# Optimization of a Liquefied Natural Gas Portfolio by SDDP techniques

J. Frédéric Bonnans

INRIA-Saclay and CMAP, École Polytechnique

Commun work with Zhihao Cen (now at Artelys)
Thibault Christel (Total)

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# Main work

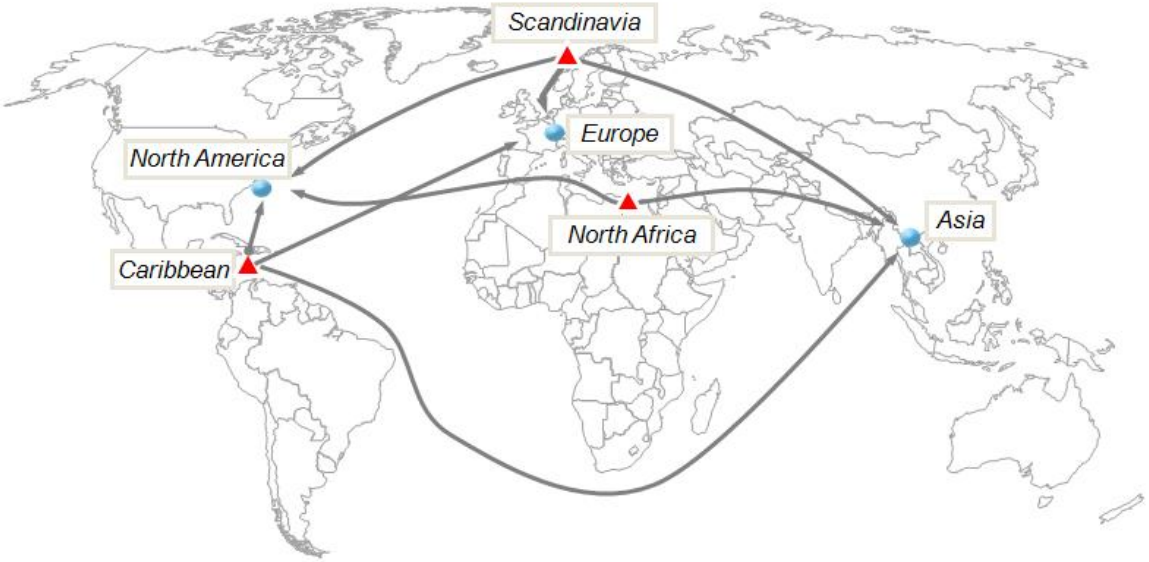
Large size stochastic, dynamic and integer optimization problem

- continuous relaxation (SP):
  - develop an algorithm (L-Shape based) for risk neutral problem and risk averse problem (CVaR);
  - study (first order) sensitivity analysis w.r.t. random process parameters;
- integer problem (SMIP):
  - propose an heuristic method based on cutting plane method;
  - highlight the difficulty (by dual programming).

# Outline

# Outline

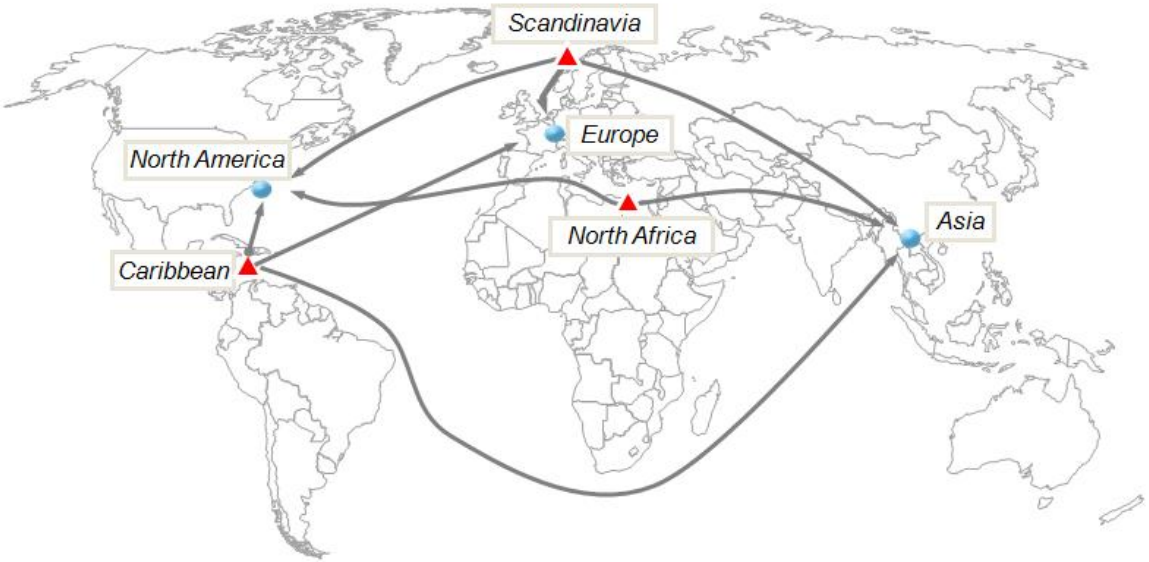
# Motivation I: LNG shipping portfolio optimization




## Basic rules

- Long term buying and selling contracts (min-max amounts per month and per year);
- Route: between two ports able to receive ships of a given size (in discrete number);
- Seller-buyer price formulas based on various (**spot**) commodities indexes
  $\Rightarrow$ uncertain income;
- Discrete decisions: how many ships on each route, each month?

# Motivation I: LNG shipping portfolio optimization




| Port | Cargo size | Annual QC. | Monthly QC. | Price formula (in $/MMBtu) |
|---|---|---|---|---|
| Caribbean | 2.9, 3.4 | [48.0, 54.0] | [0, 6.0] | NA NG – 0.1 |
| Scandinavia | 2.9 | [24.0, 30.0] | [0, 3.0] | $\begin{cases} 0.05\text{OIL} + 2.5 & \text{if OIL} \leq 75 \\ 0.07\text{OIL} + 1.0 & \text{otherwise} \end{cases}$ |
| North Africa | 2.9, 3.4 | {100.0} | [0, 12.0] | $\begin{cases} 0.9\text{NA NG} + 0.4 & \text{if NA NG} \leq 5 \\ 0.8\text{NA NG} + 0.9 & \text{otherwise} \end{cases}$ |
| North American | 2.9, 3.4 | [84.0, 88.0] | [0, 8.0] | NA NG |
| Europe | 2.9, 3.4 | [68.0, 76.0] | [0, 8.0] | EU NG |
| Asia | 2.9, 3.4 | {20.0} | [0, 4.0] | 0.08OIL – 0.8 |

# Motivation II: swing option

## Swing option

A swing option can be view as a one-dimensional portfolio: only one route.

$$P(Q_{min}, Q_{max}) = \sup\left\{\mathbb{E}\left(\sum_{j=0}^{T-1} e^{-rt_j}(S_j - K_j)q_j\right), \right. \\ \left. q_j : (\Omega, \mathcal{F}_j) \to [0,1], \sum_{j=0}^{T-1} q_j \in [Q_{min}, Q_{max}]\right\}. \tag{1}$$

## Proposition (Bardou et al. 2010)

*If* $(Q_{min}, Q_{max}) \in \mathbb{Z}^2$*, bang bang property on optimal control :* $q_j \in \{0,1\}$ *a.s.*

Existing methods:

- Monte-Carlo + Longstaff-Schwartz: Barrera-Esteve et al. 2006;
- discretization (quantization): Bardou et al. 2009;
- PDE's numerical methods: Kluge thesis 2006;
- other numerical methods for BSDE.

# Mathematical formulation

$(\boldsymbol{\xi}_t) \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$ Markov process :

$$\boldsymbol{\xi}_{t+1} = f(W_t, \boldsymbol{\xi}_t, \alpha_t) \quad t = 0, \ldots, T-1; \tag{2}$$

$\boldsymbol{\xi}_0 = \xi_0$, $(W_t) \in \mathbb{R}^d$ i.i.d., independent of $\boldsymbol{\xi}_t$; $\mathcal{F}_t = \sigma(\boldsymbol{\xi}_s, 0 \le s \le t)$.

$$\begin{aligned} \inf_{u,x} \quad & \mathbb{E}\left[\sum_{t=0}^{T-1} c_t(\boldsymbol{\xi}_t) u_t + g(\boldsymbol{\xi}_T, x_T)\right] \\ \text{s.t.} \quad & u_t : (\Omega, \mathcal{F}_t) \to \mathfrak{U}_t \subset \mathbb{R}^n; \\ & x_{t+1} = x_t + A_t u_t, x_0 = 0, \\ & x_T \in \mathfrak{X}_T \subset \mathbb{R}^m; \end{aligned} \tag{3}$$

where

- $c_t(\boldsymbol{\xi})$ Lipschitz;
- $g(\boldsymbol{\xi}, x)$ convex and l.s.c. w.r.t. $x$, Lipschitz;
- $\mathfrak{U}_t, \mathfrak{X}_T$ nonempty, convex, compact;
- $\left(\sum_{t=0}^{T-1} A_t \mathfrak{U}_t\right) \cap \mathfrak{X}_T \neq \emptyset$.

# Dynamic programming (DP.) formulation

Following the Markov property of $(x_t, \xi_t)$, define Bellman (cost-to-go) function

$$Q(t, x_t, \xi_t) := \operatorname{essinf} \left\{ \mathbb{E}\left[ \sum_{s=t}^{T-1} c_s(\xi_s) u_s + g(\xi_T, x_T) \,|\, \mathcal{F}_t \right] : \right. \\ \left. u_s : (\Omega, \mathcal{F}_s) \to \mathfrak{U}_s, x_t + \sum_{s=t}^{T-1} A_s u_s \in \mathfrak{X}_T \right\}. \tag{4}$$

The <u>dynamic programming principle</u> reads as:

$$\begin{aligned} Q(t, x_t, \xi_t) = \operatorname{essinf} \quad & c_t(\xi_t) u_t + Q(t+1, x_{t+1}, \xi_t) \\ \text{s.t.} \quad & u_t \in \mathfrak{U}_t, \\ & x_{t+1} = x_t + A_t u_t; \end{aligned} \tag{5}$$

where $Q(t+1, x_{t+1}, \xi_t) := \mathbb{E}\left[ Q(t+1, x_{t+1}, \xi_{t+1}) \mid \mathcal{F}_t \right]$, and final cost (stage $T$):

$$Q(T, x_T, \xi_T) = \begin{cases} g(\xi_T, x_T) & \text{if } x_T \in \mathfrak{X}_T, \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

# Difficulties and main solution methods / ideas

- High dimension $x_t$:
  - dual decomposition (L-Shape type method);
- Numerical conditional expectation computations:
  - space discretization (tree method, quantization tree);
  - projection (least square regression, Tsitsiklis);
  - kernel estimation...
- Integer optimization problem:
  - branch and bound method;
  - cutting plane method;
  - metaheuristic method: local search...

