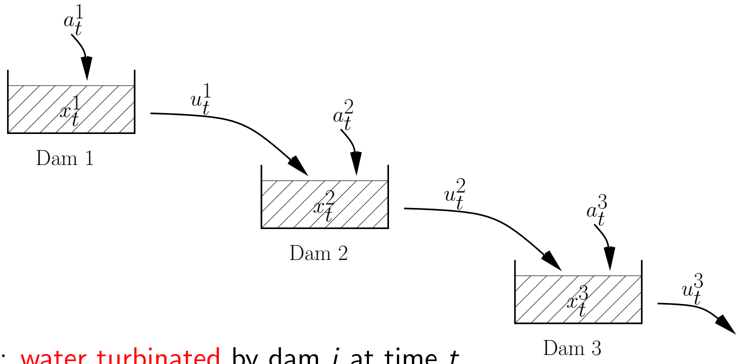
1 Examples and mathematical background
- Interconnected systems
- Optimization background
- Standard decomposition methods

2 About decomposition in stochastic optimization
- Couplings in stochastic optimization
- Dynamic programming and decomposition

3 Dual approximate dynamic programming (DADP)
- Problem formulation and price decomposition
- Subproblems resolution and coordination
- What has been really done?

4 Hydro valleys management problem
- DADP implementation for hydro valleys
- Numerical results for different valleys

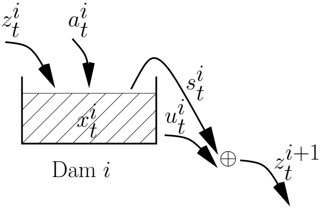
Operating scheme



u_t^i : water turbinated by dam i at time t,
x_t^i : water volume of dam i at time t,
a_t^i : water inflow at dam i at time t,
p_t^i : water price at dam i at time t,

Randomness: $w_t^i = (a_t^i, p_t^i) \quad , \quad w_t = (w_t^1, \ldots, w_t^N)$.

Dynamics and cost functions



Dam dynamics

$$x_{t+1}^i = f_t^i(x_t^i, u_t^i, w_t^i, z_t^i) \; ,$$
$$= x_t^i - u_t^i + a_t^i + z_t^i - s_t^i \; ,$$

z_t^{i+1} being the outflow of dam i:

$$z_t^{i+1} = g_t^i(x_t^i, u_t^i, w_t^i, z_t^i) \; ,$$
$$= u_t^i + \underbrace{\max\left\{0, x_t^i - u_t^i + a_t^i + z_t^i - \overline{x}^i\right\}}_{s_t^i} \; .$$

We assume that u_t^i is chosen once w_t^i is observed (HD information structure), so that $\underline{u}^i \leq u_t^i \leq \min\left\{\overline{u}^i, x_t^i + a_t^i + z_t^i - \underline{x}^i\right\}$.

Gain at time $t < T$: $L_t^i(x_t^i, u_t^i, w_t^i, z_t^i) = p_t^i u_t^i - \epsilon (u_t^i)^2$.

Final gain at time T: $K^i\left(x_T^i\right) = -a^i \min\{0, x_T^i - \widehat{x}^i\}^2$.

Stochastic optimization problem

The global optimization problem reads:

$$\max_{(\boldsymbol{X},\boldsymbol{U},\boldsymbol{Z})} \mathbb{E}\left(\sum_{i=1}^{N}\left(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t^i,\boldsymbol{Z}_t^i) + K^i(\boldsymbol{X}_T^i)\right)\right),$$

subject to:

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t^i,\boldsymbol{Z}_t^i)\ ,\ \forall i\ ,\ \forall t\ ,$$

$$\boldsymbol{U}_t^i \preceq \sigma\big(\boldsymbol{W}_0,\ldots,\boldsymbol{W}_t\big)\ ,\ \forall i\ ,\ \forall t\ ,$$

$$\boldsymbol{Z}_t^{i+1} = g_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t^i,\boldsymbol{Z}_t^i)\ ,\ \forall i\ ,\ \forall t\ .$$

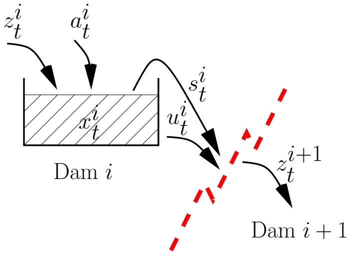
$\rightsquigarrow$ Additive structure ("cascade" model).

