The optimal harvesting problem under price uncertainty

Summer School CEA-EDF-INRIA 2012 - Stochastic Optimization

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... [We] obtain the result that uncertainty lengthens the optimal rotation. [...] Under the mean reverting price process, optimal harvesting becomes more sensitive to price level,[...] Including risk aversion completely changes the harvesting policy .

Tahvonen & Kallio (2006)

Forest economics literature

- The optimal harvesting problem with uncertainty has been considerably studied, but the vast majority of papers
  - **present numerical solutions**
  - assuming single stands, random walk price process and risk neutrality

We work with multiple stands, random walk and mean reverting price process to characterize **theoretically** the optimal harvest, risk aversion but...

we consider forest growth as a deterministic and pure aging process.

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Multi-period choice under uncertainty

- Stochastic Dynamic Programming. Or ...
- Markov Decision Process, or
- Multistage Stochastic Programming, or
- Intertemporal Consumption, or
- Life-Cycle Consumption, or...

They all want to solve the same problem: optimal decision making over time, often under uncertainty.

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Multistage stochastic programming literature

- Extensive research in portfolio selection, hydrothermal scheduling, production planning and others.
- Popular algorithms include the Nested L-Shaped (Birge '85), SDDP (Pereira and Pinto '91), Progressive Hedging (Rockafellar and Wets '91), SAA (Shapiro '03, '06), ADP (Powell '07).
- We will see that harvesting constraints are unique and are not equivalent to stocks' buy-and-sell constraints and to the hydric balance equation.
- General purpose algorithms are not readily applicable.

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The one million dollar question:

If price is uncertain,

is it better to harvest everything available now or

is it worth waiting for prices to rise???

- Before the maturity age: harvest forbidden
- After the maturity age: trees **do not** grow

State at time t

$$\mathbb{X}(t) = \begin{pmatrix} \bar{x}(t) \\ x_n(t) \\ \hline x_{n-1}(t) \\ \vdots \\ x_2(t) \\ x_1(t) \end{pmatrix}$$

$x_a(t)$: surface occupied by trees of age a at time t

$\bar{x}(t)$: surface occupied by trees beyond maturity at time t

Forestry model
Rapaport, Sraidi and Terreaux (2003)

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Order of events

$$(p(1), \mathbb{X}(1)) \rightsquigarrow c(1) \rightsquigarrow \mathbb{X}(2) \rightsquigarrow p(2) \rightsquigarrow c(2) \rightsquigarrow \mathbb{X}(3) \rightsquigarrow \cdots \rightsquigarrow c(T-1) \rightsquigarrow \mathbb{X}(T) \rightsquigarrow p(T) \rightsquigarrow c(T).$$

Dynamics

At every time t:

- Knowing $p(t)$ and $\mathbb{X}(t)$ we must choose how much to harvest: $c(t)$ $(0 \leq c(t) \leq \bar{x}(t) + x_n(t))$.

At time t, depending on $c(t)$

- Benefit $\quad c(t)p(t)$
- State $\quad \mathbb{X}(t) = \begin{pmatrix} \bar{x} \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} \longrightarrow \mathbb{X}(t+1) = \begin{pmatrix} \bar{x} + x_3 - c(t) \\ x_2 \\ x_1 \\ c(t) \end{pmatrix}$

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Dynamics

- Example: $n = 3, S = 6$:

$$\begin{pmatrix}1\\2\\1\\2\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}3\\1\\2\\0\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}0\\2\\0\\4\end{pmatrix}$$

$$c(1) = 0 \qquad\qquad c(2) = 4$$

- Total benefit: $c(1)p(1) + \delta c(2)p(2) = \delta 4\, p(2)$

  $\delta \in (0, 1)$ discount factor.

Optimization problem

Deterministic case with constant price $p(t) = \bar{p}$.

