The optimal harvesting problem under price uncertainty
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... [We] obtain the result that uncertainty lengthens the optimal rotation. [...] Under the mean reverting price process, optimal harvesting becomes more sensitive to price level,[...]. Including risk aversion completely changes the harvesting policy.

Tahvonen & Kallio (2006)
The optimal harvesting problem with uncertainty has been considerably studied, but the vast majority of papers present numerical solutions assuming single stands, random walk price process and risk neutrality. We work with multiple stands, random walk and mean reverting price process to characterize theoretically the optimal harvest, risk aversion but we consider forest growth as a deterministic and pure aging process.
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Multi-period choice under uncertainty

- Stochastic Dynamic Programming. Or ...

- Markov Decision Process, or

- Multistage Stochastic Programming, or

- Intertemporal Consumption, or

- Life-Cycle Consumption, or...

They all want to solve the same problem: optimal decision making over time, often under uncertainty.
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Multistage stochastic programming literature

- Extensive research in portfolio selection, hydrothermal scheduling, production planning and others.

- Popular algorithms include the Nested L-Shaped (Birge ’85), SDDP (Pereira and Pinto ’91), Progressive Hedging (Rockafellar and Wets ’91), SAA (Shapiro ’03, ’06), ADP (Powell ’07).

- We will see that harvesting constraints are unique and are not equivalent to stocks’ buy-and-sell constraints and to the hydric balance equation.

- General purpose algorithms are not readily applicable.
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We will see that harvesting constraints are unique and are not equivalent to stocks’ buy-and-sell constraints and to the hydric balance equation.

General purpose algorithms are not readily applicable.
The one million dollar question:

If price is uncertain,
is it better to harvest everything available now or
is it worth waiting for prices to rise???
- Before the maturity age: harvest forbidden
- After the maturity age: trees do not grow

State at time $t$

$$\mathbf{X}(t) = \begin{pmatrix} \bar{x}(t) \\ x_n(t) \\ x_{n-1}(t) \\ \vdots \\ x_2(t) \\ x_1(t) \end{pmatrix}$$

- $x_a(t)$: surface occupied by trees of age $a$ at time $t$
- $\bar{x}(t)$: surface occupied by trees beyond maturity at time $t$
- Before the maturity age: harvest forbidden
- After the maturity age: trees do not grow

State at time $t$

$$\mathbf{X}(t) = \begin{pmatrix} \bar{x}(t) \\ x_n(t) \\ x_{n-1}(t) \\ \vdots \\ x_2(t) \\ x_1(t) \end{pmatrix}$$

$x_a(t)$: surface occupied by trees of age $a$ at time $t$

$\bar{x}(t)$: surface occupied by trees beyond maturity at time $t$
Order of events

\[(p(1), X(1)) \leadsto c(1) \leadsto X(2) \leadsto p(2) \leadsto c(2) \leadsto X(3) \leadsto \cdots \leadsto c(T - 1) \leadsto X(T) \leadsto p(T) \leadsto c(T).\]
At every time $t$:

- Knowing $p(t)$ and $\mathbf{X}(t)$ we must choose how much to harvest: $c(t)$

\[ 0 \leq c(t) \leq \bar{x}(t) + x_n(t) \].

At time $t$, depending on $c(t)$

- Benefit $c(t)p(t)$

- State $\mathbf{X}(t) = \begin{pmatrix} \bar{x} \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} \rightarrow \mathbf{X}(t+1) = \begin{pmatrix} \bar{x} + x_3 - c(t) \\ x_2 \\ x_1 \\ c(t) \end{pmatrix}$
Dynamics

At every time $t$:
- Knowing $p(t)$ and $\mathbf{X}(t)$ we must choose how much to harvest: $c(t)$
  
  \[
  0 \leq c(t) \leq \overline{x}(t) + x_n(t).
  \]