Assumption. Noises $\boldsymbol{W}_0,\ldots,\boldsymbol{W}_{T-1}$ are *independent over time*.

Price decomposition

- Dualize the coupling constraints $\boldsymbol{Z}_t^{i+1} = g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^i)$. Denote by $\boldsymbol{\Lambda}_t^{i+1}$ the associated multiplier (random variable).

- Minimize the dual problem (using a gradient-like algorithm).



- At iteration k, the duality term:

  $\boldsymbol{\Lambda}_t^{i+1,(k)} \cdot \big(\boldsymbol{Z}_t^{i+1} - g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^i)\big)$,

  is the difference of two terms:

  - $\boldsymbol{\Lambda}_t^{i+1,(k)} \cdot \boldsymbol{Z}_t^{i+1} \rightsquigarrow$ dam $i+1$,
  - $\boldsymbol{\Lambda}_t^{i+1,(k)} \cdot g_t^i(\cdots) \rightsquigarrow$ dam i.

- Dam by dam decomposition for the maximization in $(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{Z})$ at $\boldsymbol{\Lambda}_t^{i+1,(k)}$ fixed.

DADP implementation

DADP approximation:

- replace the constraint $\boldsymbol{Z}_t^{i+1} - g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^i) = 0$ by its conditional expectation with respect to $\boldsymbol{Y}_t^i$:
$$\mathbb{E}\big(\boldsymbol{Z}_t^{i+1} - g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^i) \bigm| \boldsymbol{Y}_t^i\big) = 0 ,$$
- where $(\boldsymbol{Y}_0^i, \ldots, \boldsymbol{Y}_{T-1}^i)$ is a "well-chosen" information process.

The expression of the i-th subproblem becomes:

$$\max_{\boldsymbol{U}^i, \boldsymbol{Z}^i, \boldsymbol{X}^i} \mathbb{E}\Bigg(\sum_{t=0}^{T-1} \Big(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^i) + \mathbb{E}\big(\boldsymbol{\Lambda}_t^{i,(k)} \bigm| \boldsymbol{Y}_t^{i-1}\big) \cdot \boldsymbol{Z}_t^i$$
$$- \mathbb{E}\big(\boldsymbol{\Lambda}_t^{i+1,(k)} \bigm| \boldsymbol{Y}_t^i\big) \cdot g_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^i\big)\Big)$$
$$+ K^i\big(\boldsymbol{X}_T^i\big)\Bigg) .$$

A crude relaxation: $\boldsymbol{Y}_t^i \equiv \mathrm{cste}$

1. The multipliers $\boldsymbol{\Lambda}_t^{i,(k)}$ appear only in the subproblems by means of their expectations $\mathbb{E}\big(\boldsymbol{\Lambda}_t^{i,(k)}\big)$, so that each subproblem involves the 1-dimensional state variable $\boldsymbol{X}_t^i$.
2. For the gradient algorithm, the coordination task reduces to:

$$\mathbb{E}\big(\boldsymbol{\Lambda}_t^{i,(k+1)}\big) = \mathbb{E}\big(\boldsymbol{\Lambda}_t^{i,(k)}\big) - \rho_t \, \mathbb{E}\Big(\boldsymbol{Z}_t^{i+1,(k)} - g_t^i\big(\boldsymbol{X}_t^{i,(k)}, \boldsymbol{U}_t^{i,(k)}, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^{i,(k)}\big)\Big) \; .$$

3. The constraints taken into account by DADP are in fact

$$\mathbb{E}\Big(\boldsymbol{Z}_t^{i+1} - g_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^i\big)\Big) = 0 \; .$$

The solutions do not satisfy the initial almost sure constraints: need to use a heuristic method to regain admissibility.

