8/10/2015
MPRO - Stochastic Optimization

Homework to be done for the 14/10/2015 The three exercises are independent. When possible underline the final answers.

Exercise 1 (5pts). *Consider a control dynamical system following*

$$\boldsymbol{x}_{t+1} = f_t\big(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{w}_{t+1}\big) \ , \tag{1}$$

where

- *the state $\boldsymbol{x}_t$ belongs to a Cartesian product finite set $\mathbb{X} \times \cdots \times \mathbb{X} = \mathbb{X}^{n_x}$;*
- *the control $\boldsymbol{u}_t$ belongs to a Cartesian product finite set $\mathbb{U} \times \cdots \times \mathbb{U} = \mathbb{U}^{n_u}$;*
- *the noise $\boldsymbol{w}_t$ belongs to a Cartesian product finite set $\mathbb{W} \times \cdots \times \mathbb{W} = \mathbb{W}^{n_w}$.*

We suppose that $\big(\boldsymbol{w}_t\big)_{t\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables.
We consider the following optimization problem

$$\min \quad \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{w}_{t+1}\big) + K(\boldsymbol{x}_T)\Big] \tag{2a}$$

$$s.t. \quad \boldsymbol{x}_{t+1} = f_t\big(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{w}_{t+1}\big), \qquad \boldsymbol{x}_0 = x_0 \tag{2b}$$

$$\boldsymbol{u}_t \in \mathbb{U}, \qquad \boldsymbol{u}_t \preceq \sigma\big(\boldsymbol{w}_1, \dots, \boldsymbol{w}_t\big) \tag{2c}$$

1. *Write, in the form of a pseudo-code, the Dynamic Programming algorithm attached to the optimization problem.*
2. *What is the complexity of the Dynamic Programming algorithm in terms of the cardinals of the sets $\mathbb{X}^{n_x}$, $\mathbb{U}^{n_u}$, $\mathbb{W}^{n_w}$ and of the horizon T?*
3. *If the optimization problem were written on a tree (with one decision per node), how many decisions are there in total?*

Solution 1. *1.*

$V_T \equiv K$;
for $t : T-1 \to 0$ **do**
  for $x \in \mathbb{X}_t^{n_X}$ **do**
    $\underline{v} = -\infty$;
    for $u \in \mathbb{U}^{n_X}$ **do**
      $Vmoy = 0$;
      for $w \in \mathbb{W}^{n_W}$ **do**
        $Vmoy = Vmoy + \mathbb{P}(w)\big(L_t(x, u, \boldsymbol{w}_{t+1}) + V_{t+1} \circ f_t\big(x, u, \boldsymbol{w}_{t+1}\big)\big)$;
      end
      $\underline{v} = \min\Big\{\underline{v}, Vmoy\Big\}$;
    end
    $V_t(x) = \underline{v}$;
  end
end

2. *The number of operations is in* $O\big(|\mathbb{X}|^{n_X} \times |\mathbb{U}|^{n_U} \times |\mathbb{W}|^{n_W}\big)$

3. *At stage* t *there is* $\big(|\mathbb{W}|^{n_W}\big)^t$ *decisions, hence there is* $\frac{|\mathbb{W}|^{(T+1)n_W}-1}{|\mathbb{W}|^{n_W}}$.

Exercise 2 (8pts)**.** *Consider the controlled dynamic system given by*

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t + \boldsymbol{u}_t + \boldsymbol{w}_{t+1} \ ,$$

where $(\boldsymbol{w}_t)$ *is a sequence of independent, centered (i.e.* $\mathbb{E}[\boldsymbol{w}_t] = 0$*) exogeneous noises of standard-deviation* σ_t*. The control* $\boldsymbol{u}_t$ *is taken knowing the past noises* $\boldsymbol{w}_1, \dots, \boldsymbol{w}_t$ *(in particular,* $\boldsymbol{u}_0$ *is deterministic and denoted* u_0 *).*
We consider the following optimization problem

$$\min_{u_0, \boldsymbol{u}_1} \quad \mathbb{E}\Big[u_0^2 + k\boldsymbol{u}_1^2 + \boldsymbol{x}_2^2\Big] \tag{3a}$$
$$s.t. \quad x_0 \text{ given} \tag{3b}$$
$$\boldsymbol{x}_1 = x_0 + u_0 + \boldsymbol{w}_1 \tag{3c}$$
$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \boldsymbol{u}_1 + \boldsymbol{w}_2 \tag{3d}$$
$$\boldsymbol{u}_1 \preceq \boldsymbol{w}_1 \tag{3e}$$

where $k > 0$ *is a given parameter.*

1. *Identify the elements of a stochastic optimization control problem: what is the state variable? the control variable? the terminal time* T*? the final cost function* K*? the time-step cost functions* L_t*? the initial state?*

2. *Why can we apply the Dynamic Programming approach to this problem? Hence, what is the form assumed by the optimal solution?*

3. *Determine the Bellman function* V_1 *at time* 1*, and the optimal strategy* $\boldsymbol{u}_1^\sharp$ *at time* 1.

