8/10/2015
MPRO - Stochastic Optimization

Homework to be done for the 14/10/2015 The three exercises are independent. When possible underline the final answers.

Exercise 1 (5pts). *Consider a control dynamical system following*

$$\boldsymbol{x}_{t+1} = f_t\big(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{w}_{t+1}\big) \ , \tag{1}$$

where

- *the state* $\boldsymbol{x}_t$ *belongs to a Cartesian product finite set* $\mathbb{X} \times \cdots \times \mathbb{X} = \mathbb{X}^{n_x}$*;*
- *the control* $\boldsymbol{u}_t$ *belongs to a Cartesian product finite set* $\mathbb{U} \times \cdots \times \mathbb{U} = \mathbb{U}^{n_u}$*;*
- *the noise* $\boldsymbol{w}_t$ *belongs to a Cartesian product finite set* $\mathbb{W} \times \cdots \times \mathbb{W} = \mathbb{W}^{n_w}$*.*

We suppose that $\big(\boldsymbol{w}_t\big)_{t\in\mathbb{N}}$ *is a sequence of independent and identically distributed random variables.*
We consider the following optimization problem

$$\min \quad \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{w}_{t+1}\big) + K(\boldsymbol{x}_T)\Big] \tag{2a}$$

$$s.t. \quad \boldsymbol{x}_{t+1} = f_t\big(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{w}_{t+1}\big), \qquad \boldsymbol{x}_0 = x_0 \tag{2b}$$

$$\boldsymbol{u}_t \in \mathbb{U}, \qquad \boldsymbol{u}_t \preceq \sigma\big(\boldsymbol{w}_1, \dots, \boldsymbol{w}_t\big) \tag{2c}$$

1. *Write, in the form of a pseudo-code, the Dynamic Programming algorithm attached to the optimization problem.*
2. *What is the complexity of the Dynamic Programming algorithm in terms of the cardinals of the sets* $\mathbb{X}^{n_x}$*,* $\mathbb{U}^{n_u}$*,* $\mathbb{W}^{n_w}$ *and of the horizon* T*?*
3. *If the optimization problem were written on a tree (with one decision per node), how many decisions are there in total?*

Exercise 2 (8pts). *Consider the controlled dynamic system given by*

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t + \boldsymbol{u}_t + \boldsymbol{w}_{t+1} \ ,$$

where $(\boldsymbol{w}_t)$ *is a sequence of independent, centered (i.e.* $\mathbb{E}[\boldsymbol{w}_t] = 0$*) exogeneous noises of standard-deviation* σ_t*. The control* $\boldsymbol{u}_t$ *is taken knowing the past noises* $\boldsymbol{w}_1, \dots, \boldsymbol{w}_t$ *(in particular,* $\boldsymbol{u}_0$ *is deterministic and denoted* u_0 *).*
We consider the following optimization problem

$$\min_{u_0, \boldsymbol{u}_1} \quad \mathbb{E}\Big[u_0^2 + k\boldsymbol{u}_1^2 + \boldsymbol{x}_2^2\Big] \tag{3a}$$

$$s.t. \quad x_0 \text{ given} \tag{3b}$$

$$\boldsymbol{x}_1 = x_0 + u_0 + \boldsymbol{w}_1 \tag{3c}$$

$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \boldsymbol{u}_1 + \boldsymbol{w}_2 \tag{3d}$$

$$\boldsymbol{u}_1 \preceq \boldsymbol{w}_1 \tag{3e}$$

where $k > 0$ *is a given parameter.*

1. *Identify the elements of a stochastic optimization control problem: what is the state variable? the control variable? the terminal time T? the final cost function K? the time-step cost functions L_t? the initial state?*
2. *Why can we apply the Dynamic Programming approach to this problem? Hence, what is the form assumed by the optimal solution?*
3. *Determine the Bellman function V_1 at time 1, and the optimal strategy $\boldsymbol{u}_1^\sharp$ at time 1.*
4. *Determine the Bellman function V_0 at time 0, and the optimal control u_0 at time 0.*
5. *What is the optimal value of the optimization problem?*
6. *What are the optimal trajectories $t \mapsto \boldsymbol{x}_t^\sharp$ and $t \mapsto \boldsymbol{u}_t^\sharp$?*
7. *Interpret the behavior of the optimal trajectories $t \mapsto \boldsymbol{x}_t^\sharp$ and $t \mapsto \boldsymbol{u}_t^\sharp$, when $k \to +\infty$.*

Exercise 3 (7pts)**.** *You are the IT manager of a small company. You use a software for a period of 4 years. The software needs to be updated every year (else it becomes outdated). It has been bought (intalled up-to-date) on January 1st 2014, and will be changed January 1st 2018.*

- *If the software is up-to-date at the beginning of a year, its efficiency is evaluated at 100k$ for the year.*
- *If the software is outdated at the beginning of a year, it is less efficient and only accounts for 80k$.*
- *If the software is broken, reinstalling the software cost time and money and its value for one year is 0; however, the software is up-to-date at the end of the year.*
- *An outdated software has 25% chance of getting broken after one year of usage if nothing is done.*
- *When updating an outdated software there is 50% chance of breaking it (else it is up-to-date).*

Suppose you want to optimize the expected value of the software on between January 1st 2014, and January 1st 2018.

1. *Find a Controlled Markov chain representing the problem.*
2. *Solve the problem by dynamic programming: give the Bellman value function, the optimal policy, and the expected value of the software over the period. The Bellman function and policy can be presented as a table.*