Optimization problem:

$$V_1(p(1), \mathbb{X}(1)) = \begin{cases} Max_{c(1),\ldots,c(T)} & \sum_{t=1}^{T} \delta^{t-1} \bar{p}\, c(t). \\ \text{s.t.} & \text{feasibility constraints.} \end{cases}$$

- The *greedy* policy is optimal (Rapaport et al., 2003.)
- Always harvest everything available.

$$\begin{pmatrix}1\\2\\1\\2\end{pmatrix} \longrightarrow \begin{pmatrix}0\\1\\2\\3\end{pmatrix} \longrightarrow \begin{pmatrix}0\\2\\3\\1\end{pmatrix} \longrightarrow \begin{pmatrix}0\\3\\1\\2\end{pmatrix} \longrightarrow \begin{pmatrix}0\\1\\2\\3\end{pmatrix} \rightarrow \ldots$$

$c(1)=3$ $c(2)=1$ $c(3)=2$ $c(4)=3$ $c(5)=1$

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Optimization problem - risk neutral framework

- Objective function : $\mathbb{E}\left[\sum_{t=1}^{T} \delta^{t-1} p(t)c(t)\right]$

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Dynamic programming equations:

$$V_t(p(t), \mathbb{X}(t)) = \begin{cases} Max_{c(t)} & \mathbb{E}\left[p(t)c(t) + \delta V_{t+1}(\cdot,\cdot) \middle| p(t)\right] \\ \text{s.t.} & \text{feasibility constraints} \end{cases}$$

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Geometric Brownian Motion

Definition:	$dp_t = \mu p_t dt + \sigma p_t dW_t$
Drift:	$\mu \in I\!R$
Volatility:	$\sigma \in I\!R_+$
$\mathbb{E}[p(t+1)\|p(t)] =$	$p(t)e^{\mu}$

Optimal policy for GBM

Theorem

If condition $1 \geq \delta e^{\mu}$ holds, the optimal policy is Greedy.

Proof: The coefficient of c in the Bellman eq.

$$V_t(p(t), \mathbb{X}(t)) = Max_c \left\{ p(t)c + \delta \mathbb{E}_{p(t+1)} \left[V_{t+1}(p(t+1), \mathbb{X}(t+1)) \middle| p(t) \right] \right\}$$

is $$p(t)(1 - \delta e^{\mu}) \sum_{k=0}^{K} \delta^{kn} e^{(kn)\mu} \geq 0$$

Several pages in a few lines

- But what is the intuition behind this result?

- future $\leq$ present

$$\begin{aligned} \delta \mathbb{E}[p(t+1)|p(t)] &\leq p(t), t = 1, \ldots, T-1, \\ \delta p(t) e^{\mu} &\leq p(t), t = 1, \ldots, T-1, \end{aligned}$$

which is equivalent to

$$\delta e^{\mu} \leq 1.$$

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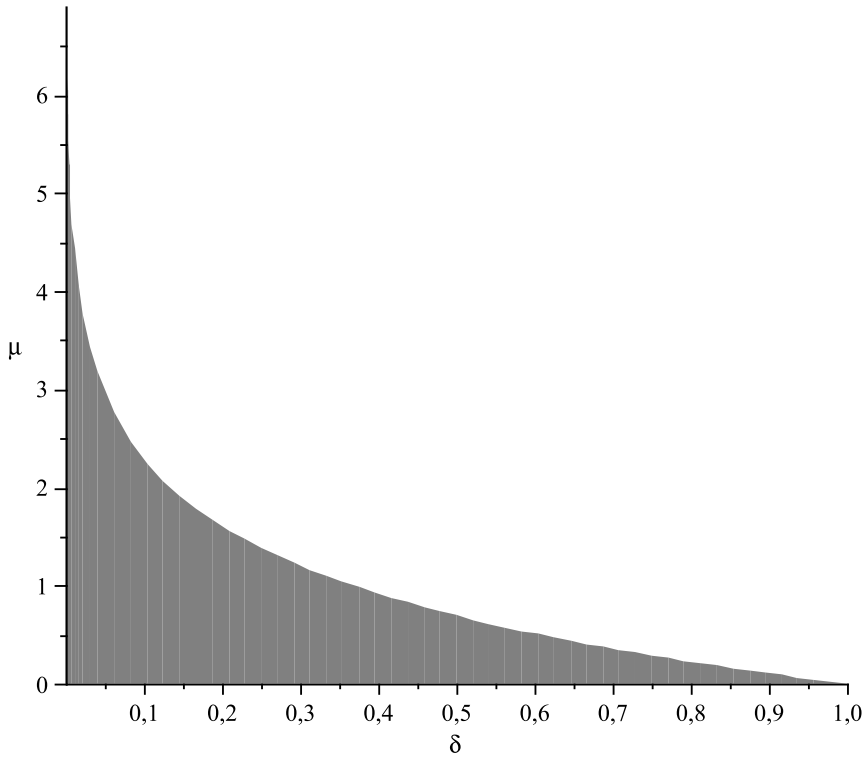