At time $t$, depending on $c(t)$
- Benefit $c(t)p(t)$
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Example: \( n = 3, S = 6: \)

\[
\begin{pmatrix}
1 \\
2 \\
1 \\
2
\end{pmatrix} \rightarrow \begin{pmatrix}
3 \\
1 \\
2 \\
0
\end{pmatrix} \rightarrow \begin{pmatrix}
0 \\
2 \\
0 \\
4
\end{pmatrix}
\]

\( c(1) = 0 \quad c(2) = 4 \)

Total benefit: \( c(1)p(1) + \delta c(2)p(2) = \delta 4 p(2) \)

\( \delta \in (0, 1) \) discount factor.
Optimization problem

Deterministic case with constant price $p(t) = \bar{p}$.

Optimization problem:

$$V_1(p(1), X(1)) = \begin{cases} \begin{array}{l} \text{Max}_{c(1), \ldots, c(T)} \sum_{t=1}^{T} \delta^{t-1} \bar{p} \: c(t). \\ \text{s.t.} \text{ feasibility constraints.} \end{array} \end{cases}$$

- The *greedy* policy is optimal (Rapaport et al., 2003.)
- Always harvest everything available.

\[
\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \rightarrow \ldots
\]

$c(1) = 3 \quad c(2) = 1 \quad c(3) = 2 \quad c(4) = 3 \quad c(5) = 1$
Optimization problem

Deterministic case with constant price \( p(t) = \bar{p} \).

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\[
V_1(p(1), X(1)) = \begin{cases} 
\max_{c(1), \ldots, c(T)} & \sum_{t=1}^{T} \delta^{t-1} \bar{p} \ c(t) \\
\text{s.t.} & \text{feasibility constraints.}
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\]

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\[
\begin{pmatrix}
1 \\
2 \\
1 \\
2
\end{pmatrix} \rightarrow 
\begin{pmatrix}
0 \\
1 \\
2 \\
3
\end{pmatrix} \rightarrow 
\begin{pmatrix}
0 \\
2 \\
3 \\
1
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\[
c(1) = 3 \\
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c(5) = 1
\]
Optimization problem - risk neutral framework

Objective function: $\mathbb{E} \left[ \sum_{t=1}^{T} \delta^{t-1} p(t)c(t) \right]$

Optimization problem:

$V_1(p(1), X(1)) = \begin{cases} 
\max_{c(1), \ldots, c(T)} & \mathbb{E} \left[ \sum_{t=1}^{T} \delta^{t-1} p(t)c(t) \right] \\
\text{s.t.} & \text{feasibility constraints}.
\end{cases}$

Dynamic programming equations:

$V_t(p(t), X(t)) = \begin{cases} 
\max_{c(t)} & \mathbb{E} \left[ p(t)c(t) + \delta V_{t+1}(\cdot, \cdot) | p(t) \right] \\
\text{s.t.} & \text{feasibility constraints}
\end{cases}$
Optimization problem - risk neutral framework

Objective function: \( \mathbb{E} \left[ \sum_{t=1}^{T} \delta^{t-1} p(t)c(t) \right] \)

Optimization problem:

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V_1(p(1), X(1)) = \begin{cases} 
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\end{cases}
\]
**Geometric Brownian Motion**

**Definition:**

\[ dp_t = \mu p_t \, dt + \sigma p_t \, dW_t \]

**Drift:**

\[ \mu \in \mathbb{IR} \]

**Volatility:**

\[ \sigma \in \mathbb{IR}_+ \]

\[ \mathbb{E}[p(t + 1)|p(t)] = p(t)e^{\mu} \]
Optimal policy for GBM

Theorem

If condition $1 \geq \delta e^\mu$ holds, the optimal policy is Greedy.