How to regain admissible policies?

We have computed N local Bellman functions V_t^i at each time t, each depending on a single state variable x^i, whereas we want a unique global Bellman function V_t depending on $(x^1, \ldots, x^N)$.

Value function approximation: form the following functions:

$$\widehat{V}_t(x^1, \ldots, x^N) = \sum_{i=1}^{N} V_t^i(x^i) \ .$$

For any (x_t, w_t) at time t, solve the one-step DP problem:

$$\max_{u,z} \ \sum_{i=1}^{N} L_t^i(x_t^i, u^i, w_t^i, z^i) + \widehat{V}_{t+1}(x_{t+1}^1, \ldots, x_{t+1}^N) \ ,$$

$$s.t. \quad x_{t+1}^i = f_t^i(x_t^i, u^i, w_t^i, z^i) \quad \text{and} \quad z^{i+1} = g_t^i(x_t^i, u^i, w_t^i, z^i) \ .$$

$\rightsquigarrow$ control value $u_t^\sharp = (u_t^{1,\sharp}, \ldots, u_t^{N,\sharp})$ to be used at (x_t, w_t).

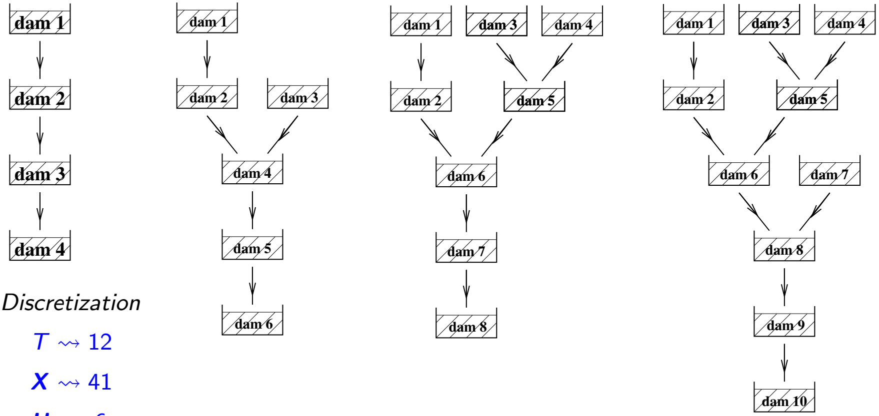
Full optimization and simulation process

- **Optimization**
  - Apply DADP and compute the cost-to-go functions V_t^i.
  - Form the approximated global Bellman functions $\widehat{V}_t$.

- **Simulation**
  - Draw a large number of noise scenarios.
  - Compute the control values along each scenario by solving the one-step DP problems involving the $\widehat{V}_t$'s, thus satisfying all the constraints of the initial problem:
    $\leadsto$ payoff value for each scenario,
    $\leadsto$ state and control trajectories.
  - Evaluate the quality of the solution: mean payoff,...

1. Examples and mathematical background
   - Interconnected systems
   - Optimization background
   - Standard decomposition methods
2. About decomposition in stochastic optimization
   - Couplings in stochastic optimization
   - Dynamic programming and decomposition
3. Dual approximate dynamic programming (DADP)
   - Problem formulation and price decomposition
   - Subproblems resolution and coordination
   - What has been really done?
4. Hydro valleys management problem
   - DADP implementation for hydro valleys
   - Numerical results for different valleys