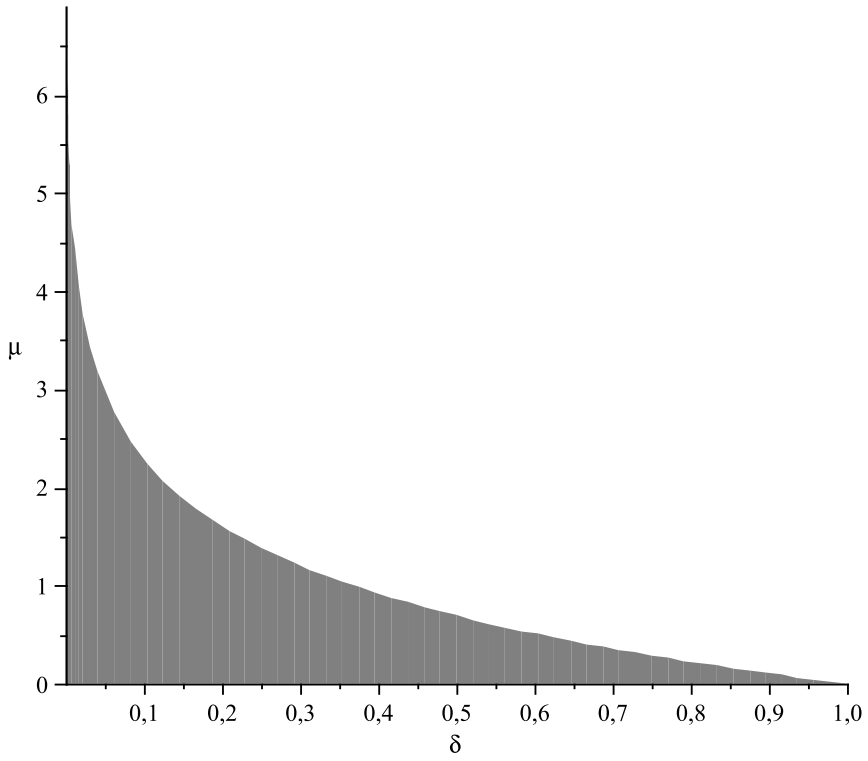
Shaded area:

$1 - \delta e^{\mu} > 0$ holds

Greedy policy is optimal

(every state & price realization)

What if $1 - \delta e^{\mu} < 0$?



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What if $1 - \delta e^{\mu} < 0$?

Another optimal policy for GBM

If $1 - \delta e^{\mu} < 0$, it is optimal to postpone the harvest.

- Harvesting is allowed only at: T, $T - n$, $T - 2n, \ldots$
- Every mature tree is cut

$$c(t) = \begin{cases} \bar{x}(t) + x_n(t) & \text{if } t = T - kn \\ 0 & \text{else} \end{cases}$$

Example with $T = 8$ and $n = 3$, $S = 6$.

$$\begin{array}{ccccccccccccccc} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} & \rightarrow & \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} & \rightarrow & \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix} & \rightarrow & \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} & \rightarrow & \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} & \rightarrow & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \end{pmatrix} & \rightarrow & \begin{pmatrix} 0 \\ 0 \\ 6 \\ 0 \end{pmatrix} & \rightarrow & \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix} \\ t = 1 & & t = 2 & & t = 3 & & t = 4 & & t = 5 & & t = 6 & & t = 7 & & t = 8 \end{array}$$

Proof, using backwards induction on t.

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Second price process

Ornstein-Uhlenbeck

Definition:	$dp_t = \eta(\bar{p} - p_t)dt + \sigma dW_t$
Equilibrium:	$\bar{p}$
Rate of mean-reversion:	$\eta \in I\!R_+$
Volatility:	$\sigma \in I\!R_+$
$\mathbb{E}[p(t+1)\|p(t)] =$	$p(t)e^{-\eta} + \bar{p}(1 - e^{-\eta})$

Several pages in a few lines again

- What if we do the same trick?

$$\begin{aligned} \delta \mathbb{E}_{|p(t)}[p(t+1)] &\leq p(t), \\ \delta[p(t)e^{-\eta} + \bar{p}(1-e^{-\eta})] &\leq p(t), \end{aligned}$$

which is equivalent to

$$\frac{p(t)}{\bar{p}} \geq \frac{\delta(1-e^{-\eta})}{(1-\delta e^{-\eta})} := r. \tag{1}$$

- Obs.: Condition depends on $p(t)$.