# Outline

# Dual programming

**Proposition**

*$Q(t, x_t, \xi_t)$ ($Q(t, x_t, \xi_{t-1})$) is convex, lower semi-continuous w.r.t. $x_t$.*

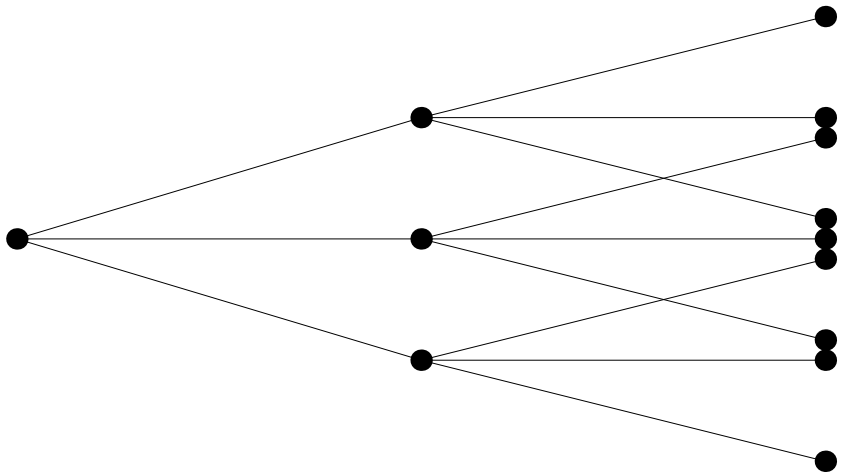
By Moreau Fenchel theorem, we define optimality cut

$$Q(t, x_t, \xi_t) = Q^{**}(t, x_t, \xi_t) \geq \langle x_t, x^* \rangle - Q^*(t, x^*, \xi_t), \quad \forall x^*.$$

$\Rightarrow$ Linear programming (very efficient).

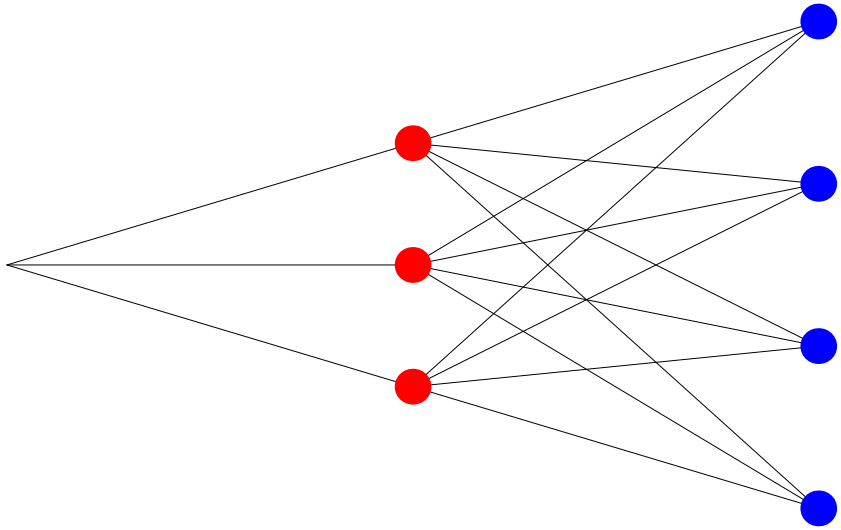
L-Shape method (Van Silke and Wets 69) for linear recourse problem based on dual programming.

# Discretization – Scenario tree



Non-combination tree for non-Markov case.

Ex: tree reduction method (Heitsch and Römisch 2003).

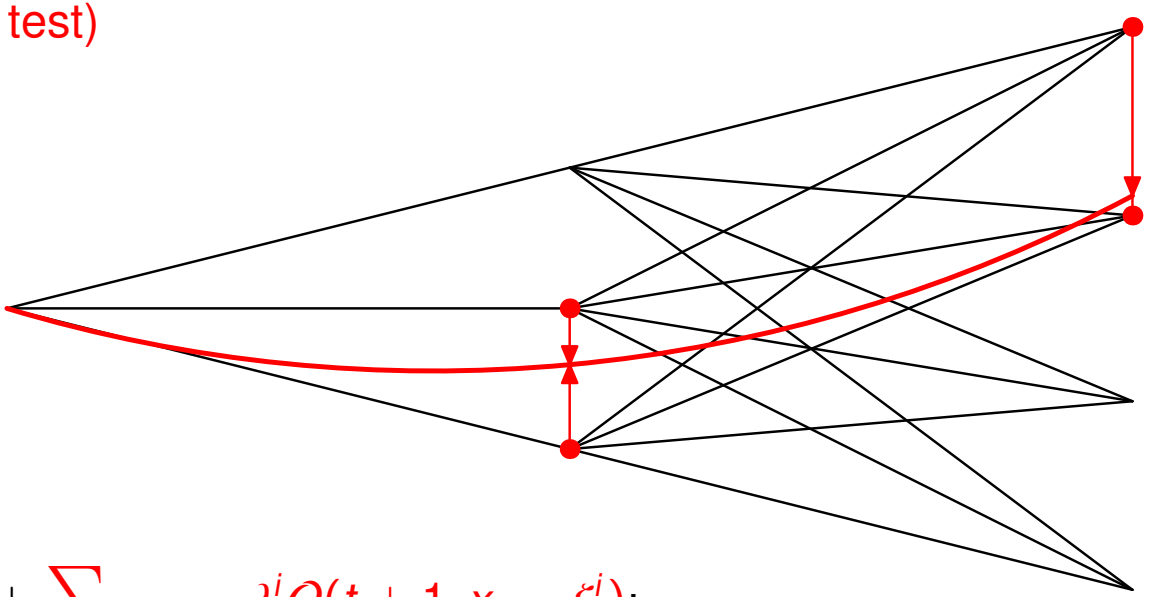


Combination tree for Markov case.

Ex: vectorial quantization tree method (Pagès et al. 2000).

# L-Shape (modified) – Forward pass

Partial sampling (out-of-sample test)

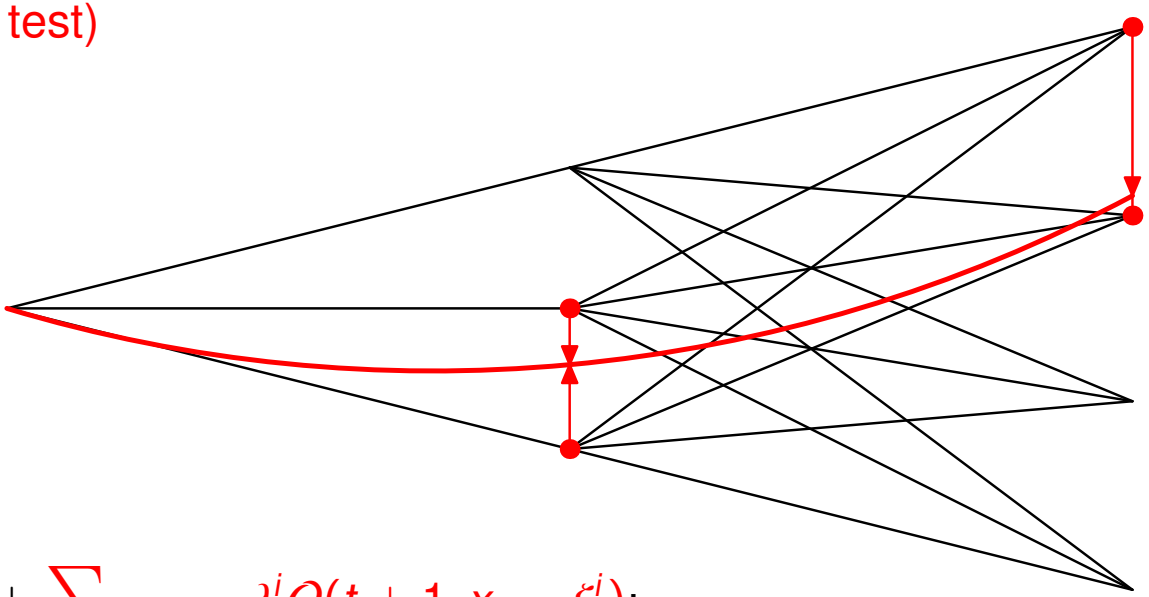


$$Q(t, x_t, \boldsymbol{\xi}_t) = \text{essinf } c_t(\boldsymbol{\xi}_t)u_t + \sum_{\xi_t^i \in \mathcal{T}(\boldsymbol{\xi}_t)} \lambda_t^i Q(t+1, x_{t+1}, \xi_t^i); \tag{7}$$

where $\mathcal{T}(\boldsymbol{\xi}_t) = \{(\xi_t^i, \lambda_t^i)\}$ is the Delaunay triangulation of $\boldsymbol{\xi}_t$.

# L-Shape (modified) – Forward pass

Partial sampling (out-of-sample test)



$$Q(t, x_t, \boldsymbol{\xi}_t) = \text{essinf } c_t(\boldsymbol{\xi}_t)u_t + \sum_{\xi_t^i \in \mathcal{T}(\boldsymbol{\xi}_t)} \lambda_t^i Q(t+1, x_{t+1}, \xi_t^i); \tag{7}$$

$$\approx \text{essinf } c_t(\boldsymbol{\xi}_t)u_t + \sum_{\xi_t^i \in \mathcal{T}(\boldsymbol{\xi}_t)} \lambda_t^i \vartheta(t+1, x_{t+1}, \xi_t^i, O_{t+1}), \tag{8}$$

$$\text{s.t. } x_{t+1} = x_t + A_t u_t,$$

(feasibility cut) $$u_t \in \mathfrak{U}_t^{ad}(x_t) = \left\{ u_t : x_t + \sum_{s=t}^{T-1} A_s u_s \in \mathfrak{X}_T, u_s \in \mathfrak{U}_s \right\},$$

(optimality cut) $$\vartheta(t+1, x_{t+1}, \xi_t^i, O_{t+1}) \geq \left[ \sum_{\xi_{t+1}^j} \hat{p}_t^{ij} (\lambda^x x_{t+1} + \lambda^0) \right], \ \forall (\lambda^x, \lambda^0) \in O_{t+1}^j;$$

where $\mathcal{T}(\boldsymbol{\xi}_t) = \{(\xi_t^i, \lambda_t^i)\}$ is the Delaunay triangulation of $\boldsymbol{\xi}_t$.