Academic case studies of increasing complexity



Discretization

$T \rightsquigarrow 12$

$\boldsymbol{X} \rightsquigarrow 41$

$\boldsymbol{U} \rightsquigarrow 6$

$\boldsymbol{W} \rightsquigarrow 10$

4-Dams 6-Dams 8-Dams 10-Dams

Optimal values and computational times

Valley	**4-Dams**	**6-Dams**	**8-Dams**	**10-Dams**
DP CPU time	*1.6 10^3'*	$\sim$ *10^8'*	$\sim \infty$	$\sim \infty$
DP value	3743	N.A.	N.A.	N.A.
$\mathrm{SDDP_c}$ value	3742	7026	11834	17069
$\mathrm{SDDP_c}$ CPU time	*5*'	*7*'	*9*'	*50*'
Valley	**4-Dams**	**6-Dams**	**8-Dams**	**10-Dams**

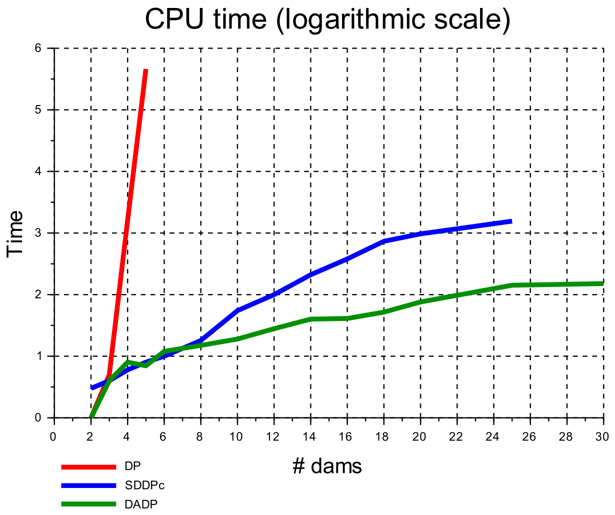
Table: Results obtained by DP and $\mathrm{SDDP_c}$[4]

Valley	**4-Dams**	**6-Dams**	**8-Dams**	**10-Dams**
DADP CPU time	*7*'	*12*'	*17*'	*24*'
DADP value	3667	6816	11573	16760
Gap with $\mathrm{SDDP_c}$	$-$**2.0%**	$-$**3.0%**	$-$**2.2%**	$-$**1.8%**

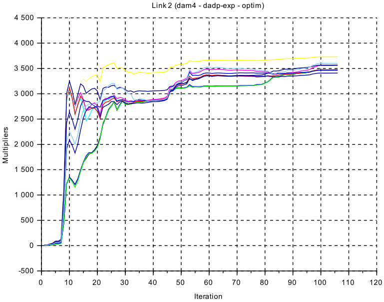
Table: Results obtained by DADP "Expectation"

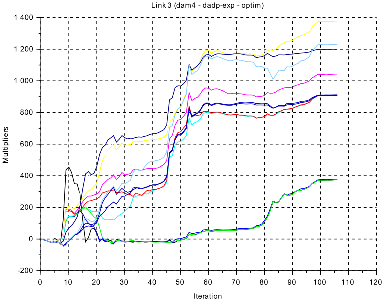
[4] The SDDP method will be explained in detail next Monday.

CPU time summary



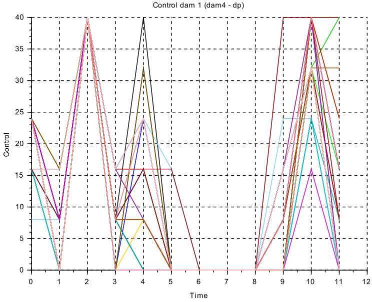
4-Dams in detail: DADP convergence



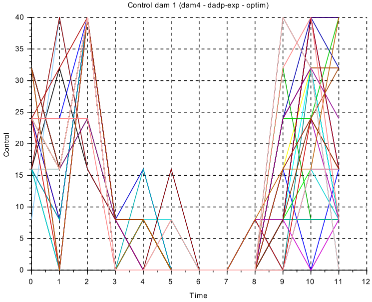


Multipliers convergence (dam1↔dam2 and dam2↔dam3)