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Solution for the O-U: our best result

Theorem

If there is t such that $p(t) \geq r\bar{p}$, then $c^*(t) = \bar{x}(t) + x_n(t)$

- If $p(t) \geq r\bar{p}$ the optimal decision **at that particular time** t is to harvest everything available
- If $p(t) < r\bar{p}$, then $c^*(t) = ?$

Numerical experiments:

- We use $r\bar{p}$ as a *reservation price*.
- Results within 5% of the optimum for some parameter values.

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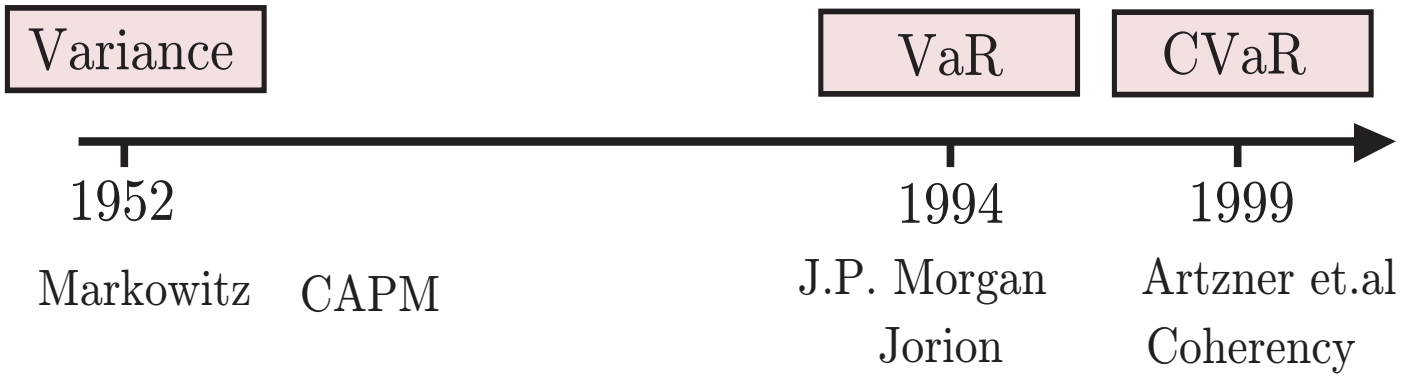
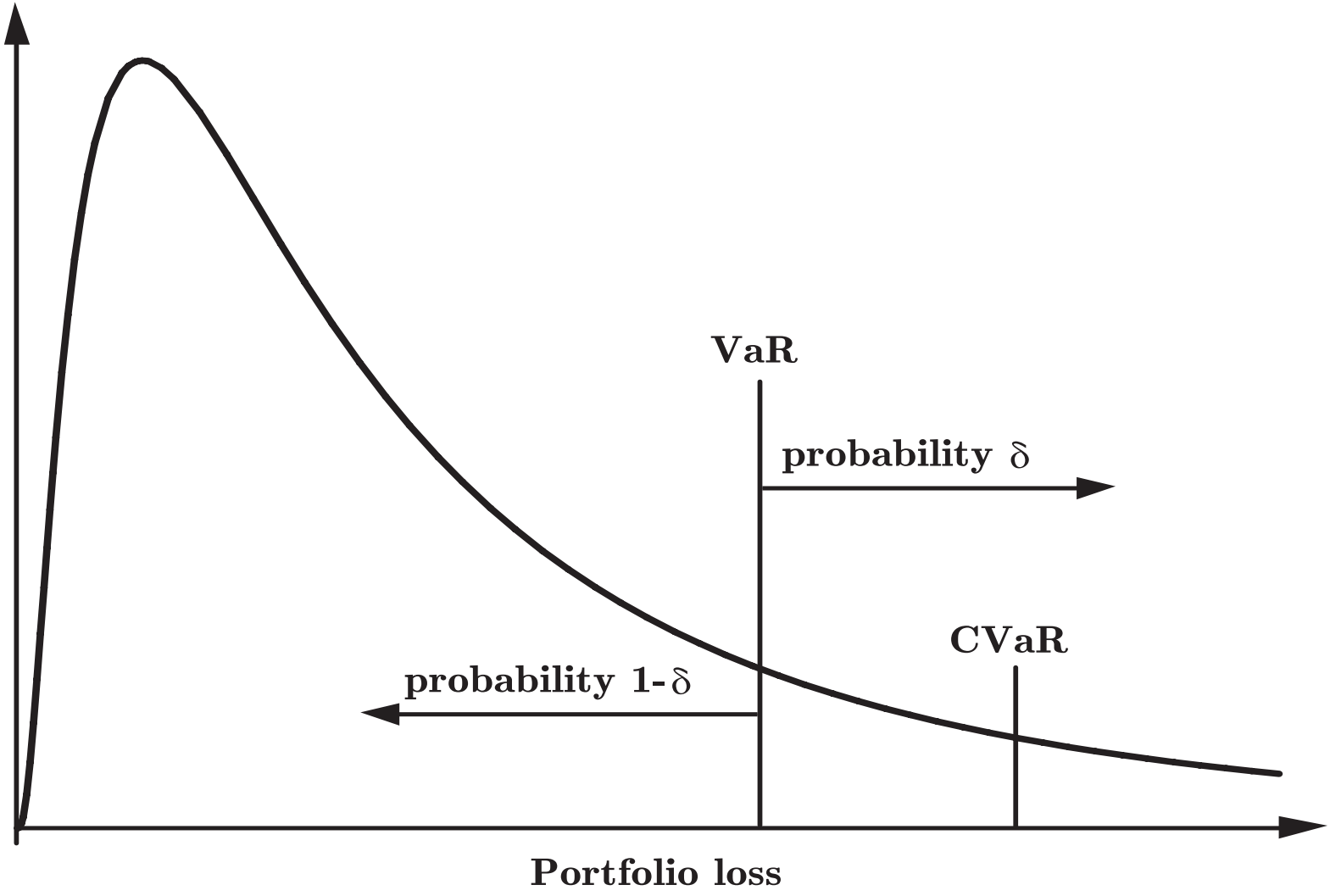
Risk measures