# Algorithm I

## Forward pass (upper bound)

Step 1 : **Initialize**

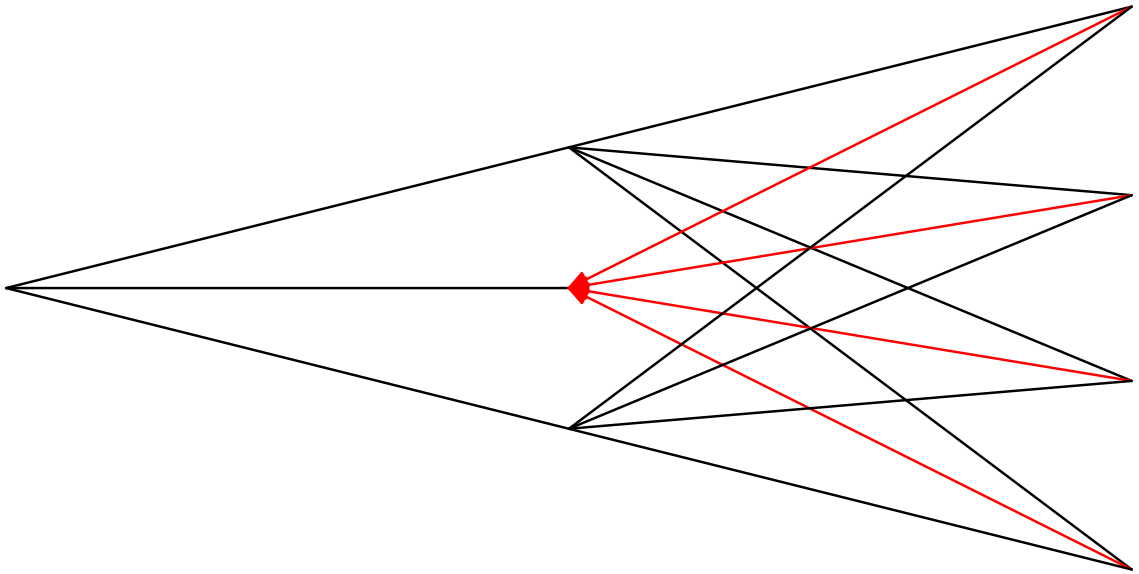
the optimality cuts on tree;
the first stage control $u_0$.

Step 2 : **Forward pass**

Simulate $M_f$ random process following (2).
For $m = 1, \ldots, M_f$ do:
for $t = 1, \ldots, T-1$ do:
calculate the Delaunay triangle $\mathcal{T}(\xi_t^m)$,
solve subproblem (8);
compute $v^m$ optimal value of scenario $\xi^m$.
Compute the empirical statistic :

$$\overline{v} = \frac{1}{M_f} \sum_{m=1}^{M_f} v^m \text{ and } s = \frac{1}{M_f} \sqrt{\sum_{m=1}^{M_f} (v^m - \overline{v})^2}.$$

# L-Shape – Backward pass



$$Q(t, x_t, \xi_t^i) = \text{essinf } c_t(\xi_t^i)u_t + Q(t+1, x_{t+1}, \xi_t^i); \tag{9}$$

$$\approx \text{essinf } c_t(\xi_t^i)u_t + \vartheta(t+1, x_{t+1}, \xi_t^i, O_{t+1}), \tag{10}$$

$$\text{s.t. } x_{t+1} = x_t + A_t u_t, \qquad (\leftarrow \text{dual value } \lambda^x \in O_t^i)$$

(feasibility cut) $$u_t \in \mathfrak{U}_t^{ad}(x_t) = \left\{ u_t : x_t + \sum_{s=t}^{T-1} A_s u_s \in \mathfrak{X}_T, u_s \in \mathfrak{U}_s \right\},$$

(optimality cut) $$\vartheta(t+1, x_{t+1}, \xi_t^i, O_{t+1}) \geq \left[ \sum_{\xi_{t+1}^j} \hat{p}_t^{ij} (\lambda^x x_{t+1} + \lambda^0) \right], \ \forall (\lambda^x, \lambda^0) \in O_{t+1}^j.$$

# Algorithm II

## Backward pass (lower bound)

Step 3 : **Backward pass**

For $t = T - 1, \ldots, 1$ do:

for $m = 1, \ldots, M_b$ do:

solve subproblem (10) on all vertices in $\Gamma_t$;

compute new optimality cuts and add them $O_t$.

$t = 0$:

solve subproblem (10), $\Rightarrow$ backward value $\underline{v}^{it}$.

Step 4 : **Check stop condition**

If $\underline{v}^{it} \in [\overline{v} - \varrho s, \overline{v} + \varrho s]$ and $|\underline{v}^{it} - \underline{v}^{it-1}| \leq \epsilon |\underline{v}^{it}|$

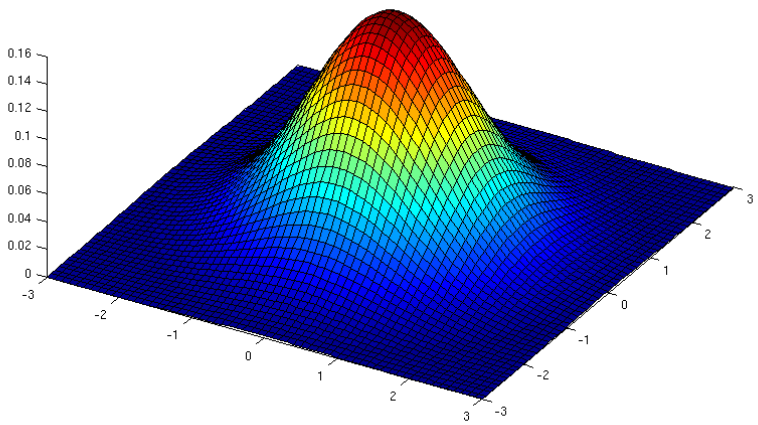
STOP;

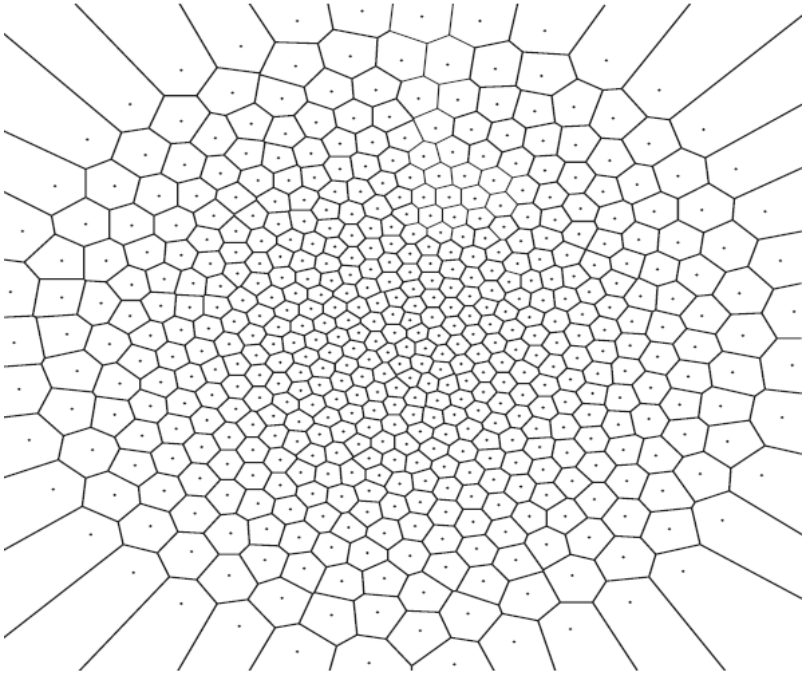
else

go to Step 2.

where $\varrho > 0$ is a parameter. In standard Monte-Carlo method, $\varrho = 1.96$ (corresponds to 95% of confidence interval).

# Discretization – vectorial quantization method



Optimal dual quantization : $\inf\left\{ \|F_p^D(\boldsymbol{\xi},\Gamma)\|_p \ : \ \Gamma = \left\{\xi^1,\dots,\xi^N\right\} \subset \mathbb{R}^d \right\},$

$$(F_p^D(\boldsymbol{\xi},\Gamma))^p = \min_{\lambda\in\mathbb{R}^N}\Big\{ \sum_{i=1}^N \lambda_i|\boldsymbol{\xi}-\xi^i|^p \ : \\ \sum_{i=1}^N \lambda_i\xi^i = \mathrm{proj}_{\overline{\mathrm{conv}}(\Gamma)}(\boldsymbol{\xi}), \sum_{i=1}^N \lambda_i = 1, \lambda \geq 0\Big\}. \tag{11}$$

combination tree

# Converge result and stability of optimal value

## Proposition

*The upper bound (forward value) and the lower bound (backward value) converge if total quantized points number* $N = \sum_{t=0}^{T} N_t \to \infty$.