4-Dams in detail: control trajectories

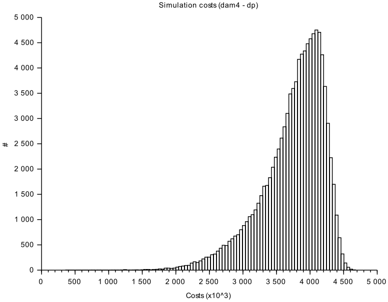


DP: dam 1 trajectories

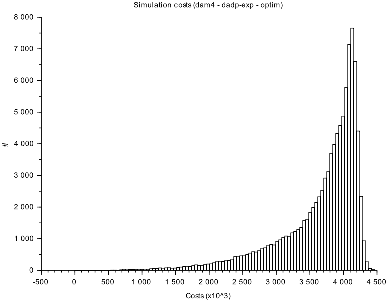


DADP: dam 1 trajectories

4-Dams in detail: payoff distributions

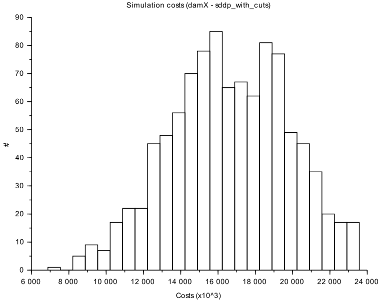


DP payoff distribution

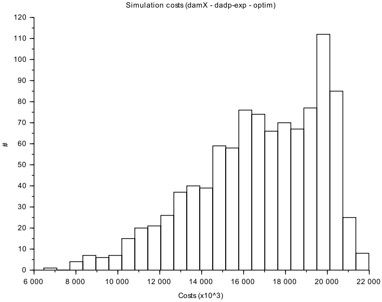


DADP payoff distribution

10-Dams in detail: payoff distribution

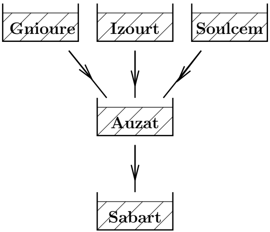


SDDP payoff distribution



DADP payoff distribution

Two “true” valleys

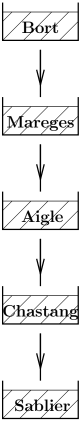


Discretization

$T \rightsquigarrow 12$, $\boldsymbol{W} \rightsquigarrow 10$

realistic grids for $\boldsymbol{U}$ and $\boldsymbol{X}$

Vicdessos



Dordogne

Results

Valley	**Vicdessos**	**Dordogne**
$\mathrm{SDDP_c}$ CPU time	*9*'	*17*'
$\mathrm{SDDP_c}$ value	2244	22145

Table: Results obtained by $\mathrm{SDDP_c}$

Valley	**Vicdessos**	**Dordogne**
DADP CPU time	*9*'	*210*'
DADP value	2237	21652
Gap with $\mathrm{SDDP_c}$	$-\mathbf{0.3\%}$	$-\mathbf{2.2\%}$

Table: Results obtained by DADP "Expectation"

Conclusions and perspectives

Conclusions for DADP

- Fast numerical convergence of the method.
- Near-optimal results even when using a “crude” relaxation.
- Method that can be used for very large valleys

General perspectives

- Apply to more complex topologies (smart grids).
- Connection with other decomposition methods.
- Theoretical study.

C. Strugarek.
Approches variationnelles et contributions en optimisation stochastique.
Thèse de doctorat, Ecole Nationale des Ponts et Chausées, 2006.

P. Girardeau.
Résolution de grands problèmes en optimisation stochastique dynamique.
Thèse de doctorat, Université Paris-Est, 2010.

J.-C. Alais.
Risque et optimisation pour le management d'énergies.
Thèse de doctorat, Université Paris-Est, 2013.

V. Leclère.
Contributions aux méthodes de décomposition en optimisation stochastique.
Thèse de doctorat, Université Paris-Est, 2014.

K. Barty, P. Carpentier, G. Cohen and P. Girardeau,
Price decomposition in large-scale stochastic optimal control.
arXiv, math.OC 1012.2092, 2010.