How bad is bad? Conditional Value-at-Risk



CVaR definiton

- Conditional Value-at-Risk, Average Value-at-Risk, Expected Tail Loss and Expected Shortfall are all the same thing!

- Formally, we define $\mathrm{CVaR}_\alpha[X] = \inf_{t\in\mathbb{R}} \left\{ t + \frac{1}{\alpha}\mathbb{E}\left[X - t\right]_+ \right\} =$ (Cont. case) $= \mathbb{E}\left[X \mid X > \mathrm{VaR}_\alpha\right]$.

- The Value-at-Risk:

$$\mathrm{VaR}_\alpha[X] = \inf\{x : \mathbb{P}(X \leq x) \geq 1 - \alpha\}, \quad \alpha \in (0, 1).$$

- What happens when we incorporate the CVaR in our forestry model?

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Coherent risk measures

1) $\rho(X + c) = \rho(X) + c$.

2) $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$.

3) $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \geq 0$.

4) $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

A risk measure that satisfies axioms 1) – 4) is called *coherent*.

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A risk measure that satisfies axioms 1) – 4) is called *coherent*.

Theorem - CVaR under GBM

Risk averse case, GBM

If prices evolve according to a GBM and condition

$$\delta e^{\mu} \frac{1}{\alpha} \Phi(\Phi^{-1}(\alpha) - \sigma) \leq 1$$

holds, the greedy policy is optimal.

Observation: If the condition is not satisfied the optimal policy is analogous to the one we obtained for the risk neutral case.

