The L-Shaped Method

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Linear 2-stage stochastic program

Consider the following problem

$$
\begin{aligned}
\min_{u_0, \mathbf{u}_1} \quad & \mathbb{E}\left[c^\top u_0 + \mathbf{q}^\top \mathbf{u}_1\right] \\
s.t. \quad & Au_0 = b, \quad u_0 \geq 0 \\
& \mathbf{T}u_0 + \mathbf{W}\mathbf{u}_1 = \mathbf{h}, \quad \mathbf{u}_1 \geq 0, \qquad \mathbb{P} - a.s. \\
& u_0 \in \mathbb{R}^n, \quad \sigma(\mathbf{u}_1) \subset \sigma(\underbrace{\mathbf{q}, \mathbf{T}, \mathbf{W}, \mathbf{h}}_{\xi})
\end{aligned}
$$

Which we rewrite

$$
\begin{aligned}
\min_{u_0 \geq 0} \quad & c^\top u_0 + \mathbb{E}\left[Q(u_0, \xi)\right] \\
s.t. \quad & Au_0 = b
\end{aligned}
$$

with

$$
\begin{aligned}
Q(u_0, \xi) := \min_{u_1 \geq 0} \quad & q_\xi^\top u_1 \\
s.t. \quad & W_\xi u_1 = h_\xi - T_\xi u_0
\end{aligned}
$$

Linear 2-stage stochastic program: extensive formulation

The associated extensive formulation reads

$$\begin{aligned} \min \quad & c^\top u_0 + \sum_{s=1}^{S} \pi^s \, q^s \cdot u_1^s \\ s.t. \quad & Au_0 = b, \quad u_0 \geq 0 \\ & T^s u_0 + W^s u_1^s = h^s, \quad u_1^s \geq 0 \, , \quad \forall s \in [\![1, S]\!] \end{aligned}$$

Which we rewrite

$$\begin{aligned} \min_{u_0} \quad & c^\top u_0 + \sum_{s=1}^{S} \pi^s Q^s(u_0) \\ s.t. \quad & Au_0 = b, \quad u_0 \geq 0 \end{aligned}$$

with

$$Q^s(u_0) := \min_{u_1 \geq 0} \quad q^s \cdot u_1 \qquad s.t. \quad W^s u_1 = h^s - T^s u_0$$

Relatively complete recourse

- We assume here relatively complete recourse (RCR) (without this assumption, we would need feasability cuts, as explained in a forthcoming slide)
- Here, relatively complete recourse means that, for $u_0 \geq 0$,

$$Au_0 = b \quad \Longrightarrow \quad Q_s(u_0) < +\infty \;, \;\; \forall s \in [\![1, S]\!]$$

  that is,

$$Au_0 = b \Rightarrow \left\{ u_1^s \geq 0 \,\middle|\, T^s u_0 + W^s u_1^s = h^s \right\} \neq \emptyset \;, \;\; \forall s \in [\![1, S]\!]$$

- Interpret what RCR means

The value functions are polyhedral

- Recall that

$$Q^s(u_0) := \min_{u_1^s \in \mathbb{R}^n} \quad q^s \cdot u_1^s$$
$$s.t. \quad W^s u_1^s = h^s - T^s u_0, \quad u_1^s \geq 0$$

- can also be written
(through strong duality by relatively complete recourse assumption)

$$(D_{u_0}) \quad Q^s(u_0) = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot \left(h^s - T^s u_0\right)$$
$$s.t. \quad (W^s)^\top \lambda^s \leq q^s$$

- Let P be the polyhedron $\left\{\lambda^s \mid (W^s)^\top \lambda^s \leq q^s\right\}$ of admissible dual value (independent of u_0), and $ext(P)$ the (finite) set of its extremal point. We have that $Q^s(u_0) = \max_{\lambda^s \in ext(P)} \lambda^s \cdot \left(h^s - T^s u_0\right)$, and thus Q^s is polyhedral.

Decomposition of linear 2-stage stochastic program

We rewrite the extended formulation as

$$\begin{aligned} \min \quad & c^\top u_0 + \sum_s \pi^s \theta^s \\ s.t. \quad & Au_0 = b, \quad u_0 \geq 0 \\ & \theta^s \geq Q^s(u_0) \qquad\qquad u_0 \in \mathbb{R}^n , \ \forall s \in [\![1, S]\!] \end{aligned}$$

- As $Q^s(u_0)$ is a polyhedral function of u_0, $\theta^s \geq Q^s(u_0)$ can be rewritten as $\theta \geq \alpha_k^s \cdot u_0 + \beta_k^s \ , \ \forall k \in K^s$
- and the decomposition approach consists in constructing iteratively cut coefficients α_k^s and β_k^s

Obtaining (optimality) cuts

- Recall that

$$Q^s(u_0) := \min_{u_1^s \in \mathbb{R}^n} \quad q^s \cdot u_1^s$$
$$\text{s.t.} \quad W^s u_1^s = h^s - T^s u_0, \quad u_1^s \geq 0$$

- can also be written
(through strong duality by relatively complete recourse assumption)

$$(D_{u_0}) \quad Q^s(u_0) = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot \left(h^s - T^s u_0\right)$$
$$\text{s.t.} \quad (W^s)^\top \lambda^s \leq q^s$$

Obtaining (optimality) cuts II

- Let $\lambda^s_{u_0}$ be an optimal solution of the linear program

$$(D_{u_0}) \quad Q^s(u_0) = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot \big(h^s - T^s u_0\big) \qquad s.t. \quad (W^s)^\top \lambda^s \leq q^s$$

- Considering another control u_0', we have

$$(D_{u_0'}) \quad Q^s(u_0') = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot \big(h^s - T^s u_0'\big) \qquad s.t. \quad (W^s)^\top \lambda^s \leq q^s$$