Proof: The coefficient of $c$ in the Bellman eq.

$$V_t(p(t), \bar{X}(t)) = \text{Max}_c \{ p(t)c + \delta \mathbb{E}_{p(t+1)} [V_{t+1}(p(t+1), \bar{X}(t+1)|p(t)] \}$$

is

$$p(t)(1 - \delta e^\mu) \sum_{k=0}^{K} \delta^k n e^{(kn)\mu} \geq 0$$
But what is the intuition behind this result?

\[
\delta \mathbb{E}[p(t+1) | p(t)] \leq p(t), \quad t = 1, \ldots, T - 1, \\
\delta p(t)e^{\mu} \leq p(t), \quad t = 1, \ldots, T - 1,
\]

which is equivalent to

\[
\delta e^{\mu} \leq 1.
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\begin{align*}
\delta \mathbb{E}[p(t + 1) | p(t)] &\leq p(t), \ t = 1, \ldots, T - 1, \\
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But what is the intuition behind this result?

future $\leq$ present

$$\delta \mathbb{E}[p(t + 1)|p(t)] \leq p(t), \ t = 1, \ldots, T - 1,$$
$$\delta p(t)e^{\mu} \leq p(t), \ t = 1, \ldots, T - 1,$$

which is equivalent to

$$\delta e^{\mu} \leq 1.$$
Shaded area:

\[ 1 - \delta e^\mu > 0 \] holds

**Greedy policy is optimal**

(every state & price realization)

What if \( 1 - \delta e^\mu < 0 \)?
Shaded area:

$1 - \delta e^\mu > 0$ holds

**Greedy policy is optimal**

(every state & price realization)

What if $1 - \delta e^\mu < 0$?
Another optimal policy for GBM

If $1 - \delta e^\mu < 0$, it is optimal to **postpone** the harvest.

- Harvesting is allowed **only** at: $T$, $T - n$, $T - 2n$, ...
- Every mature tree is cut

$$c(t) = \begin{cases} 
\bar{x}(t) + x_n(t) & \text{if } t = T - kn \\
0 & \text{else}
\end{cases}$$

Example with $T = 8$ and $n = 3$, $S = 6$.

\[
\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{pmatrix}
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$t = 1$ $t = 2$ $t = 3$ $t = 4$ $t = 5$ $t = 6$ $t = 7$ $t = 8$

Proof, using backwards induction on $t$. 
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\( t = 1 \quad t = 2 \quad t = 3 \quad t = 4 \quad t = 5 \quad t = 6 \quad t = 7 \quad t = 8 \)

Proof, using backwards induction on \( t \).
Ornstein-Uhlenbeck

Definition: \[ dp_t = \eta (\bar{p} - p_t) dt + \sigma dW_t \]

Equilibrium: \[ \bar{p} \]

Rate of mean-reversion: \[ \eta \in \mathbb{IR}_+ \]

Volatility: \[ \sigma \in \mathbb{IR}_+ \]

\[ \mathbb{E}[p(t+1)|p(t)] = p(t)e^{-\eta} + \bar{p}(1 - e^{-\eta}) \]
What if we do the same trick?

\[ \delta \mathbb{E}_{t}[p(t+1)] \leq p(t), \]
\[ \delta[p(t)e^{-\eta} + \bar{p}(1 - e^{-\eta})] \leq p(t), \]

which is equivalent to

\[ \frac{p(t)}{\bar{p}} \geq \frac{\delta(1 - e^{-\eta})}{(1 - \delta e^{-\eta})} := r. \]  

Obs.: Condition depends on \( p(t) \).
What if we do the same trick?

\[
\delta E[p(t)[p(t+1)] \leq p(t),
\]
\[
\delta[p(t)e^{-\eta} + \bar{p}(1 - e^{-\eta})] \leq p(t),
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Obs.: Condition depends on \( p(t) \).
Theorem

If there is $t$ such that $p(t) \geq r\bar{p}$, then $c^*(t) = \bar{x}(t) + x_n(t)$

- If $p(t) \geq r\bar{p}$ the optimal decision at that particular time $t$ is to harvest everything available
- If $p(t) < r\bar{p}$, then $c^*(t) = ?$

Numerical experiments:

- We use $r\bar{p}$ as a reservation price.
- Results within 5% of the optimum for some parameter values.
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Risk measures

1952
Markowitz  CAPM

1994
J.P. Morgan  Jorion

1999
Artzner et.al  Coherency

Variance
VaR  CVaR
How bad is bad? Conditional Value-at-Risk

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The optimal harvesting problem under price uncertainty
Conditional Value-at-Risk, Average Value-at-Risk, Expected Tail Loss and Expected Shortfall are all the same thing!