## Theorem

*Assume that the solution set is nonempty and bounded. Then,* $\exists\, L > 0$ *such that:*

$$|\mathrm{val}(\xi) - \mathrm{val}(\tilde{\xi})| \leq L(\|F_p^D(\xi, \Gamma)\|_p + D_f(\xi, \hat{\xi})) \tag{12}$$

*where*

$$D_f(\xi, \hat{\xi}) := \inf_{u \in S(\xi)} \sum_{t=0}^{T-1} \|u_t - \mathbb{E}[u_t | \hat{\mathcal{F}}_t]\|_q \tag{13}$$

*where* $(\tilde{\xi}_t)$ *is the finite state Markov chain built based on the quantization tree,* $(\hat{\xi}_t)$ *is the quantization process,* $\hat{\mathcal{F}}_t = \sigma(\hat{\xi}_s : 0 \leq s \leq t)$*;* $p^{-1} + q^{-1} = 1$*;* $\mathrm{val}(\cdot)$ *(resp.* $S(\cdot)$*) is the optimal value (resp. the optimal solution set).*

# Convergence rate of distortion

## Dual quantization case

**Theorem (Pagès and Wilbertz 10)**

*Assume that* $\xi \in L^{p+\eta}(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{R}^d)$ *for some* $\eta > 0$. *Then*

$$\lim_{N\to\infty}\left(N^{\frac{p}{d}} \inf_{\#(\Gamma)\le N} \|F_p^D(\xi,\Gamma)\|_p^p\right) = J_{p,d}\|\varphi\|_{\frac{d}{d+p}} \tag{14}$$

*where* $\mu(d\xi) = \varphi(\mu)\cdot\lambda_d(d\xi) + \upsilon$ *(*$\lambda_d$ *Lebesgue measure on* $\mathbb{R}^d$*), and* $\upsilon \perp \lambda_d$. *The constant* $J_{p,d}$ *corresponds to the case of the uniform distribution on* $[0,1]^d$ *(or any Borel set of Lebesgue measure* 1*).*

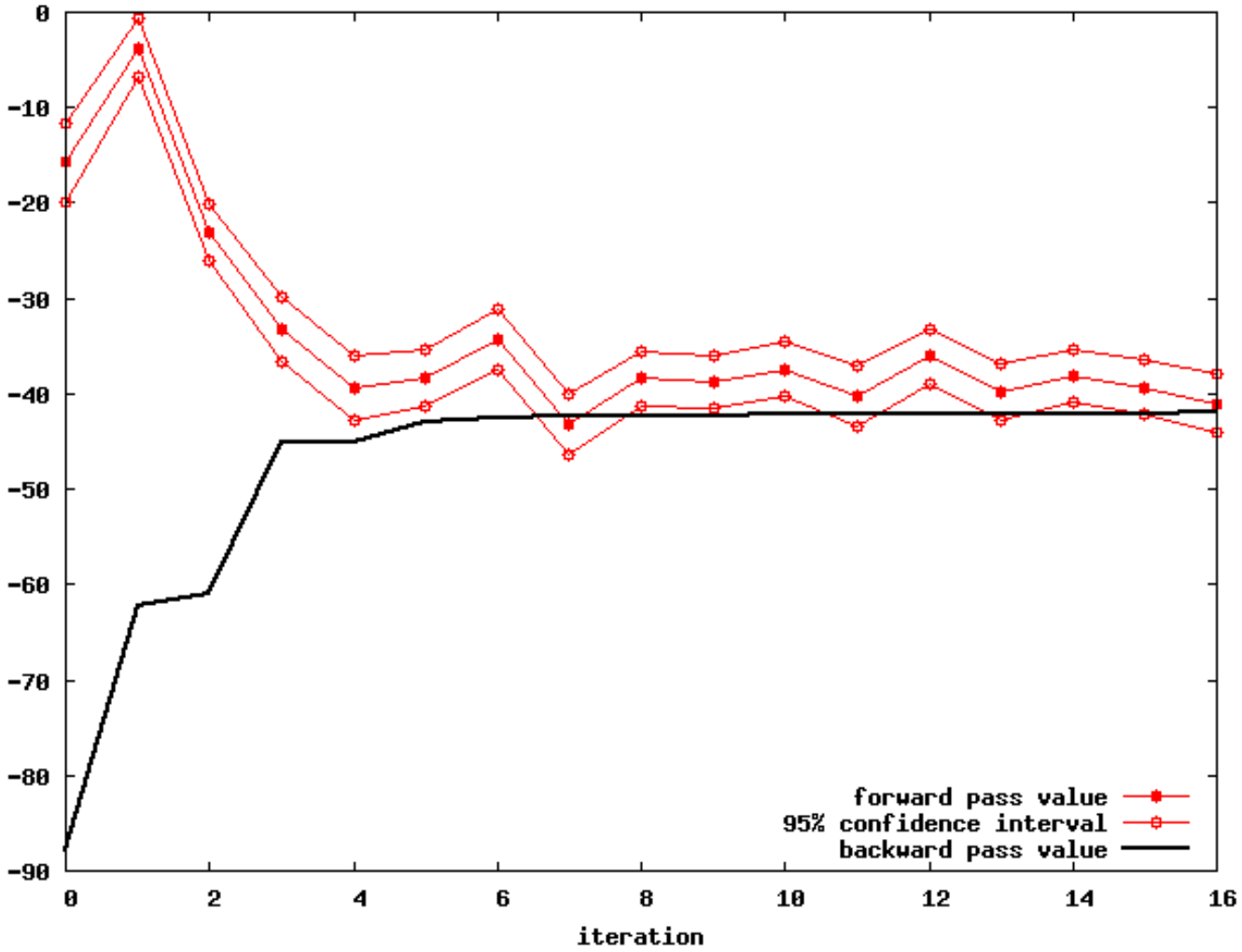
<u>Quantization tree</u>: $\inf_{\#(\Gamma)\le N} \|F_p^D(\xi,\Gamma)\|_p = O(N^{-1/d})$ where $N = \sum_{t=0}^{T} N_t$.

# Result on LNG portfolio continuous relaxation problem

Quantization size: $N = 36000$;

Algorithm parameter: $M_f = 3000$ to $6000$, $M_b = 8$ to $12$;

Process parameters $\sigma_1 = \sigma_2 = \sigma_3 = 40\%$; $\rho_{12} = 0.7, \rho_{13} = 0.2, \rho_{23} = 0.4$.



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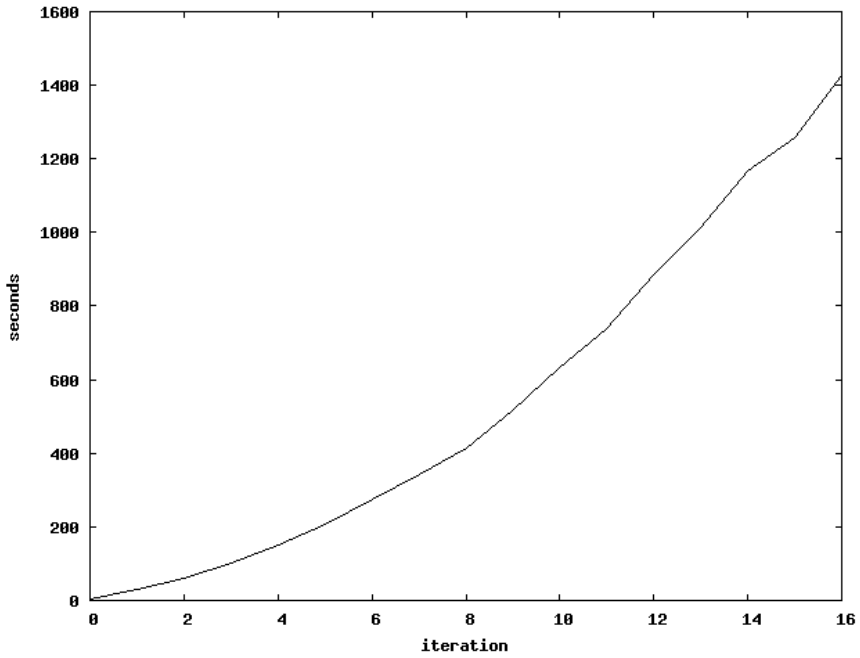
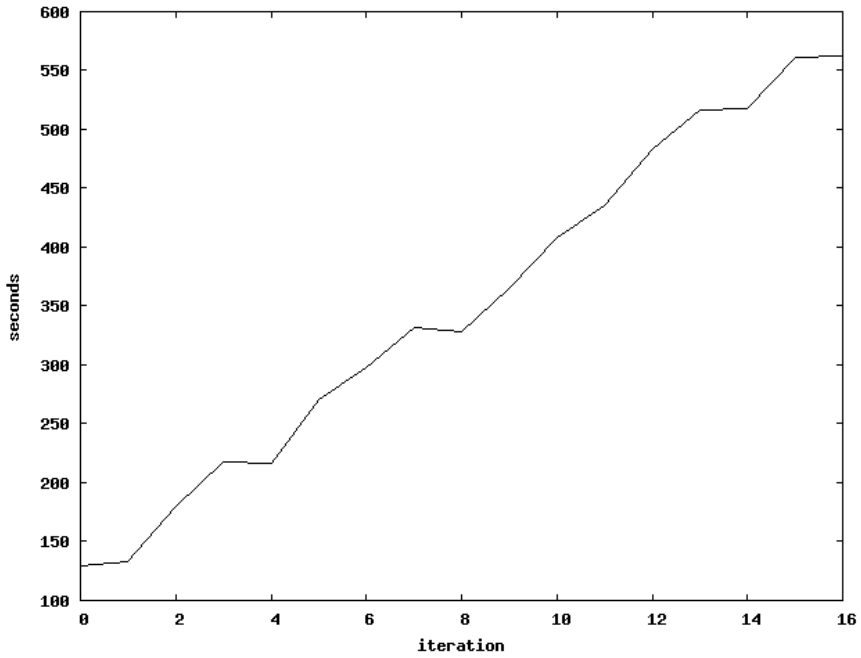



forward time-consumption




backward time-consumption

# Outline

# Markov process (price) model

A discrete-time one factor model:

$$\begin{aligned} \ln \frac{(F^t_{s+1})^i}{(F^t_s)^i} &= \sigma^i W^i_s - \frac{1}{2}(\sigma^i)^2, && i = 1, \dots, d; \\ \xi_t{}^i = (F^t_t)^i &= (F^t_0)^i \exp\left( \sum_{s=0}^{t-1} \sigma^i W^i_s - \frac{1}{2}(\sigma^i)^2 t \right), && i = 1, \dots, d; \end{aligned} \tag{15}$$

where

- $F^t_s, s \leq t$ the forward contract price at time $s$ with maturity $t$;
- $W^i_t$ follows $\mathcal{N}(0,1)$, $\mathrm{corr}(W^i_t, W^j_t) = \rho_{ij}$.

# Sensitivity analysis

## Problem

$$\begin{aligned} \inf \quad & \mathbb{E}\left[\sum_{t=0}^{T-1} c_t(\xi_t) u_t\right] \\ \text{s.t.} \quad & u_t : (\Omega, \mathcal{F}_t) \to \mathfrak{U}_t \subset \mathbb{R}^n; \\ & x_{t+1} = x_t + A_t u_t, x_0 = 0, \\ & x_T \in \mathfrak{X}_T \subset \mathbb{R}^m. \end{aligned} \tag{16}$$

no final cost function $g$, the criterion is linear w.r.t. $(u_t)$.

- $v^*(F_0, \sigma)$ the optimal value function;
- $U^*(F_0, \sigma)$ the optimal solution set.

## Sensitivity estimates (first order)

- $\delta$ : sensitivity w.r.t. $F_0$;
- $\nu$ : sensitivity w.r.t. $\sigma$.