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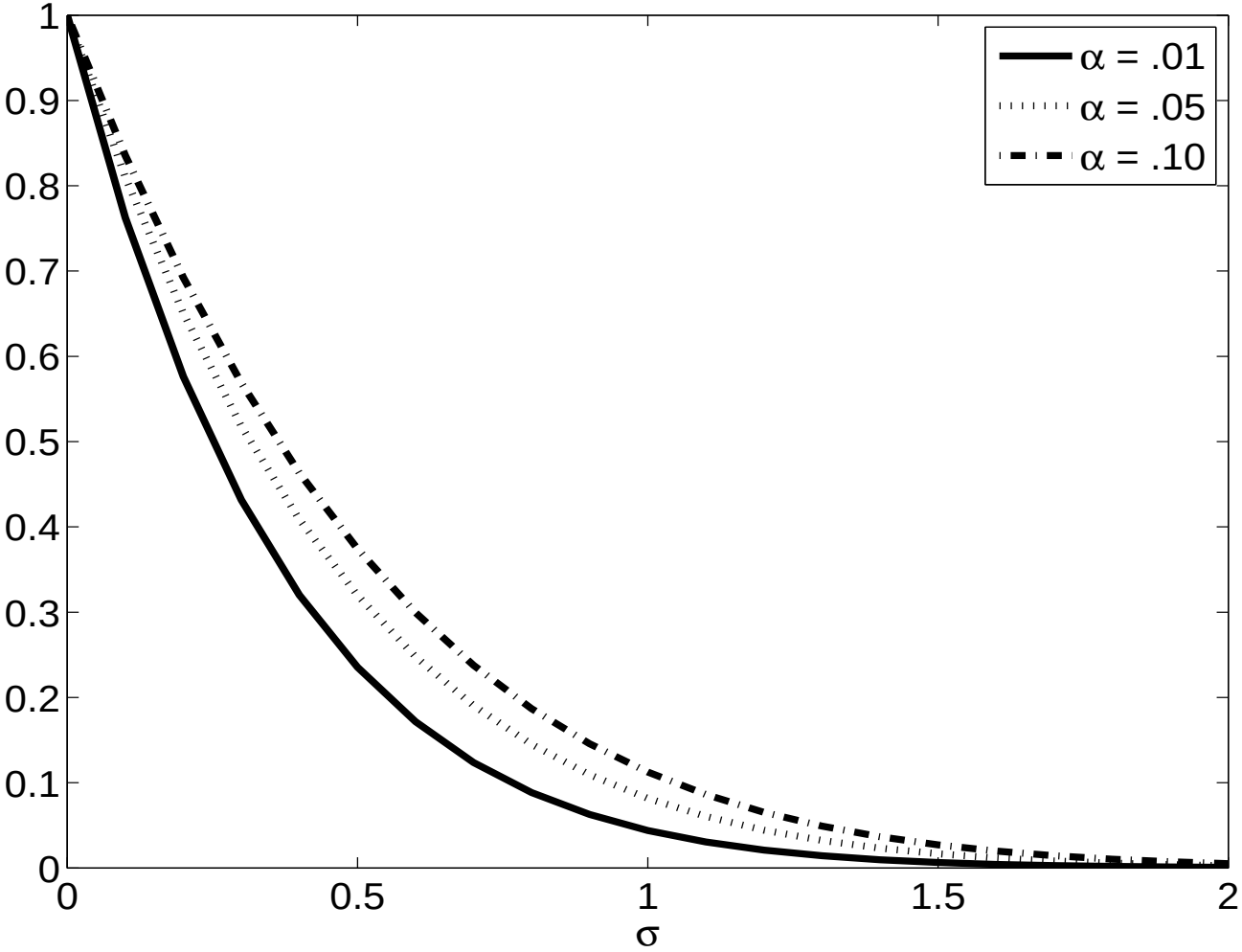
Sensitivity analysis



Theorem - CVaR under O-U

Risk averse case, O-U

If prices evolve according to an O-U process and

$$p(t) \geq \bar{p}r - \frac{\delta}{1-\delta e^{-\eta}} \frac{\sigma}{\sqrt{2\pi}} \sqrt{\frac{(1-e^{-2\eta})}{2\eta}} \frac{e^{-z_\alpha^2/2}}{\alpha} = p_r^{O-U}$$

holds at time t, the greedy policy is optimal at time t.

Some observations:

- When $\eta \mapsto \infty \Rightarrow$, the reservation price goes to $\delta\bar{p} \Rightarrow$ Greedy policy is optimal.
- When $\alpha \to 1 \Rightarrow$, reservation prices coincide.
- When $\sigma \downarrow$, reservation prices coincide.
- When $\sigma \uparrow$, reservations prices are smaller than $\bar{p}r$.

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Present and future research

- Multistage least cost plus damage
- Inclusion of forest fires
- Include diameter as a state variable, instead of (or in addition to) age
- Growth as a stochastic process and consider natural mortality

Merci!