Formally, we define $\text{CVaR}_\alpha [X] = \inf_{t \in \mathbb{R}} \left\{ t + \frac{1}{\alpha} \mathbb{E} [X - t]_+ \right\} = \mathbb{E} [X | X > \text{VaR}_\alpha]$. (Cont. case)

The Value-at-Risk:

$$\text{VaR}_\alpha [X] = \inf \{ x : \mathbb{P}(X \leq x) \geq 1 - \alpha \}, \quad \alpha \in (0, 1).$$

What happens when we incorporate the CVaR in our forestry model?
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\]

What happens when we incorporate the CVaR in our forestry model?
1) \( \rho(X + c) = \rho(X) + c \).

2) \( X \leq Y \Rightarrow \rho(X) \leq \rho(Y) \).

3) \( \rho(\lambda X) = \lambda \rho(X) \) for \( \lambda \geq 0 \).

4) \( \rho(X + Y) \leq \rho(X) + \rho(Y) \).

A risk measure that satisfies axioms 1) – 4) is called coherent.
Coherent risk measures

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A risk measure that satisfies axioms 1) – 4) is called *coherent*. 
Theorem - CVaR under GBM

Risk averse case, GBM

If prices evolve according to a GBM and condition

$$\delta e^{\mu} \frac{1}{\alpha} \Phi(\Phi^{-1}(\alpha) - \sigma) \leq 1$$

holds, the greedy policy is optimal.

Observation: If the condition is not satisfied the optimal policy is analogous to the one we obtained for the risk neutral case.
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Sensitivity analysis

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The optimal harvesting problem under price uncertainty
Theorem - CVaR under O-U

Risk averse case, O-U

If prices evolve according to an O-U process and

$$p(t) \geq \bar{p}r - \frac{\delta}{1 - \delta e^{-\eta}} \frac{\sigma}{\sqrt{2\pi}} \sqrt{\frac{(1 - e^{-2\eta})}{2\eta}} \frac{e^{-z^2/2\alpha}}{\alpha} = pr_{O-U}$$

holds at time $t$, the greedy policy is optimal at time $t$.

Some observations:

- When $\eta \to \infty \Rightarrow$, the reservation price goes to $\delta \bar{p} \Rightarrow$ Greedy policy is optimal.
- When $\alpha \to 1 \Rightarrow$, reservation prices coincide.
- When $\sigma \downarrow$, reservation prices coincide.
- When $\sigma \uparrow$, reservations prices are smaller than $\bar{p}r$. 
Risk averse case, O-U

If prices evolve according to an O-U process and

\[ p(t) \geq \bar{pr} - \frac{\delta}{1 - \delta e^{-\eta}} \frac{\sigma}{\sqrt{2\pi}} \sqrt{\frac{(1 - e^{-2\eta})}{2\eta}} \frac{e^{-z^2/2}}{\alpha} = p_r^{O-U} \]

holds at time \( t \), the greedy policy is optimal at time \( t \).

Some observations:

- When \( \eta \to \infty \Rightarrow \), the reservation price goes to \( \delta \bar{p} \Rightarrow \) Greedy policy is optimal.
- When \( \alpha \to 1 \Rightarrow \), reservation prices coincide.
- When \( \sigma \downarrow \), reservation prices coincide.
- When \( \sigma \uparrow \), reservations prices are smaller than \( \bar{pr} \).
Present and future research

- Multistage least cost plus damage
- Inclusion of forest fires
- Include diameter as a state variable, instead of (or in addition to) age
- Growth as a stochastic process and consider natural mortality
Merci!