# Differentiability and sensitivity estimates

By Danskin's theorem

**Corollary**

*If $\mu(\xi(F_0,\sigma)) \ll \lambda^{Lebesgue}$, then the optimal function $v^*(F_0,\sigma)$ is Fréchet differentiable at almost every point $(F_0,\sigma)$.*

Thus, at almost every point $(F_0,\sigma)$, with one optimal solution $u^* \in U^*$:

$$\begin{aligned} \delta(dF_0) &:= Dv^*(F_0,\sigma;dF_0,0) = \mathbb{E}\left[\sum_{t=0}^{T-1} D_F c_t dF_0^t \cdot u_t^*\right]; \\ \nu(d\sigma) &:= Dv^*(F_0,\sigma;0,d\sigma) = \mathbb{E}\left[\sum_{t=0}^{T-1} D_\sigma c_t d\sigma \cdot u_t^*\right]. \end{aligned} \tag{17}$$

# Discretized problem

A sequence of optimal quantization $\hat{\xi}^m$, $\lim_{m\to\infty} N^m = \infty$:

## Discretized problem I – quantized problem

$$
\begin{aligned}
v_Q^m(F_0,\sigma) = \min \quad & \mathbb{E}\left[\sum_{t=0}^{T-1} c_t(\xi_t) u_t^m\right] \\
\text{s.t.} \quad & u_t^m : (\Omega, \sigma(\hat{\xi}_t^m, x_t^m)) \to \mathfrak{U}_t, \\
& x_{t+1}^m = x_t^m + A_t u_t^m, \quad x_0^m = 0, \\
& x_T^m \in \mathfrak{X}_T^m \quad \text{a.s.}
\end{aligned}
\tag{18}
$$

## Discretized problem II – scenario problem

$$
\begin{aligned}
v_S^m(F_0,\sigma) = \min \quad & \mathbb{E}\left[\sum_{t=0}^{T-1} c_t(\hat{\xi}_t^m) u_t^m\right] \\
\text{s.t.} \quad & u_t^m : (\Omega, \sigma(\hat{\xi}_t^m, x_t^m)) \to \mathfrak{U}_t, \\
& x_{t+1}^m = x_t^m + A_t u_t^m, \quad x_0^m = 0, \\
& x_T^m \in \mathfrak{X}_T^m \quad \text{a.s.}
\end{aligned}
\tag{19}
$$

Both problems are in finite dimension.

# Differentiability and approximate formula

## Lemma

*If $\mu(\xi(F_0,\sigma)) \ll \lambda^{Lebesgue}$, then the optimal functions of the discretized problems $v_Q^{*,m}(F_0,\sigma)$ and $v_S^{*,m}(F_0,\sigma)$ are Fréchet differentiable at almost every point $(F_0,\sigma)$.*

Thus, at almost every point $(F_0,\sigma)$, with one optimal solution $u_Q^{*,m} \in U_Q^{*,m}$:

$$\begin{aligned} \delta_Q^m(dF_0) &= Dv_Q^{*,m}(F_0,\sigma;dF_0,0) = \mathbb{E}\left[\sum_{t=0}^{T-1} D_F c_t dF_0^t \cdot (u_Q^{*,m})_t\right]; \\ \nu_Q^m(d\sigma) &= Dv_Q^{*,m}(F_0,\sigma;0,d\sigma) = \mathbb{E}\left[\sum_{t=0}^{T-1} D_\sigma c_t d\sigma \cdot (u_Q^{*,m})_t\right]. \end{aligned} \tag{20}$$

Same result for scenario problem $\delta_S^m, \nu_S^m$.

# Convergence result

## Lemma

$$\begin{aligned} u_Q^{*,m'} &\rightharpoonup u^* \\ u_S^{*,m''} &\rightharpoonup u^* \end{aligned} \qquad \text{in } L^2(\Omega, (\mathcal{F}_t), \mathbf{P}; \mathbb{R}^{n\times T}) \tag{21}$$

*where*

- $u_Q^{*,m'}$ *is a subsequence of optimal strategy of quantized problem* (18)*;*
- $u_S^{*,m''}$ *is a subsequence of optimal strategy of scenario problem* (19)*.*

## Corollary

*At almost every point* $(F_0, \sigma)$*, we have*

$$\begin{aligned} \delta_Q^{m'} &\to \delta; \quad \nu_Q^{m'} \to \nu; \\ \delta_S^{m''} &\to \delta; \quad \nu_S^{m''} \to \nu; \end{aligned} \tag{22}$$

*where*

- $\delta_Q^{m'}$ *and* $\nu_Q^{m'}$ *are sensitivity estimates of quantized problem* (18)*;*
- $\delta_S^{m''}$ *and* $\nu_S^{m''}$ *are sensitivity estimates of scenario problem* (19)*.*

# Swing option

- Price formula : $c_t(\xi_t) = K_t - \xi_t$;
- Strike : $K_t = F_0^t$;
- Process parameters : $T = 50$; $\sigma = 30\%/\sqrt{T} = 0.042$;
- Forward price : $F_0^t = 1 + 0.2\sin(2\pi \cdot t/T)$;
- Swing parameters : $Q_{min} = 20$; $Q_{max} = 30$.

## Comparison methods

| | sensitivity estimate | cond. expectation | dp. method |
|---|---|---|---|
| I | Danskin | quantization tree | L-Shape |
| II | Danskin | quantization tree | discretization + dp. |
| III | Danskin | PDE | discretization + dp. |
| IV | Finite difference | PDE | discretization + dp. |

# Sensitivity result I

## Optimal value

| | tree+L-Shape | | | tree+discret. | pde.+discret. |
|---|---|---|---|---|---|
| | ub. | $\sigma$(ub.) | lb. | | |
| $v^*$ | -2.41646 | 0.105759 | -2.3741 | -2.37197 | -2.36187 |

## Sensitivity $\nu$

| | tree+L-Shape | | tree+discret. | | pde.+discret. | | fd.+ pde. |
|---|---|---|---|---|---|---|---|
| | mean | std. dev. | mean | std. dev. | mean | std. dev. | |
| $\nu$ | -32.181 | 0.695 | -32.198 | 0.698 | -32.207 | 0.700 | -35.558 |

# Sensitivity result II

## Sensitivity $\delta$

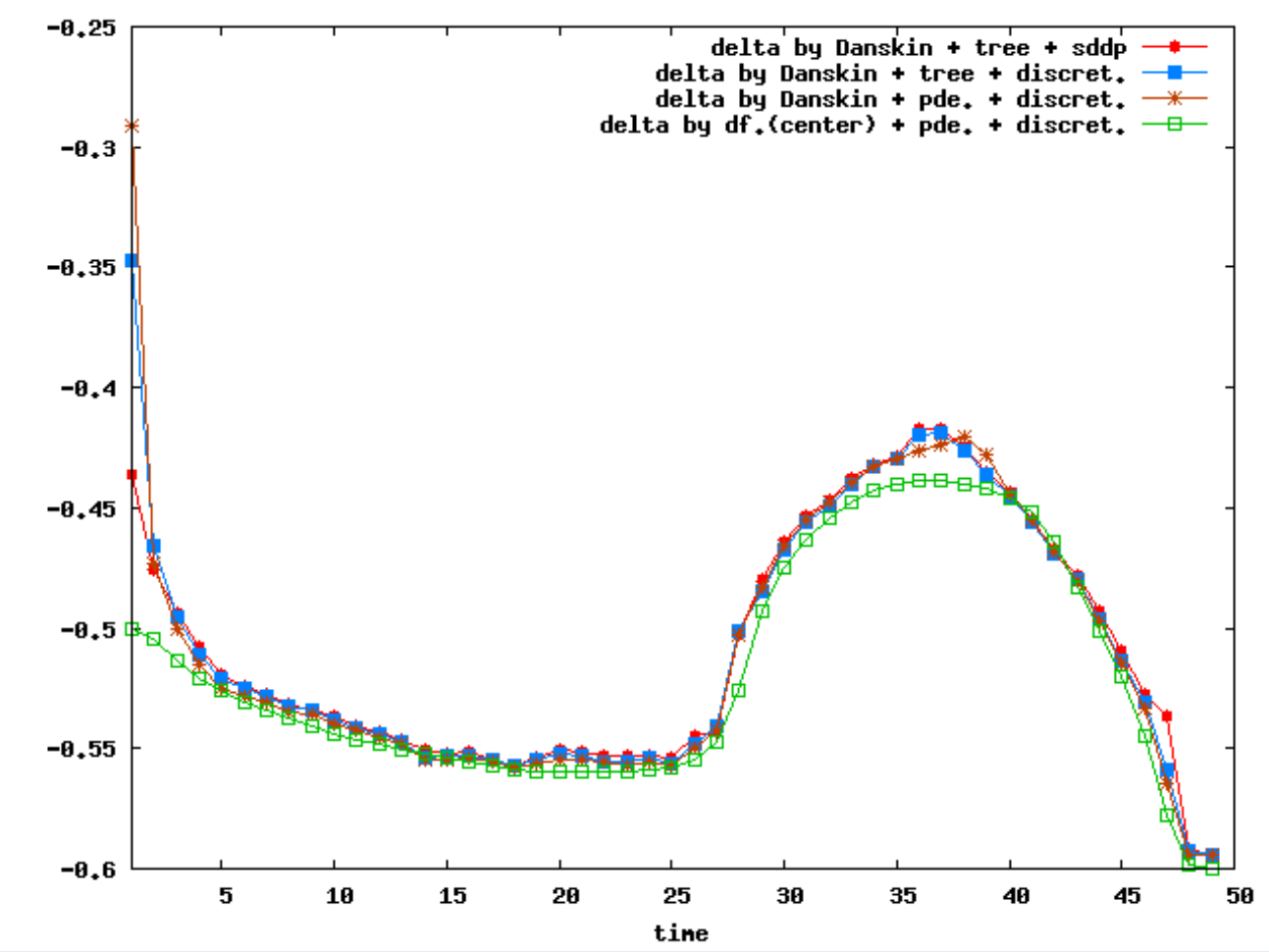



Figure: Sensitivity values with respect to $F_0$ obtained by 4 methods.

# Outline

# Example

## Example

$$
\begin{aligned}
Q(x_0) = \min \quad & \mathbb{E}[\xi u_0 + \theta] \\
\text{s.t.} \quad & u_0 \in [0,3] \cap \mathbb{Z}, \\
& x_1 = x_0 + 2u_0, \\
& x_1 \leq 6, \\
& \theta \geq \frac{x_1 - 3}{2}, \\
& \theta \geq -\frac{x_1 - 3}{2}.
\end{aligned} \tag{23}
$$

where $\xi = \{-2, 0.5\}$ that each takes probability 0.5;
$x_0 \in \{0, 1, 2, 3\}$.