4. *Determine the Bellman function* V_0 *at time* 0*, and the optimal control* u_0 *at time* 0.

5. *What is the optimal value of the optimization problem?*

6. *What are the optimal trajectories* $t \mapsto \boldsymbol{x}_t^\sharp$ *and* $t \mapsto \boldsymbol{u}_t^\sharp$*?*

7. *Interpret the behavior of the optimal trajectories* $t \mapsto \boldsymbol{x}_t^\sharp$ *and* $t \mapsto \boldsymbol{u}_t^\sharp$*, when* $k \to +\infty$.

Solution 2.

1. *The state variable is* $\boldsymbol{x}_t$*, the control variable* $\boldsymbol{u}_t$*, the terminal time* $T = 2$*, the final cost function* $K(x) = kx^2$*, the time-step cost functions* $L_1(x, u, w) = x^2$*,* $L_0(x, u, w) = 0$*, the initial state* $x_0 = 0$.

2. *The noises are independent, hence we can apply the DP approach and the optimal control are function of the current state.*

3. *By DP we have*

$$
\begin{aligned}
V_1(x) &= \min_{u\in\mathbb{R}} \mathbb{E}\big[ku^2 + (x+u+\boldsymbol{w}_2)^2\big] \\
&= \min_{u\in\mathbb{R}} (1+k)u^2 + x^2 + 2xu + 2(x+u)\mathbb{E}\big[\boldsymbol{w}_2\big] + \mathbb{E}\big[\boldsymbol{w}_2^2\big] && \textit{by linearity of } \mathbb{E} \\
&= \min_{u\in\mathbb{R}} (1+k)u^2 + x^2 + 2xu + \sigma^2 && \textit{as } \mathbb{E}\big[\boldsymbol{w}_2\big] = 0 \\
&= -\frac{1}{1+k}x^2 + x^2 + \sigma^2 && \textit{with } u^\sharp = -\frac{x}{1+k} \\
&= \frac{k}{1+k}x^2 + \sigma^2
\end{aligned}
$$

and $\boldsymbol{u}_1^\sharp = -\frac{\boldsymbol{x}_1}{1+k}$.

4. *By DP we have with* $\kappa = \frac{k}{1+k}$

$$
\begin{aligned}
V_0(x) &= \min_{u\in\mathbb{R}} \mathbb{E}\big[u^2 + \kappa(x+u+\boldsymbol{w}_2)^2\big] + \sigma^2 \\
&= \min_{u\in\mathbb{R}} (1+\kappa)u^2 + \kappa x^2 + 2\kappa xu + (1+\kappa)\sigma^2 \\
&= -\frac{\kappa^2}{1+\kappa}x^2 + x^2 + \sigma^2 && \textit{with } u^\sharp = -\frac{\kappa x}{1+\kappa}
\end{aligned}
$$

and $u_0^\sharp = -\kappa x_0/(1+\kappa)$. *The optimal value is given by* $V_0(x_0)$.

5. *If* k *tends to* ∞ *the second control is extrmely costly and thus the optimal second control is* 0.

Exercise 3 (7pts)**.** *You are the IT manager of a small company. You use a software for a period of 4 years. The software needs to be updated every year (else it becomes outdated). It has been bought (intalled up-to-date) on January 1st 2014, and will be changed January 1st 2018.*

- *If the software is up-to-date at the beginning of a year, its efficiency is evaluated at 100k$ for the year.*
- *If the software is outdated at the beginning of a year, it is less efficient and only accounts for 80k$.*
- *If the software is broken, reinstalling the software cost time and money and its value for one year is 0; however, the software is up-to-date at the end of the year.*
- *An outdated software has* 25% *chance of getting broken after one year of usage if nothing is done.*
- *When updating an outdated software there is* 50% *chance of breaking it (else it is up-to-date).*

Suppose you want to optimize the expected value of the software on between January 1st 2014, and January 1st 2018.

1. *Find a Controlled Markov chain representing the problem.*
2. *Solve the problem by dynamic programming: give the Bellman value function, the optimal policy, and the expected value of the software over the period. The Bellman function and policy can be presented as a table.*