- As $\lambda^s_{u_0}$ is admissible for (D_{u_0}), it is also admissible for $(D_{u_0'})$, hence

$$Q^s(u_0') \geq \lambda^s_{u_0} \cdot \big(h^s - T^s u_0'\big)$$

Obtaining (optimality) cuts III

- To sum up we have seen that, for any admissible first stage solution, we can construct an exact cut for Q^s
  by solving the dual of the second stage problem
- More precisely, lettting $u_0^k \geq 0$ be such that $Au_0^k = b$, and λ_k^s be an optimal dual solution, we set

$$\alpha_k^s := -(T^s)^\top \lambda_k^s \quad \text{and} \quad \beta_k^s := (\lambda_k^s)^\top h^s$$

- and we get

$$\begin{cases} Q^s(u_0') \geq \alpha_k^s \cdot u_0' + \beta_k^s \ , & \forall u_0' \geq 0 \ , \ \ Au_0' = b \\ \\ Q^s(u_0^k) = \alpha_k^s \cdot u_0^k + \beta_k^s & \end{cases}$$

L-shaped method (multi-cut version)

1. Start with a collection of $K \times S$ cuts, such that $Q^s(u_0) \geq \alpha_k^s \cdot u_0 + \beta_k^s$
2. Solve the master problem, with optimal primal solution u_0^{K+1}

$$
\begin{aligned}
\min_{u_0 \geq 0} \quad & c^\top u_0 + \sum_{s=1}^{S} \pi^s \theta^s \\
s.t. \quad & A u_0 = b \\
& \theta^s \geq \alpha_k^s u_0 + \beta_k^s \,, \qquad \forall k \in [\![1, K]\!] \,, \ \forall s \in [\![1, S]\!]
\end{aligned}
$$

3. Solve S "slave" (dual) problems, with optimal dual solution λ_{K+1}^s

$$
\begin{aligned}
Q^s(u_0^{K+1}) = \max_{\lambda^s \in \mathbb{R}^m} \quad & \lambda^s \cdot \left(h^s - T^s u_0^{K+1}\right) \\
s.t. \quad & W^s \cdot \lambda^s \leq q^s
\end{aligned}
$$

4. Construct S new cuts with

$$
\alpha_{K+1}^s := -(T^s)^\top \lambda_{K+1}^s, \qquad \beta_{K+1}^s := h^s \cdot \lambda_{K+1}^s
$$

L-shaped method (multi-cut version): bounds

At any iteration of the L-shaped method, we can easily determine upper and lower bounds of the original problem

Upper bound. As u_0^K is an admissible first stage solution, and $Q^s(u_0^K)$ is the value of a slave problem, thus the value of the admissible solution u_0^k is simply given by

$$UB = c^\top u_0^K + \sum_{s=1}^{S} \pi^s Q^s(u_0^K)$$

Lower bound. As $Q_K^s(u_0) \geq \max_{k \leq K} \alpha_k^s \cdot u_0 + \beta_k^s$, the value of the master problem is always a lower bound over the value of the SP problem

$$LB = c^\top u_0^K + \sum_{s=1}^{S} \pi^s \theta_K^s$$

L-shaped method (single-cut version)

1. Start with a collection of K cuts, such that $Q(u_0) \geq \alpha_k \cdot u_0 + \beta_k$
2. Solve the master problem, with optimal primal solution u_0^{K+1}

$$
\begin{aligned}
\min_{u_0 \geq 0} \quad & c^\top u_0 + \theta \\
s.t. \quad & A u_0 = b \\
& \theta \geq \alpha_k u_0 + \beta_k \qquad \forall k \in [\![1, K]\!]
\end{aligned}
$$

3. Solve S slave dual problems, each with optimal dual solution λ_{K+1}^s

$$
\begin{aligned}
\max_{\lambda^s \in \mathbb{R}^m} \quad & \lambda^s \cdot \left(h^s - T^s u_0^{K+1}\right) \\
s.t. \quad & W^s \cdot \lambda^s \leq q^s
\end{aligned}
$$

4. Construct a new cut with

$$
\alpha_{K+1} := -\sum_{i=1}^{S} \pi^s \left(T^s\right)^\top \lambda^s \,, \quad \beta_{K+1} := \sum_{i=1}^{S} \pi^s \, h^s \cdot \lambda^s
$$

Feasibility cuts (complementary material)

- The relatively complete recourse (RCR) property, here $Q(u_0) < +\infty$, is equivalent to the property that the effective domain $dom(Q)$ of the convex function Q is the whole space
- However, without the relatively complete recourse assumption, we still have that Q is polyhedral, thus so is the effective domain $dom(Q)$
- Without RCR, we need to add feasibility cuts in the following way
  - If $Q^s(u_0^k) = +\infty$, there exists an unbounded ray of the dual problem

$$\begin{aligned} \max_{\lambda^s \in \mathbb{R}^m} \quad & \lambda^s \cdot \big(h^s - T^s u_0^k\big) \\ s.t. \quad & W^s \cdot \lambda^s \leq q^s \end{aligned}$$

    more precisely, there exists a vector $\overline{\lambda}^k$ such that, for all $t \geq 0$, we have $W^s \cdot t\overline{\lambda}^k \leq q^s$
  - Then, for u_0 to be admissible, we need that

$$\overline{\lambda}^k \cdot \big(h^s - T^s u_0\big) \leq 0$$

    which is a feasibility cut

Convergence

Theorem

In the linear case, the L-Shaped algorithm terminates in finitely many steps, and yields the optimal solution

The proof is done by noting that only finitely many cuts can be added, and that not being able to add a cut proves that the algorithm has converged