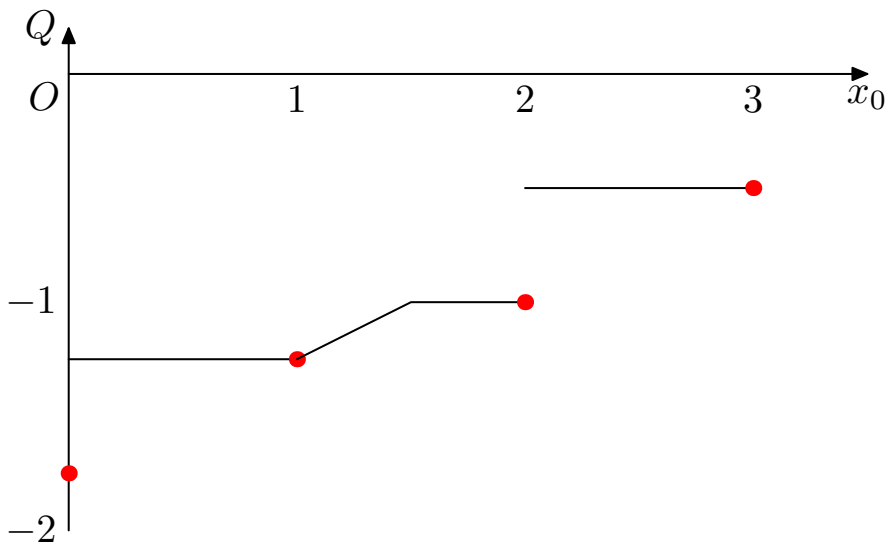


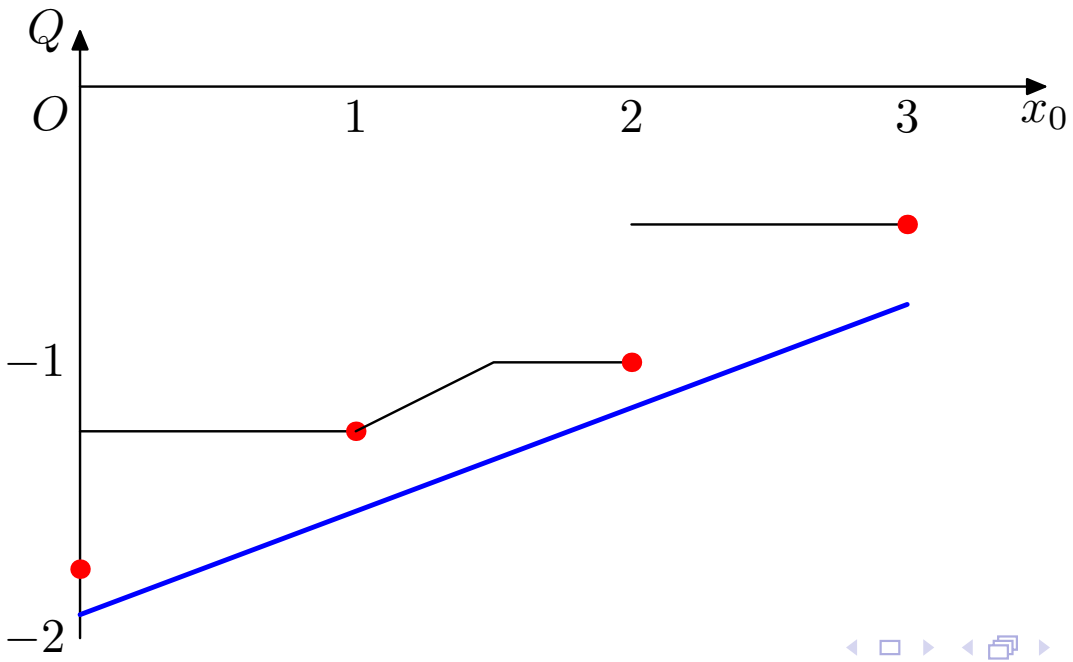

Figure: Bellman value function

$Q(t, x_t, \xi_t)$ (resp $Q(t, x_t, \xi_{t-1})$) is generally non-convex, lower semicontinuous w.r.t. $x_t$.

# First heuristic method (Birge and Louveaux)

Compute the "optimal" integer solution using the Bellman function of the continuous (relaxation) problem:

$$\begin{aligned} \min \quad & c_t(\xi_t)u_t + Q^{cont}(t+1, x_{t+1}, \xi_t) \\ \text{s.t.} \quad & x_{t+1} = x_t + A_t u_t, \\ & u_t \in \mathfrak{U}_t^{int,ad}(x_t) = \left\{ u_t : x_t + \sum_{s=t}^{T-1} A_s u_s \in \mathfrak{X}_T, u_s \in \mathfrak{U}_s \cap \mathbb{Z}^n \right\}, \\ \text{(optimality cut)} \quad & Q^{cont}(t+1, x_{t+1}, \xi_t) \geq \overline{\lambda}^x x_{t+1} + \overline{\lambda}^0, \ \forall (\overline{\lambda}^x, \overline{\lambda}^0) \in \mathcal{O}_{t+1}(\xi_t). \end{aligned}$$



# Cutting plane method

## Theorem

*The mixed integer program*

$$\min\{cx + hy : (x, y) \in S\} \text{ where } S = \left\{x \in \mathbb{Z}_+^n, y \in \mathbb{R}_+^m : Ax + Gy \leq b\right\} \quad \text{(MIP)}$$

*and the linear program*

$$\min\{cx + hy : (x, y) \in \overline{\mathrm{conv}}(S)\} \quad \text{(LP)}$$

*have same optimal value. Furthermore, if* $(x^*, y^*)$ *is an optimal solution of* (MIP)*, then it is an optimal solution of* (LP)*.*

Objective : build $\overline{\mathrm{conv}}(S)$.

- lifting and projecting : Balas, Ceria and Cornuéjols, Sherali and Adams, etc.;
- Gomory cut: Gomory, Balas et al., etc.;
- disjunctive cut: Balas et al., Sen et al., etc.;
- Fenchel cut: Boyd, Ntaimo, etc.;
- other methods.

# Feasible set and notations

$$\mathfrak{U}_t^{int,ad}(\hat{x}_t) = \Big\{ u_t : u_s \in \mathfrak{U}_s \cap \mathbb{Z}^n, \\ x_t + \sum_{s=t}^{T-1} A_s u_s \in \mathfrak{X}_T \Big\};$$

**Example**

$\hat{x}_0 = 1$:

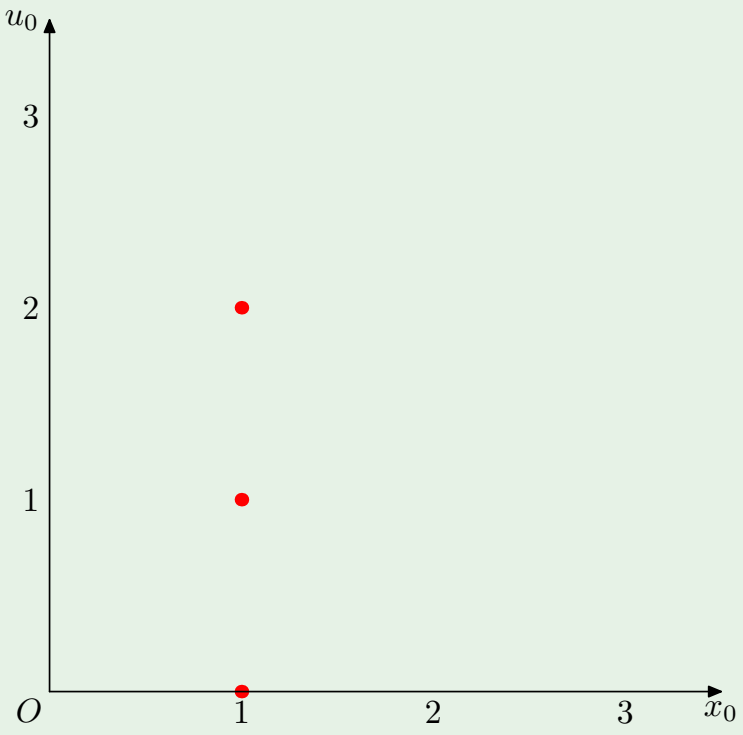



Figure: feasibility set

# Feasible set and notations

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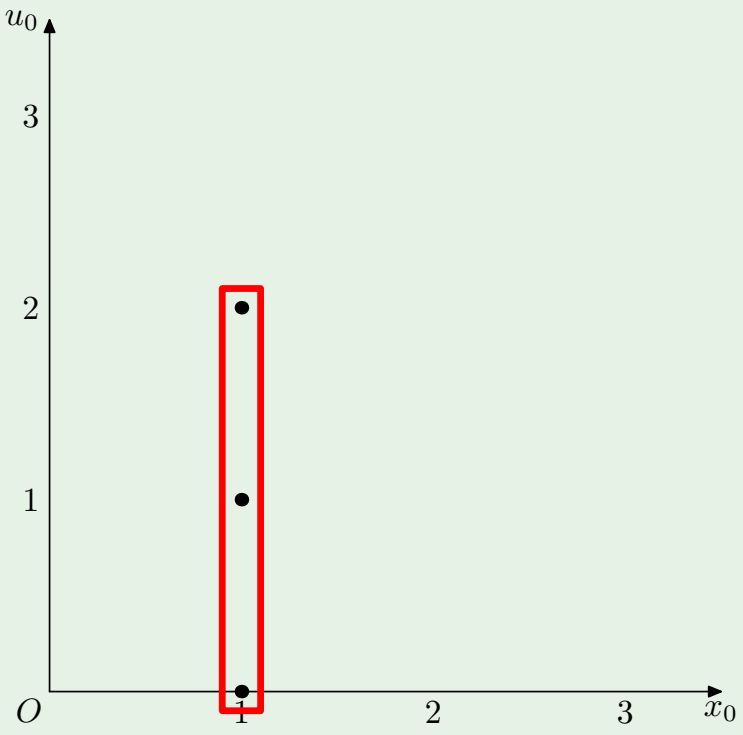
$$\overline{\mathrm{conv}}\,\mathfrak{U}_t^{int,ad}(\hat{x}_t);$$

**Example**

$\hat{x}_0 = 1$:

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$$\overline{\mathrm{conv}}\,\mathfrak{U}_t^{int,ad}(\hat{x}_t);$$

$$(\mathfrak{X},\mathfrak{U})_t^{int,ad} = \Big\{(x_t,u_t) : u_s \in \mathfrak{U}_s \cap \mathbb{Z}^n, \quad \sum_{t=0}^{T-1} A_t u_t \in \mathfrak{X}_T, x_t = \sum_{s=0}^{t-1} A_s u_s\Big\};$$

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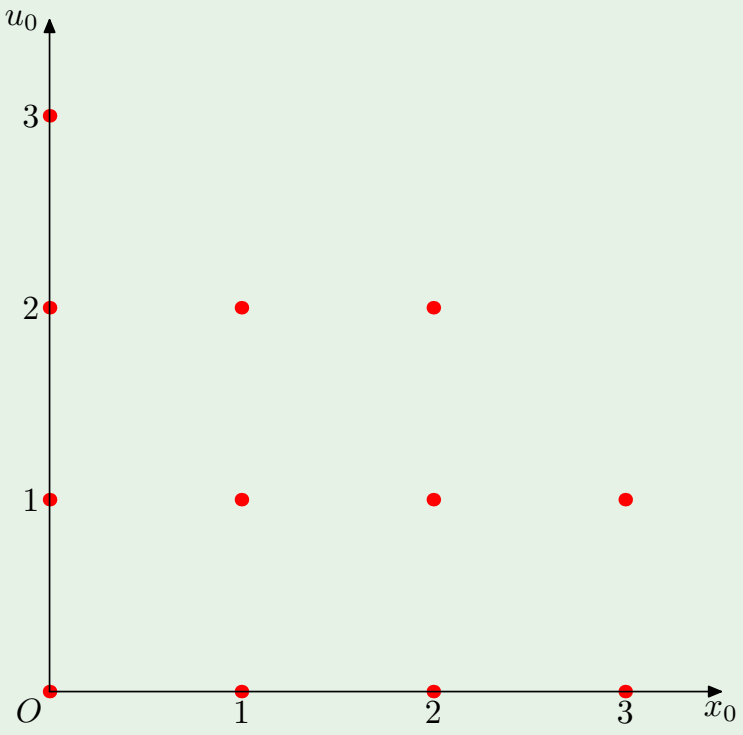
$\hat{x}_0 = 1$:

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$$\mathfrak{U}_t^{int,ad}(\hat{x}_t) = \Big\{u_t : u_s \in \mathfrak{U}_s \cap \mathbb{Z}^n, \quad x_t + \sum_{s=t}^{T-1} A_s u_s \in \mathfrak{X}_T\Big\};$$

$$\overline{\mathrm{conv}}\mathfrak{U}_t^{int,ad}(\hat{x}_t);$$

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$$\overline{\mathrm{conv}}(\mathfrak{X},\mathfrak{U})_t^{int,ad};$$

## Example

$\hat{x}_0 = 1$:

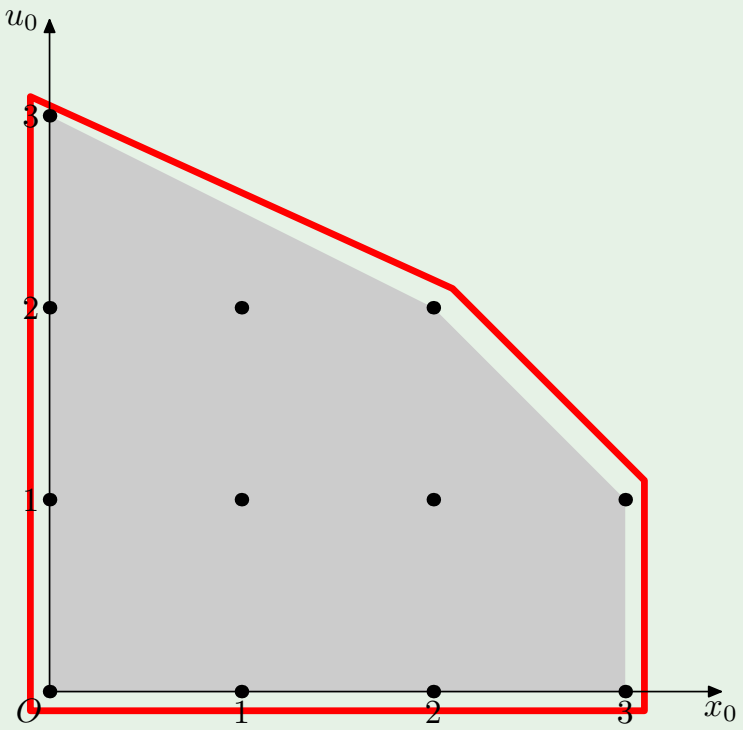


Figure: feasibility set

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$$\mathfrak{U}_t^{int,ad}(\hat{x}_t) = \Big\{u_t : u_s \in \mathfrak{U}_s \cap \mathbb{Z}^n, \quad x_t + \sum_{s=t}^{T-1} A_s u_s \in \mathfrak{X}_T\Big\};$$

$$\overline{\mathrm{conv}}\mathfrak{U}_t^{int,ad}(\hat{x}_t);$$

$$(\mathfrak{X},\mathfrak{U})_t^{int,ad} = \Big\{(x_t,u_t) : u_s \in \mathfrak{U}_s \cap \mathbb{Z}^n, \quad \sum_{t=0}^{T-1} A_t u_t \in \mathfrak{X}_T, x_t = \sum_{s=0}^{t-1} A_s u_s\Big\};$$

$$\overline{\mathrm{conv}}(\mathfrak{X},\mathfrak{U})_t^{int,ad};$$

$$\overline{\mathfrak{U}}_t^{int,ad}(\hat{x}_t) = \mathrm{proj}_{U_t}(\overline{\mathrm{conv}}(\mathfrak{X},\mathfrak{U})_t^{int,ad} \cap \{x_t = \hat{x}_t\});$$

## Example

$\hat{x}_0 = 1$:

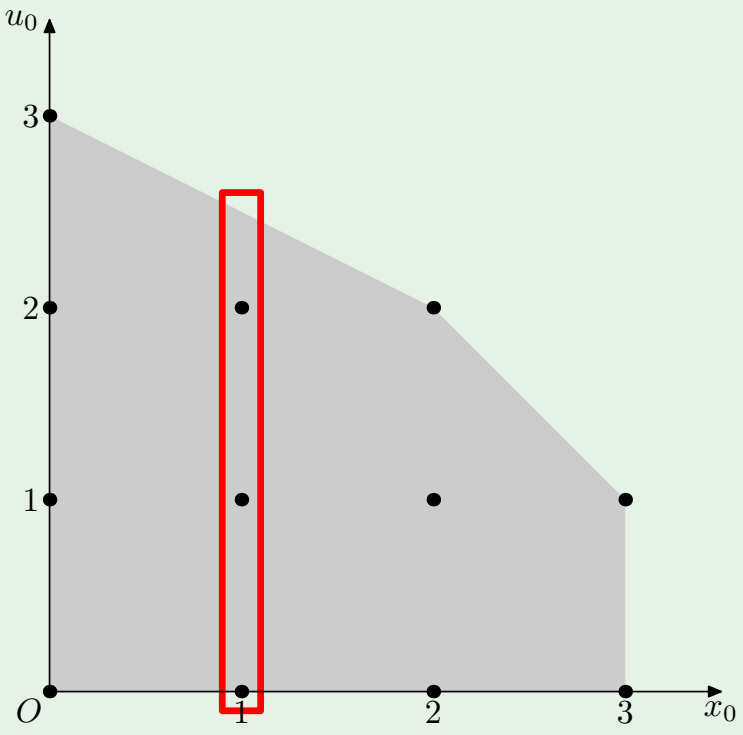


Figure: feasibility set

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$$\mathfrak{U}_t^{int,ad}(\hat{x}_t) = \Big\{u_t : u_s \in \mathfrak{U}_s \cap \mathbb{Z}^n, \quad x_t + \sum_{s=t}^{T-1} A_s u_s \in \mathfrak{X}_T\Big\};$$

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$$\overline{\mathrm{conv}}(\mathfrak{X},\mathfrak{U})_t^{int,ad};$$

$$\overline{\mathfrak{U}}_t^{int,ad}(\hat{x}_t) = \mathrm{proj}_{U_t}(\overline{\mathrm{conv}}(\mathfrak{X},\mathfrak{U})_t^{int,ad} \cap \{x_t = \hat{x}_t\});$$

$$\overline{\mathrm{conv}}\mathfrak{U}_t^{int,ad}(\hat{x}_t) \subset \overline{\mathfrak{U}}_t^{int,ad}(\hat{x}_t)$$

In general, the inclusion is strict.

## Example

$\hat{x}_0 = 1$:

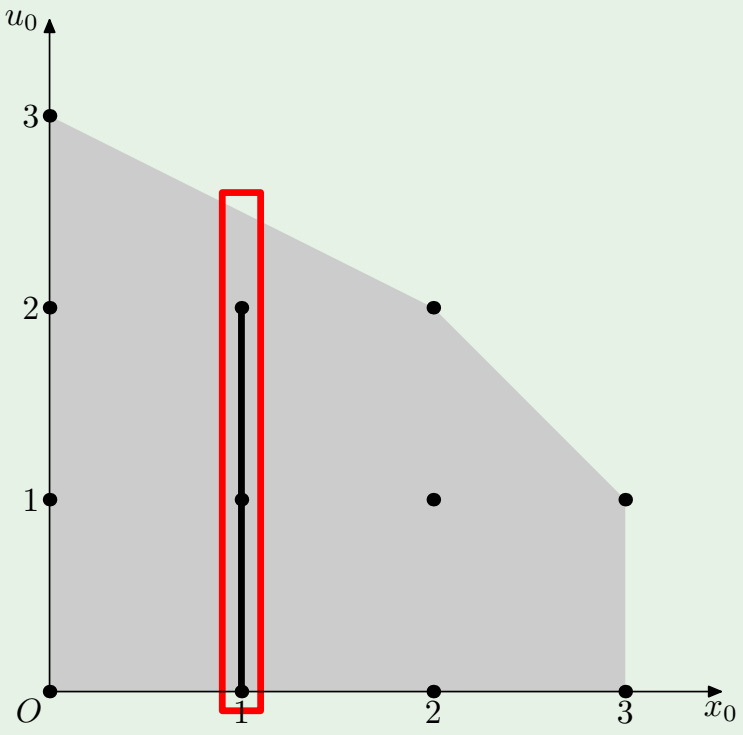
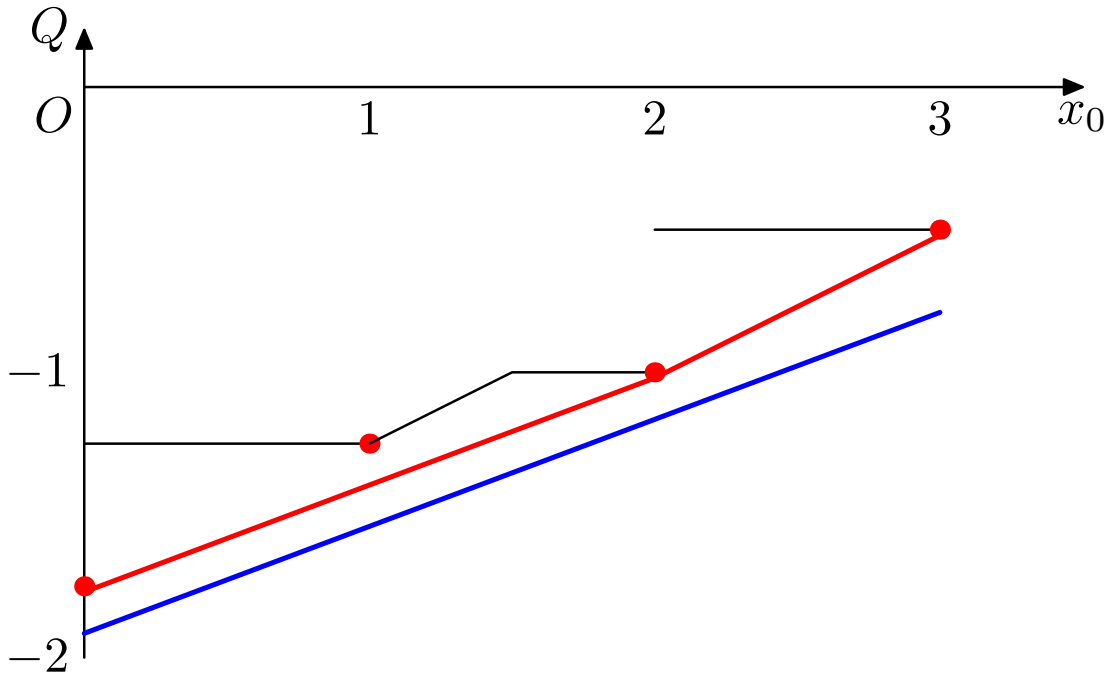



Figure: feasibility set

# Limitation of dual type algorithm

Using dual type algorithm, the best approximation is the convex envelop.




## Objective

- Approximate the convex envelop as good as possible.
- It is impossible to provide the exact optimal solution / optimal value.

# Algorithm

We solve the continuous relaxation problem with the cutting plane generated:

$$
\begin{array}{lll}
\min & c_t(\xi_t)u_t + \vartheta & \\
\text{s.t.} & x_{s+1} = x_s + A_s u_s, s = t, \dots, T-1, & \\
 & u_s \in \mathfrak{U}_s, x_T \in \mathfrak{X}_T, & \\
\text{(feasibility cut)} & \lambda_s^x x_s + \lambda_s^u u_s \leq \lambda_s^0, \quad \forall (\lambda_s^x, \lambda_s^u, \lambda_s^0) \in \mathcal{I}_s, s = t, \dots, T-1, & \\
\text{(optimality cut)} & \vartheta \geq \overline{\lambda}^x x_{t+1} + \overline{\lambda}^u u_t + \overline{\lambda}^0, \forall (\overline{\lambda}^x, \overline{\lambda}^u, \overline{\lambda}^0) \in \mathcal{O}_{t+1}(\xi_t); & (24)
\end{array}
$$

- If $u_t^*$ is integer, OK.
- Otherwise, study where find the $u_t^* \in U_t$.
  - $u_t^* \in \overline{\text{conv}}\mathfrak{U}_t^{int,ad}(x_t)$;
  - $u_t^* \in \overline{\mathfrak{U}}_t^{int,ad}(\hat{x}_t) \setminus \overline{\text{conv}}\mathfrak{U}_t^{int,ad}(\hat{x}_t)$;
  - $u_t^* \notin \overline{\mathfrak{U}}_t^{int,ad}(x_t)(\Leftrightarrow (x_t, u_t^*) \notin \overline{\text{conv}}(\mathfrak{X}, \mathfrak{U})_t^{int,ad})$.

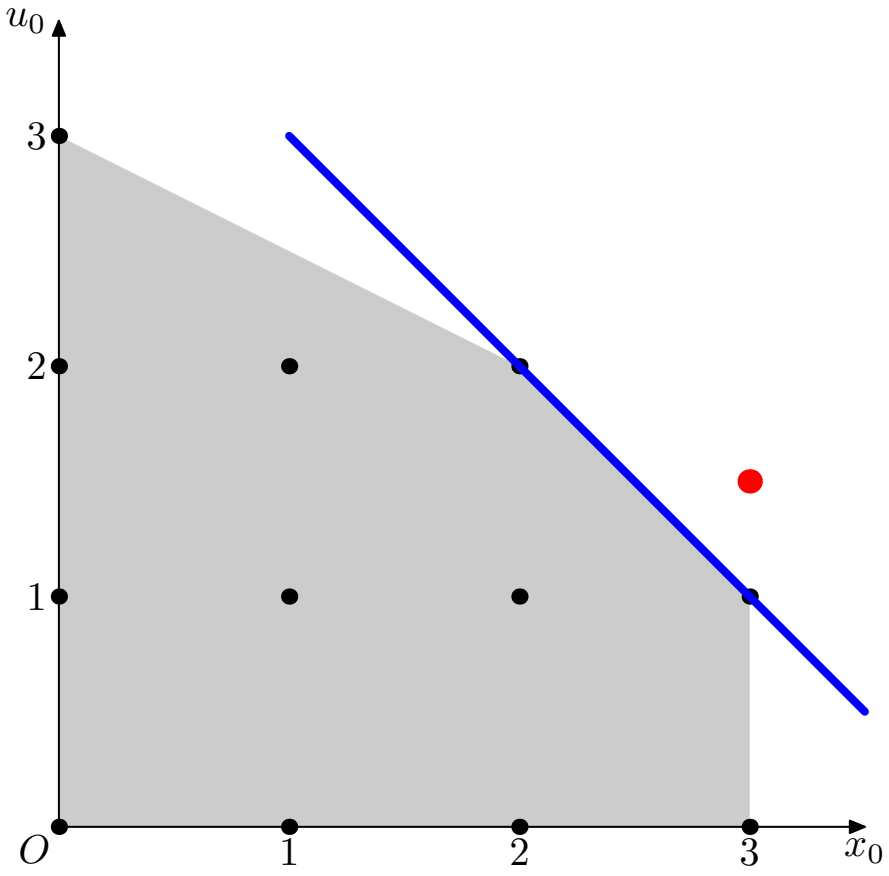
# Algorithm (cont.)

Case III : $u_t^* \notin \overline{\mathfrak{U}}_t^{int,ad}(x_t)(\Leftrightarrow (x_t, u_t^*) \notin \overline{\mathrm{conv}}(\mathfrak{X}, \mathfrak{U})_t^{int,ad})$.

## Example

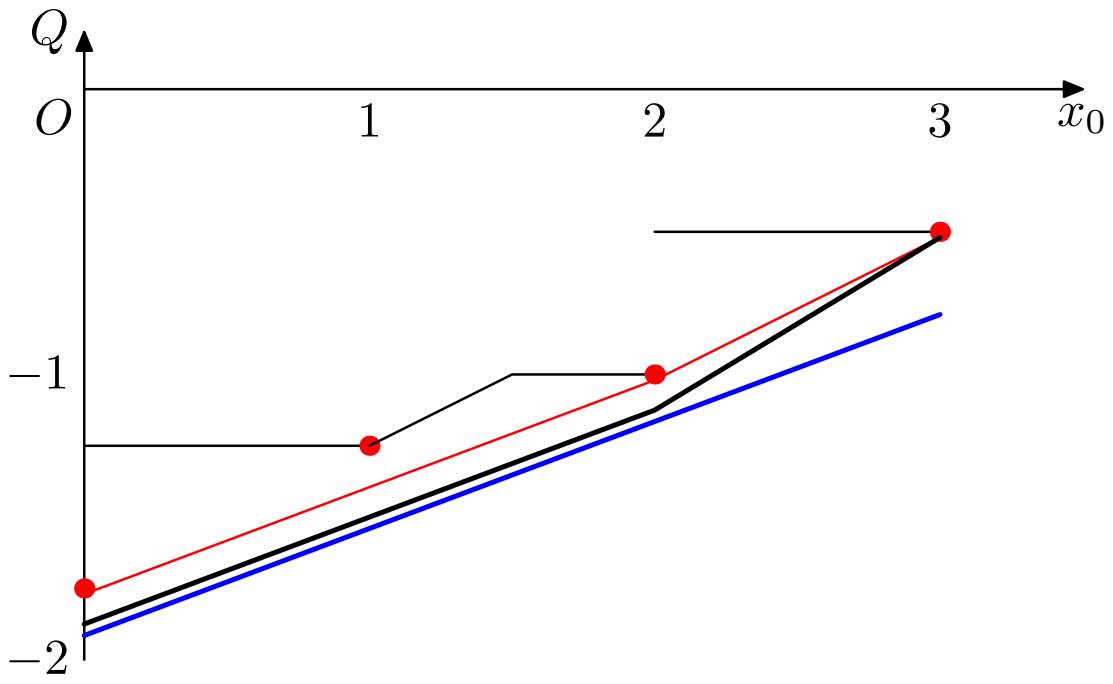
$\xi = -2$, $x_0 = 3$:

$$\begin{aligned} \min \quad & -2u_0 + \theta \\ \text{s.t.} \quad & u_0 \leq 3, x_1 \leq 6, \\ & x_1 = 2u_0 + 3, \\ & \theta \geq \frac{x_1 - 3}{2}, \\ & \theta \geq -\frac{x_1 - 3}{2}. \end{aligned}$$




Integer cutting plane can be generated.

# Improvement on example




- Red line: convex envelop of Bellman function of integer problem;
- Blue line: Bellman function of continuous relaxation problem;
- Black line: approximation by cutting plane method.

# Numerical result of LNG portfolio

$T = 6$

| | u.b. | $\sigma$(u.b.) | l.b. | det. counterpart value |
|---|---|---|---|---|
| Continuous | -31.9767 | 0.638715 | -32.3395 | -2.20492 |
| Integer (heuristic) | -13.8319 | 0.417892 | | 3.0384 |
| Integer (cut) | -17.0725 | 1.44108 | -31.8612 | 3.0384 |

Table: Comparison of optimal value of continuous problem and integer problem

That is the end !